

SYMMETRIC POLYNOMIAL MATRICES AND VANDERMONDE MATRIX

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The factorizations of the elementary symmetric polynomial matrix and complete symmetric polynomial matrix are obtained by using the recurrence relations among their elements. The factorizations of Vandermonde matrix and its inverse are obtained by the elementary symmetric polynomial matrix and complete symmetric polynomial matrix. A connection between the inverse and transpose of the Vandermonde matrix is established.

Key words : Symmetric function; Pascal matrix; Stirling matrix; Vandermonde matrix; elementary symmetric polynomial matrix; complete symmetric polynomial matrix

1. INTRODUCTION

Let n be a positive integer. For integers $1 \leq r \leq (n+1)$, the r th elementary symmetric function or polynomial on the variables set $X = \{x_0, x_1, \dots, x_n\}$ is defined by (see [9, 15])

$$e_r(x_0, x_1, \dots, x_n) = \sum_{0 \leq k_1 < k_2 < \dots < k_r \leq n} x_{k_1} x_{k_2} \dots x_{k_r},$$

and the r th complete symmetric function or polynomial on the variables set $X = \{x_0, x_1, \dots, x_n\}$ is defined by

$$h_r(x_0, x_1, \dots, x_n) = \sum_{0 \leq k_1 \leq k_2 \leq \dots \leq k_r \leq n} x_{k_1} x_{k_2} \dots x_{k_r}.$$

We set $e_0(x_0, x_1, \dots, x_n) = 1$, $h_0(x_0, x_1, \dots, x_n) = 1$.

The generating function for the e_r is

$$E(t) = \sum_{r=0}^{n+1} e_r(x_0, x_1, \dots, x_n) t^r = \prod_{i=0}^n (1 + x_i t).$$

The generating function for the h_r is

$$H(t) = \sum_{r=0}^{\infty} h_r(x_0, x_1, \dots, x_n) t^r = \prod_{i=0}^n (1 - x_i t)^{-1}.$$

From $E(-t)H(t) = 1$ we have

$$\sum_{r=0}^m (-1)^{m-r} e_{m-r}(x_0, x_1, \dots, x_n) h_r(x_0, x_1, \dots, x_n) = 0, \quad (1)$$

for all $m \geq 1$.

It is not hard to obtain the following Lemma [11, 18]:

Lemma 1 — The elementary and complete symmetric functions satisfy the following recurrence relations

$$e_r(x_0, x_1, \dots, x_n) = e_r(x_0, x_1, \dots, x_{n-1}) + x_n e_{r-1}(x_0, x_1, \dots, x_{n-1}), \quad (2)$$

$$h_r(x_0, x_1, \dots, x_n) = h_r(x_0, x_1, \dots, x_{n-1}) + x_n h_{r-1}(x_0, x_1, \dots, x_{n-1}). \quad (3)$$

We denote $e_r^{(n+1)}(x_0, x_1, \dots, x_n) = e_r(x_0, x_1, \dots, x_n)$ and $h_r^{(n+1)}(x_0, x_1, \dots, x_n) = h_r(x_0, x_1, \dots, x_n)$. Then Eq. (2) and Eq. (3) can be rewritten $e_r^{(n+1)} = e_r^{(n)} + x_n e_{r-1}^{(n)}$ and $h_r^{(n+1)} = h_r^{(n)} + x_n h_{r-1}^{(n)}$. Repeated applications of (2) yields $e_r(x_0, x_1, \dots, x_n) = e_r(x_0, x_1, \dots, x_{n-1}) + x_n e_{r-1}(x_0, x_1, \dots, x_{n-2}) + x_n x_{n-1} e_{r-2}(x_0, x_1, \dots, x_{n-3}) + \dots + x_n x_{n-1} \dots x_{n-r+2} e_1(x_0, x_1, \dots, x_{n-r}) + x_n x_{n-1} \dots x_{n-r+1}$.

Similarly, repeated applications of (3) yields $h_r(x_0, x_1, \dots, x_n) = h_r(x_0, x_1, \dots, x_{n-1}) + x_n h_{r-1}(x_0, x_1, \dots, x_{n-1}) + x_n^2 h_{r-2}(x_0, x_1, \dots, x_{n-1}) + \dots + x_n^{r-1} h_1(x_0, x_1, \dots, x_{n-1}) + x_n^r$.

Thus, we have the following lemma:

Lemma 2 — The elementary and complete symmetric functions satisfy the recurrence,

$$e_r^{(n+1)} = \sum_{k=0}^r \left(\prod_{i=1}^k x_{n-i+1} \right) e_{r-k}^{(n-k)}, \quad (4)$$

$$h_r^{(n+1)} = \sum_{k=0}^r x_n^k h_{r-k}^{(n)}, \quad (5)$$

where empty product denotes 1.

