Indian J. pure appl. Math., **40**(6): 353-355, December 2009

ON AN EXTENSION OF ITÔ′S THEOREM ON CONJUGACY CLASS SIZES

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(*Received* 16 *September* 2008; *after final revision* 10 *September* 2009;
accepted 15 *September* 2009)

Let G be a finite group. We extend Itô′s theorem on conjugacy class sizes which asserts that if 1 and $m > 1$ are the only lengths of conjugacy classes of a group G, then $G = P \times A$, where $P \in \mathrm{Syl}_p(G)$ and A is abelian. In particular, then m is a power of p. We show that if 1 and $m > 1$ are the only lengths of conjugacy classes of elements of primary and biprimary orders of a group G, then $G = P \times A$, where $P \in \mathrm{Syl}_p(G)$ and A is abelian. In particular, then m is a power of p.

Key words : Conjugacy class sizes; nilpotent groups; finite groups

1. INTRODUCTION

All groups considered in this paper are finite. If G is a group, then x^G denotes the conjugacy class containing x, $|x^G|$ the length of x^G(following Baer [1]) we call $\mathrm{Ind}_G(x) = |x^G| = |G : C_G(x)|$, the index of x in G). An element x is called primary element, if $|x| = p^a$, where p is a prime and a is an integer. Similarly, An element x is called biprimary element, if $|x| = p^a q^b$, where

[0]The research of the authors is supported by the National Natural Science Foundation of China (10771132), SGRC (GZ310), the Research Grant of Shanghai University and Shanghai Leading Academic Discipline Project (J50101).

p, q are primes and a, b are integers. The rest of our notation and terminology are standard. The reader may refer to ref. [2].

It is well known that there is a strong relation between the structure of a group and the lengths of its conjugacy classes and there exist several results studying the structure of a group under some arithmetical conditions on its conjugacy class lengths. For example, Itô shows in [3] that if the lengths of the conjugacy classes of a group G are $\{1, m\}$, then G is nilpotent, $m = p^a$ for some prime p and $G = P \times A$, with P a Sylow p-subgroup of G and $A \subseteq Z(G)$. The same author in [4] shows that G is solvable if the conjugacy class lengths of G are $\{1, n, m\}$. On the other hand, some other authors replace conditions for all conjugacy classes by conditions referring to only some conjugacy classes to investigate the structure of a finite group. For instance, Baer in [1] proves that a group G is solvable if its elements of prime power order have also prime power index. In [5], Shirong Li proves that G is solvable if the finite group G has exactly two conjugacy class lengths of elements of prime power order of G. Recently in [6], Berkovich and Kazarin prove that if indices of all elements of primary or biprimary orders of a non-abelian group G are powers of primes, then one and only one of the following holds: (a) $G = P \times A$, where $P \in \mathrm{Syl}_p(G)$ is non-abelian and A is abelian. (b) $G = F \times A$, where A is abelian, F is a nonnilpotent biprimary Hall subgroup of G with abelian Sylow subgroups. In this paper, we will replace conditions for all conjugacy classes by conditions referring to only some conjugacy classes to generalize the above It$\hat{o}'$s theorem in [3]. We put our emphasis on conjugacy class sizes of all elements of primary and biprimary orders of G and obtain the following main result.

Theorem — *If 1 and $m > 1$ are the only lengths of conjugacy classes of elements of primary and biprimary orders of a group G, then $G = P \times A$, where $P \in Syl_p(G)$ and A is abelian. In particular, then m is a power of p.*

2. The Proof of Main Result

In order to prove our main theorem, we need the following important lemma:

Lemma 2.1 — Let G be a group. Then the following two conditions are equivalent:

(i) 1 and $m > 1$ are the only lengths of conjugacy class of elements of primary and biprimary orders of G;

(ii) 1 and $m > 1$ are the only lengths of conjugacy class of G.

Proof : (i)$\Longrightarrow$(ii)

Let a be any p-element of index m and b be any q-element of $C_G(a)$, where $q \neq p$. Notice that $C_G(ab) = C_G(a) \bigcap C_G(b) \subseteq C_G(a)$ and since m is the largest conjugacy class size of elements of primary and biprimary orders of G, then $C_G(ab) = C_G(a)$, thus $C_G(a) \subseteq C_G(b)$. This implies that $b \in Z(C_G(a))$.

Now let x be any non-central element of G and write $x = x_1x_2 \cdots x_s, s \geq 3$, where the order of each x_i is a power of a prime $p_i(p_i \neq p_j$, if $i \neq j)$ and the x_i commute pairwise. As x is a non-central element of G, we know that there at least exists an element x_i such that x_i is non-central. Without the loss of generality, we can assume x_1 is non-central. Since $C_G(x) = C_G(x_1x_2 \cdots x_s) = C_G(x_1) \bigcap C_G(x_2 \cdots x_s) = C_G(x_1) \bigcap C_G(x_2) \bigcap \cdots \bigcap C_G(x_s) \subseteq C_G(x_1)$, we have that $x_i \in Z(C_G(x_1))$ by the previous argument, $i = 2, \cdots, s$. Hence we get that $C_G(x_1) \leq C_G(x_i), i = 2, \cdots, s$. Thus $C_G(x) = C_G(x_1x_2 \cdots x_s) = C_G(x_1) \bigcap C_G(x_2 \cdots x_s) = C_G(x_1) \bigcap C_G(x_2) \bigcap \cdots \bigcap C_G(x_s) = C_G(x_1)$. It follows that the conjugacy class size of x is equal to the conjugacy class size of x_1, that is, m.

(ii)$\Longrightarrow$(i) It is obvious.

The Proof of Theorem — By Lemma 2.1 the theorem is true.

ACKNOWLEDGEMENT

The authors are very grateful to the referee who read the manuscript carefully and provided a lot of valuable suggestions an useful comments. It should be said that we could not have polished the final version of this paper well without their outstanding efforts.

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