*Indian J. pure appl. Math.,* **40**(1): 29-34, February 2009


# 2-FROBENIUS $\mathbb{Q}$-GROUPS

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(*Received* 5 *July* 2008; *after final revision* 26 *November* 2008; *accepted* 4 *December* 2008)

A finite group $G$ whose all irreducible characters are rational is called a $\mathbb{Q}$-group. In this note we obtain some results concerning the structure of a 2-Frobenius $\mathbb{Q}$-group.

**Key words : Frobenius group; $\mathbb{Q}$-group; solvable group**

## 1. INTRODUCTION

Let $G$ be a finite group. $G$ is called a rational group or a $\mathbb{Q}$-group if every complex irreducible character of $G$ is rational valued. Equivalently $G$ is a $\mathbb{Q}$-group if and only if every $x$ in $G$ is conjugate to $x^m$, where $m \in \mathbb{N}$ and $(o(x), m) = 1$. This means that for every $x \in G$ of order $n$, we have $N_G(\langle x\rangle)/C_G(\langle x\rangle) \cong \mathrm{Aut}(\langle x\rangle)$, where $\mathrm{Aut}(\langle x\rangle)$ is a group of order $\varphi(n)$ where $\varphi$ is the Euler function. The symmetric group $\mathbb{S}_n$ and the Weyl groups of complex Lie algebras are examples of $\mathbb{Q}$-groups [2]. $\mathbb{Q}$-groups have been studied extensively; but classifying finite $\mathbb{Q}$-groups is still an open research problem. It has been shown in [5] that if $G$ is a solvable $\mathbb{Q}$-group then $\pi(G) \subseteq \{2, 3, 5\}$, where $\pi(G)$ denotes the set of prime divisors of $|G|$. The structure of Frobenius $\mathbb{Q}$-groups have been described in [3]. It is proved in [7] that if $G$ is a solvable $\mathbb{Q}$-group then its Sylow 5-subgroup is normal and elementary abelian.

A finite group $G$ is called a 2-Frobenius group if it has a normal series $1 \lhd H \lhd K \lhd G$, where $G/H$ and $K$ are Frobenius groups with kernels $K/H$ and $H$, respectively, see [6]. Note that 2-Frobenius groups are always solvable.

Throughout this note, $G$ is always a 2-Frobenius $\mathbb{Q}$-group with normal series $1 \lhd H \lhd K \lhd G$. Suppose that $L/H$ and $M$ are Frobenius complements for $G/H$ and $K$, respectively. Therefore we have

$$K = HM, \quad H \cap M = 1, \quad K/H \cong M \tag{1}$$

$$G/H = (K/H)(L/H), \;\; (L/H) \cap (K/H) = 1, \;\; (G/H)/(K/H) \cong L/H. \tag{2}$$

By (1) we have $|K| = |M|\,|H|$ and by (2) $|G/H| = |(K/H)|\,|(L/H)|$. Since $|K/H| = |M|$, we have $|G/H| = |M|\,|L/H|$. Therefore $|G| = |M|\,|L|$. Also we have $M \cap L \subseteq K \cap L = H$, therefore $M \cap L \subseteq M \cap H = 1$ and we have

$$G = ML \quad , \quad M \cap L = 1. \tag{3}$$

Since $M$ is a Frobenius complement for $K$ we have $N_K(M) = M$, hence $M$ is not normal in $G$.

In this paper our aim is to find the structure of $G$. More precisely we will prove that if $G$ is a 2-Frobenius $\mathbb{Q}$-group, then there is a normal subgroup $N$ of $G$ such that $G/N \cong \mathbb{S}_4$.

## 2. Proof of The Main Result

Let $G$ be a 2-Frobenius $\mathbb{Q}$-group. Then $G$ is solvable, as 2-Frobenius groups are always solvable. By ([5], Corollary 2) we have $\pi(G) \subseteq \{2, 3, 5\}$. Suppose that $|G| = 2^a \cdot 3^b \cdot 5^c$.