The Pascal matrix and its various generalizations have been studied by many authors in the last two decades (see [1-3]). The Stirling matrices are also developed in recent years (see [1, 4, 5, 16-18]). The $n + 1$ by $n + 1$ Pascal matrix $\mathcal{P}_n$ is defined by [2, 20]

$$\mathcal{P}_n(i, j) = \begin{cases} \binom{i}{j}, & \text{if } n \geq i \geq j \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

The Stirling matrix of the first kind f_n and the Stirling matrix of the second kind $\mathcal{S}_n$ are defined respectively in [4, 16] as

$$f_n(i, j) = \begin{cases} s(i, j), & \text{if } n \geq i \geq j \geq 0, \\ 0, & \text{otherwise,} \end{cases}$$

$$\mathcal{S}_n(i, j) = \begin{cases} S(i, j), & \text{if } n \geq i \geq j \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

The elementary symmetric polynomial matrix and complete symmetric polynomial matrix are introduced as unified form of the Pascal matrix, Stirling matrices, q -Pascal matrix, and q -Stirling matrices [8, 11, 12], which are all lower triangular matrices. In [14, 19], the factorization of the elementary symmetric polynomial matrix and complete symmetric polynomial matrix are obtained by applying row generating functions or column generating functions. In [7], we have given a simpler alternative approach to the LU factorization and 1-banded factorization of the Vandermonde matrix by using symmetric functions.

In this paper, the factorizations of the elementary symmetric polynomial matrix and complete symmetric polynomial matrix are obtained by using the recurrence relations among their elements. The factorizations of the Pascal matrices, Stirling matrices, q -Pascal matrices, and q -Stirling matrices may be obtained as special cases of the symmetric polynomial matrices. The factorizations of Vandermonde matrix and its inverse are obtained by the elementary symmetric polynomial matrix and complete symmetric polynomial matrix. A connection between the inverse and transpose of the Vandermonde matrix is established.

2. THE ELEMENTARY SYMMETRIC POLYNOMIAL MATRIX AND COMPLETE SYMMETRIC POLYNOMIAL MATRIX

The $(n + 1) \times (n + 1)$ elementary symmetric polynomial matrix and complete symmetric polynomial matrix are defined as follows, respectively

$$(\mathcal{E}_n[x_0, x_1, \dots, x_n])(i, j) = \begin{cases} (-1)^{i-j} e_{i-j}(x_0, x_1, \dots, x_{i-1}), & \text{if } 0 \leq j \leq i \leq n, \\ 0, & \text{otherwise,} \end{cases}$$

$$(\mathcal{H}_n[x_0, x_1, \dots, x_n])(i, j) = \begin{cases} h_{i-j}(x_0, x_1, \dots, x_j), & \text{if } 0 \leq j \leq i \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

From (1), we derive that the matrix $\mathcal{E}_n[x_0, x_1, \dots, x_n]$ and $\mathcal{H}_n[x_0, x_1, \dots, x_n]$ are inverses of each other. In [14] and [19], factorizations of $\mathcal{E}_n[x_0, x_1, \dots, x_n]$ and $\mathcal{H}_n[x_0, x_1, \dots, x_n]$ are derived by applying generating functions. Now we deduce the factorizations using the recurrence relations given in Lemma 1 and Lemma 2.

Define the $(n+1) \times (n+1)$ matrices $\mathcal{F}_n, \mathcal{K}_n, \tilde{\mathcal{E}}_{n-1}[x_1, x_2, \dots, x_n]$, and $\tilde{\mathcal{H}}_{n-1}[x_0, x_1, \dots, x_{n-1}]$ by

$$\mathcal{F}_n(i, j) = \begin{cases} 1, & \text{if } i = j, \\ -x_0, & \text{if } i = j + 1, \\ 0, & \text{otherwise,} \end{cases} \quad \mathcal{K}_n(i, j) = \begin{cases} 1, & \text{if } i = j, \\ x_{i-1}, & \text{if } i = j + 1, \\ 0, & \text{otherwise,} \end{cases}$$

$$\tilde{\mathcal{E}}_{n-1}[x_1, x_2, \dots, x_n] = \begin{pmatrix} 1 & 0 \\ 0 & \mathcal{E}_{n-1}[x_1, x_2, \dots, x_n] \end{pmatrix},$$

$$\text{and } \tilde{\mathcal{H}}_{n-1}[x_0, x_1, \dots, x_{n-1}] = \begin{pmatrix} 1 & 0 \\ 0 & \mathcal{H}_{n-1}[x_0, x_1, \dots, x_{n-1}] \end{pmatrix}.$$

By using the recurrences of the elementary and complete symmetric functions (2) and (3), we can obtain the following theorem.

Theorem 1 — *The $(n+1) \times (n+1)$ elementary symmetric polynomial matrix $\mathcal{E}_n[x_0, x_1, \dots, x_n]$ and complete symmetric polynomial matrix $\mathcal{H}_n[x_0, x_1, \dots, x_n]$ can be factorized as*

$$\mathcal{E}_n[x_0, x_1, \dots, x_n] = \tilde{\mathcal{E}}_{n-1}[x_1, x_2, \dots, x_n] \mathcal{F}_n,$$

$$\mathcal{H}_n[x_0, x_1, \dots, x_n] = \tilde{\mathcal{H}}_{n-1}[x_0, x_1, \dots, x_{n-1}] \mathcal{K}_n.$$

Repeated applications of Theorem 1 yields the following result (see also [14, 19]).

Theorem 2 —

$$\mathcal{E}_n[x_0, x_1, \dots, x_n] = \mathcal{F}_n^{(n)} \mathcal{F}_n^{(n-1)} \dots \mathcal{F}_n^{(2)} \mathcal{F}_n^{(1)},$$

$$\mathcal{H}_n[x_0, x_1, \dots, x_n] = \mathcal{K}_n^{(n)} \mathcal{K}_n^{(n-1)} \dots \mathcal{K}_n^{(2)} \mathcal{K}_n^{(1)},$$

where $\mathcal{F}_n^{(1)} = \mathcal{F}_n, \mathcal{K}_n^{(1)} = \mathcal{K}_n$, and for $k > 1$, $\mathcal{F}_n^{(k)}$ and $\mathcal{K}_n^{(k)}$ are defined by,

$$\mathcal{F}_n^{(k)}(i, j) = \begin{cases} 1, & \text{if } i = j, \\ -x_{k-1}, & \text{if } i = j + 1 \text{ and } i \geq k, \\ 0, & \text{otherwise,} \end{cases} \quad \mathcal{K}_n^{(k)}(i, j) = \begin{cases} 1, & \text{if } i = j, \\ x_{i-k}, & \text{if } i = j + 1 \text{ and } i \geq k, \\ 0, & \text{otherwise.} \end{cases}$$

From (4) and (5), we get another factorization for the elementary symmetric polynomial matrix $\mathcal{E}_n[x_0, x_1, \dots, x_n]$ and complete symmetric polynomial matrix $\mathcal{H}_n[x_0, x_1, \dots, x_n]$, respectively.

Theorem 3 —

$$\mathcal{E}_n[x_0, x_1, \dots, x_n] = M_n \tilde{\mathcal{E}}_{n-1}[x_0, x_1, \dots, x_{n-1}], \quad (6)$$

$$\mathcal{H}_n[x_0, x_1, \dots, x_n] = \mathcal{N}_n \tilde{\mathcal{H}}_{n-1}[x_1, x_2, \dots, x_n]. \quad (7)$$

where $\mathcal{M}_n$ and $\mathcal{N}_n$ are defined by,

$$\mathcal{M}_n(i, j) = \begin{cases} 1, & \text{if } i = j, \\ (-1)^{i-j} \prod_{p=j}^{i-1} x_p, & \text{if } i > j, \\ 0, & \text{otherwise,} \end{cases} \quad \mathcal{N}_n(i, j) = \begin{cases} 1, & \text{if } i = j, \\ x_0^{i-j}, & \text{if } i > j, \\ 0, & \text{otherwise.} \end{cases}$$

PROOF : We first show that (i, j) entries on both sides of (6) are equal. Since the product of two lower triangular matrices is again lower triangular, it suffices to consider $i \geq j$. Using (4), we obtain

$$e_r(x_0, x_1, \dots, x_i) = \sum_{k=r}^{2r} \prod_{p=i+k+1-2r}^i x_p e_{k-r}(x_0, x_1, \dots, x_{i+k-2r-1}), \text{ for any } i \geq 1.$$

From the last equation, we get the (i, j) entry of the matrix $\mathcal{M}_n \tilde{\mathcal{E}}_{n-1}[x_0, x_1, \dots, x_{n-1}]$ is $\sum_{k=j}^i \mathcal{M}_n(i, k) \tilde{\mathcal{E}}_{n-1}[x_0, x_1, \dots, x_{n-1}](k, j) = \sum_{k=j}^i (-1)^{i-k} \prod_{p=k}^{i-1} x_p (-1)^{k-j} e_{k-j}(x_0, x_1, \dots, x_{k-2}) = (-1)^{i-j} \sum_{k=j}^i \prod_{p=k}^{i-1} x_p e_{k-j}(x_0, x_1, \dots, x_{k-2}) = (-1)^{i-j} \sum_{r=i-j}^{2(i-j)} \prod_{t=2j-i+k}^{i-1} x_t e_{r-i+j}(x_0, x_1, \dots, x_{2j-i+k-2}) = (-1)^{i-j} e_{i-j}(x_0, x_1, \dots, x_{i-1})$, which is equal to the (i, j) entry of the matrix on the left side of (6).