$G$ is a $\mathbb{Q}$-group and quotients of $\mathbb{Q}$-groups are $\mathbb{Q}$-groups, therefore $G/H$ is a Frobenius $\mathbb{Q}$-group. By [3] Frobenius kernel of a Frobenius $\mathbb{Q}$-group can only be an elementary abelian $p$-group when $p = 3$ or $5$ and its Frobenius complement is isomorphic to $\mathbb{Z}_2$ or $Q_8$. Since $G/H$ is a Frobenius group with kernel $K/H$ and complement $L/H$, it follows that $K/H$ is an elementary abelian $p$-group for $p = 3$ or $5$ and $L/H$ is isomorphic to $\mathbb{Z}_2$ or $Q_8$.

*Lemma* 1 — Let $G$ be a 2-Frobenius $\mathbb{Q}$-group. Then $K/H \cong \mathbb{Z}_3$ and $L/H \cong \mathbb{Z}_2$.

PROOF : First we prove that $K/H \cong \mathbb{Z}_3$. Since $K/H \cong M$, it is sufficient to prove that $M$ is isomorphic to $\mathbb{Z}_3$. Suppose that $M$ is an elementary abelian $5$-group. Since $K$ is a Frobenius group with kernel $H$ and complement $M$ we have $(|M|, |H|) = 1$. Therefore $5 \nmid |H|$ (since we assumed that $5 \mid |M|$). Now $L/H$ is a 2-group, hence $5 \nmid |L|$. By (3) $G = ML$, hence $M$ is a Sylow 5-subgroup of $G$. Since $G$ is a solvable $\mathbb{Q}$-group, by ([7], Theorem 1.1) its Sylow 5-subgroup is normal and elementary abelian. Therefore $M$ is normal in $G$, a contradiction because $N_K(M) = M \neq K$. This contradiction shows that $M$ is an elementary abelian 3-group.

Assume that $|M| = 3^t$. By ([4], Theorem 3.1) any Sylow $p$-subgroup of a Frobenius complement is cyclic for odd prime $p$. Since $M$ is a Frobenius complement for $K$, its Sylow 3-subgroup is cyclic. Therefore $M$ is cyclic and elementary abelian, so $|M| = 3$. Therefore $M \cong \mathbb{Z}_3$. By the paragraph before the lemma $L/H \cong \mathbb{Z}_2$ or $Q_8$. If $L/H \cong Q_8$ then by ([4], Theorem 3.1) we have $|L/H| \mid (|K/H| - 1)$, therefore $8 \mid 3 - 1 = 2$, a contradiction. Thus $L/H$ can not be isomorphic to $Q_8$ and hence $L/H \cong \mathbb{Z}_2$. ■

*Lemma* 2 — If $G$ is a 2-Frobenius $\mathbb{Q}$-group with Sylow 3-subgroup $M$, then $N_G(M) \cong \mathbb{S}_3$ and $C_G(M) = M$.

PROOF : By Lemma 1 $M$ is a cyclic group of order 3. Now $K \lhd G$ and $M$ is a Sylow 3-subgroup of $K$. By using Frattini argument we have $G = KN_G(M)$. Since $K \cap N_G(M) = M$ and $K = MH$, by computing the order of $KN_G(M)$ we deduce $|N_G(M)| = |M| \times 2 = 6$. Now $M$ is a cyclic subgroup of a $\mathbb{Q}$ -group $G$, hence $N_G(M)/C_G(M) \cong Aut(M)$. Since $M \cong \mathbb{Z}_3$, we have that $|\mathrm{Aut}(M)| = \varphi(3) = 2$. So $|C_G(M)| = 3$ and $N_G(M)$ is a non-cyclic group of order 6, therefore $N_G(M) \cong \mathbb{S}_3$. ■

*Corollary* 1 — Let $G$ be a 2-Frobenius group. If $G$ is a $\mathbb{Q}$-group, then its center is trivial.

PROOF : Let $Z(G)$ denote the center of $G$. By ([9], p. 14, Corollary 14) the center of a $\mathbb{Q}$-group is an elementary abelian 2-group. As $Z(G) \subseteq C_G(M)$ is an elementary abelian 2-group and $|C_G(M)| = 3$ (Lemma 2), we have that $Z(G) = 1$. ■

*Lemma* 3 — Let $G$ be a 2-Frobenius $\mathbb{Q}$-group. Then $G' = K$ and $G$ has an irreducible character of degree 2.