From (5), for $i \geq j$, the (i, j) entry of the matrix $\mathcal{N}_n \tilde{\mathcal{H}}_{n-1}[x_1, x_2, \dots, x_n]$ is $\sum_{k=j}^i \mathcal{N}_n(i, k) \tilde{\mathcal{H}}_{n-1}[x_1, x_2, \dots, x_n](k, j) = \sum_{k=j}^i x_0^{i-k} h_{k-j}(x_1, x_2, \dots, x_j) = h_{i-j}(x_0, x_1, x_2, \dots, x_j)$, which equals the (i, j) entry of the matrix on the left side of (7). This completes the proof. $\square$

The following result can be obtained by repeated applying Theorem 3.

Theorem 4 —

$$\mathcal{E}_n[x_0, x_1, \dots, x_n] = \mathcal{M}_n^{(1)} \mathcal{M}_n^{(2)} \dots \mathcal{M}_n^{(n-1)} \mathcal{M}_n^{(n)},$$

$$\mathcal{H}_n[x_0, x_1, \dots, x_n] = \mathcal{N}_n^{(1)} \mathcal{N}_n^{(2)} \dots \mathcal{N}_n^{(n-1)} \mathcal{N}_n^{(n)},$$

where $\mathcal{M}_n^{(1)} = \mathcal{M}_n$, $\mathcal{N}_n^{(1)} = \mathcal{N}_n$, and for $k \geq 2$, $\mathcal{M}_n^{(k)}$ and $\mathcal{N}_n^{(k)}$ are defined by,

$$\mathcal{M}_n^{(k)}(i, j) = \begin{cases} 1, & \text{if } i = j, \\ (-1)^{i-j} \prod_{p=j-k+1}^{i-k} x_p, & \text{if } i > j \text{ and } j \geq k-1, \\ 0, & \text{otherwise;} \end{cases}$$

$$\mathcal{N}_n^{(k)}(i, j) = \begin{cases} 1, & \text{if } i = j, \\ x_{k-1}^{i-j}, & \text{if } i > j \text{ and } j \geq k-1, \\ 0, & \text{otherwise.} \end{cases}$$

Example 1 — $\mathcal{E}_3[x_0, x_1, x_2, x_3]$

$$\begin{aligned}
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ -e_1(x_0) & e_0(x_0) & 0 & 0 \\ e_2(x_0, x_1) & -e_1(x_0, x_1) & e_0(x_0, x_1) & 0 \\ -e_3(x_0, x_1, x_2) & e_2(x_0, x_1, x_2) & -e_1(x_0, x_1, x_2) & e_0(x_0, x_1, x_2) \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ -x_0 & 1 & 0 & 0 \\ x_0x_1 & -(x_0 + x_1) & 1 & 0 \\ -x_0x_1x_2 & x_0x_1 + x_0x_2 + x_1x_2 & -(x_0 + x_1 + x_2) & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -x_1 & 1 & 0 \\ 0 & x_1x_2 & -(x_1 + x_2) & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ -x_0 & 1 & 0 & 0 \\ 0 & -x_0 & 1 & 0 \\ 0 & 0 & -x_0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -x_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -x_1 & 1 & 0 \\ 0 & 0 & -x_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ -x_0 & 1 & 0 & 0 \\ 0 & -x_0 & 1 & 0 \\ 0 & 0 & -x_0 & 1 \end{pmatrix} \\
&\mathcal{H}_3[x_0, x_1, x_2, x_3] = \begin{pmatrix} h_0(x_0) & 0 & 0 & 0 \\ h_1(x_0) & h_0(x_0, x_1) & 0 & 0 \\ h_2(x_0) & h_1(x_0, x_1) & h_0(x_0, x_1, x_2) & 0 \\ h_3(x_0) & h_2(x_0, x_1) & h_1(x_0, x_1, x_2) & h_0(x_0, x_1, x_2, x_3) \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ x_0 & 1 & 0 & 0 \\ x_0^2 & x_0 + x_1 & 1 & 0 \\ x_0^3 & x_0^2 + x_0x_1 + x_1^2 & x_0 + x_1 + x_2 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & x_0 & 1 & 0 \\ 0 & x_0^2 & x_0 + x_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ x_0 & 1 & 0 & 0 \\ 0 & x_1 & 1 & 0 \\ 0 & 0 & x_2 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & x_0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & x_0 & 1 & 0 \\ 0 & 0 & x_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ x_0 & 1 & 0 & 0 \\ 0 & x_1 & 1 & 0 \\ 0 & 0 & x_2 & 1 \end{pmatrix}.
\end{aligned}$$