PROOF : Suppose that $G$ is a $\mathbb{Q}$-group. Since $G/K \cong L/H$, by Lemma 1 $L/H \cong \mathbb{Z}_2$, hence $G/K \cong \mathbb{Z}_2$, therefore $G/K$ is abelian and by definition $G' \subseteq K$, hence $|G : G'| \geq 2$.

Now we use the theory of blocks investigated in [1] to deduce that $G' = K$ and obtain some important fact about $G$. By Lemma 1 a Sylow 3-subgroup $M$ of $G$ has order 3, hence 3 divides $|G|$ to the first power. Since $|C_G(M)| = 3$ and $|N_G(M)/C_G(M)| = 2$, the principal block $B_0(3)$ is the only block containing irreducible characters of $G$ whose degree are prime to 3. By ([1], p. 416), $B_0(3)$ contains three irreducible characters and because $|G/G'| \geq 2$ at least two of the irreducible characters contained in $B_0(3)$ are of degree 1. If $B_0 = \{\chi_1, \chi_2, \chi_3\}$ with $\chi_1(1) = \chi_2(1) = 1$, then by ([1], p. 417) there exist $\delta_2, \delta_3 = \pm 1$ such that $1 + \delta_2\chi_2(1) + \delta_3\chi_3(1) = 0$, where $\delta_2\chi_2(1) \equiv 1(\mathrm{mod} 3)$ and $\delta_3\chi_3(1) \equiv -2(\mathrm{mod} 3)$. From this equation it follows that $\delta_2 = 1$ and $\delta_3 = -1$ which implies $\chi_3(1) = 2$. Therefore $G$ has only 2 irreducible characters of degree 1 implying

$|G : G'| = 2$. From $|K| = \frac{1}{2}|G| = |G'|$ it follows that $G' = K$ and the lemma is proved. ■

*Lemma* 4 — Let $G$ be a 2-Frobenius $\mathbb{Q}$-group. Then $5 \nmid |G|$.

PROOF : Let $O_p(G)$ denotes the unique largest normal $p$-subgroup of $G$. We will prove that $O_2(G) = H_2$, where $H_2$ is a Sylow 2-subgroup of $H$. Since $H$ is a Frobenius kernel for $K$, we have that $H$ is nilpotent, and we deduce that $H_2$ is a characteristic subgroup of $H$. Since $H$ is normal in $G$, it follows that $H_2$ is normal in $G$ and by definition of $O_2(G)$ we deduce that $H_2 \subseteq O_2(G)$. Suppose that $H_2 \subsetneqq O_2(G)$. Since $|G : H| = 6$, $|G_2 : H_2| = 2$, where $G_2$ is a Sylow 2-subgroup of $G$. Since $H_2 \subsetneqq O_2(G)$, hence $O_2(G) = G_2$. Since $O_2(G)$ is normal in $G$, it follows that a Sylow 2-subgroup of $G$ is normal in $G$, a contradiction because solvable $\mathbb{Q}$-groups have self-normalizing Sylow 2-subgroups ([9], p. 16, Proposition 16). This contradiction shows that $H_2 = O_2(G)$. Now assume that $5 \mid |G|$ and $x$ be an element of order 5. By ([9], p. 18, Proposition 18), $\varphi(o(x))$ divide $|G : O_2(G)|$. Since $H_2 = O_2(G)$, we have $|G : O_2(G)| = |G : H_2| = 2 \cdot 3 \cdot 5^c$. Also we have $\varphi(o(x)) = \varphi(5) = 4$. Now we should have $4 = 2^2 | 2 \cdot 3 \cdot 5^c$ which is a contradiction. Therefore $5 \nmid |G|$. ■

*Corollary* 2 — Let $G$ be a 2-Frobenius $\mathbb{Q}$-group. Then $|G| = 2^a \cdot 3$ where $a$ is an odd integer.

PROOF : By using Lemma 4 we have $|G| = 2^a \cdot 3$. Now we prove that $a$ is an odd integer. Let $n_3(G)$ denote the number of Sylow 3-subgroups of $G$. By Sylow theorem $n_3(G) \equiv 1 \pmod 3$. Since $M$ is a Sylow 3-subgroup of $G$, and $|N_G(M)| = 6$, it follows that $n_3(G) = |G : N_G(M)| = 2^{a-1}$. Hence $2^{a-1} \equiv 1 \pmod 3$ and this occurs if and only if $a$ is odd. ■

*Corollary* 3 — Let $G$ be a 2-Frobenius $\mathbb{Q}$-group. Then $G'' = K' = H$.

PROOF : We have $G' = K$(Lemma 3) and $|G| = 2^a \cdot 3$ (Corollary 2). Hence $|K| = 2^{a-1} \cdot 3$. Since $M$ is a Sylow 3-subgroup of $K$ and $|M| = 3$, $C_K(M) = M$ and $|N_K(M) : M| = 1$, by using ([8], p. 122, Problem 7.6) we conclude that $K$ has at most $3 = 1 + \dfrac{3-1}{1}$ irreducible characters with degree not divisible by 3. Since degree of every linear character of $K$ is not divisible by 3, we have that $K$ has at most 3 linear characters, so $|K : K'| \leq 3 \quad (*)$. By Lemma 1 $K/H \cong \mathbb{Z}_3$, therefore $K' \subseteq H$ and $|K : K'| \geq 3$. Combining this relation with $(*)$ we obtain $|K : K'| = 3$ and since $K' \subseteq H$, it follows that $G'' = K' = H$. ■

***Theorem* 1** — *If $G$ is a 2-Frobenius $\mathbb{Q}$-group, then $G$ has a normal 2-subgroup $N$ such that $G/N \cong \mathbb{S}_4$.*

PROOF : By Lemma 3 $G$ has an irreducible character of degree 2. We may assume that $G$ has a normal subgroup $A$ such that $G/A$ is a primitive linear group of degree 2. By ([8], p. 257), the index of the center of $G/A$ in $G/A$ has order 12, 24 or 60. This implies that $G$ has a normal subgroup $N$ such that $|G/N| = 12, 24, 60$. By Lemma 4 we have $5 \nmid |G|$, hence $|G/N| = 12$ or 24. But it is easy to prove that the only $\mathbb{Q}$-groups of order 12 and 24 are $\mathbb{A}_4$ and $\mathbb{S}_4$, respectively. Since $\mathbb{A}_4$ does not have an irreducible character of degree $2$, the only possibility is $G/N \cong \mathbb{S}_4$ and the theorem is proved. ∎

*Corollary* 4 — The subgroup $N$ in Theorem 1 is contained in $H$.

PROOF : First we show that $N \subset K$. Since $K = G'$ (Lemma 3), we show that $N \subset G'$. So, suppose that $N$ is not contained in $G'$. Then $NG'$ is a normal subgroup of $G$ containing $G'$ properly. Since index of $G'$ in $G$ is 2, it follows that $G = NG'$. So $S_4 \cong G/N = NG'/N$ is isomorphic to a subgroup of $A_4$, which is a contradiction. Hence $N \subset K$.

Now we prove that $N \subset H$. $NH$ being a product of two normal subgroups of $K$ is a normal subgroup of $K$. If $N$ is not contained in $H$, then $H$ is proper in $NH$. Since $|K/H|$ is 3 (Lemma 1), this gives that $K = NH$. Since $N$ is a 2-group (Theorem 1), $K/H \cong NH/H \cong N/(N \cap H)$ is a 2-group. This contradiction completes the proof.

*Corollary* 5 — Let $G$ be a 2-Frobenius $\mathbb{Q}$-group with a normal series $1 \lhd H \lhd K \lhd G$. If $H$ is a minimal normal subgroup of $G$, then $G \cong \mathbb{S}_4$.

PROOF : By Theorem 1 there is a normal 2-subgroup $N$ of $G$ such that $G/N \cong S_4$. Since $|G : N| = 24$ and $|G : H| = 6$ and $N \subseteq H$ (Corollary 4), hence $N$ is a proper subgroup of $H$. By hypothesis $H$ is a minimal normal subgroup of $G$, therefore $N$ is trivial subgroup and we deduce that $G \cong S_4$.


## ACKNOWLEDGEMENT

The authors would like to thank the referee for carefully reading the earlier version of this paper and making useful suggestions.


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