Example 2 — For $n = 3$,

$$\mathcal{E}_3[x_0, x_1, x_2, x_3] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -x_0 & 1 & 0 & 0 \\ x_0x_1 & -x_1 & 1 & 0 \\ -x_0x_1x_2 & x_1x_2 & -x_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -x_0 & 1 & 0 \\ 0 & x_0x_1 & -(x_0 + x_1) & 1 \end{pmatrix}$$

$$\begin{aligned}
 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ -x_0 & 1 & 0 & 0 \\ x_0x_1 & -x_1 & 1 & 0 \\ -x_0x_1x_2 & x_1x_2 & -x_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -x_0 & 1 & 0 \\ 0 & x_0x_1 & -x_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -x_0 & 1 \end{pmatrix} \\
 &\mathcal{H}_3[x_0, x_1, x_2, x_3] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ x_0 & 1 & 0 & 0 \\ x_0^2 & x_0 & 1 & 0 \\ x_0^3 & x_0^2 & x_0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & x_1 & 1 & 0 \\ 0 & x_1^2 & x_1 + x_2 & 1 \end{pmatrix} \\
 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ x_0 & 1 & 0 & 0 \\ x_0^2 & x_0 & 1 & 0 \\ x_0^3 & x_0^2 & x_0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & x_1 & 1 & 0 \\ 0 & x_1^2 & x_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & x_2 & 1 \end{pmatrix}.
 \end{aligned}$$

It is not hard to see that the Pascal matrix $\mathcal{P}_n[x]$ and its inverse $\mathcal{P}_n^{-1}[x]$ are special cases of the complete symmetric polynomial matrix $\mathcal{H}_n[x_0, x_1, \dots, x_n]$ and elementary symmetric polynomial matrix $\mathcal{E}_n[x_0, x_1, \dots, x_n]$ with $x_i = x$ for $i = 0, 1, \dots, n$, respectively; and the Stirling matrices f_n and S_n are special cases of the elementary symmetric polynomial matrix and complete symmetric polynomial matrix with $x_i = i$ for $i = 0, 1, \dots, n$, respectively. Many of the triangular factorizations of Pascal matrices and Stirling matrices in [2, 4, 20] are special cases of our Theorem 2 or Theorem 4.

The q -binomial coefficient $\binom{n}{r}_q$ is defined by

$$\binom{n}{r}_q = \frac{[n][n-1] \cdots [n-r+1]}{[r][r-1] \cdots [1]}$$

for $n \geq r \geq 1$, and has the value 1 when $r = 0$ and the value zero otherwise, where the parameter q is a positive real number and $[r]$ denotes a q -integer, defined by

$$[r] = \begin{cases} \frac{1-q^r}{1-q}, & q \neq 1, \\ r, & q = 1. \end{cases}$$

Note that this reduces to the usual binomial coefficient when we set $q = 1$. The $(n+1) \times (n+1)$ q -Pascal matrix $\mathcal{Q}_n$ (see [11]) is defined by $\mathcal{Q}_n(i, j) = \binom{i}{j}_q$, and the elements of the matrix $\mathcal{Q}_n^{-1}$ is $\mathcal{Q}_n^{-1}(i, j) = (-1)^{i-j} q^{\binom{i-j}{2}} \binom{i}{j}_q$. The q -Pascal matrix and its inverse are obtained by substituting $x_i = q^i$ in the elementary symmetric polynomial matrix $\mathcal{E}_n[x_0, x_1, \dots, x_n]$ and complete symmetric polynomial matrix $\mathcal{H}_n[x_0, x_1, \dots, x_n]$, i.e., $\mathcal{Q}_n = \mathcal{H}_n[1, q, q^2, \dots, q^n]$ and $\mathcal{Q}_n^{-1} = \mathcal{E}_n[1, q, q^2, \dots, q^n]$. From Theorem 2 and Theorem 4, $\mathcal{Q}_n$ and $\mathcal{Q}_n^{-1}$ can be factorized into products of 1-banded and lower triangular matrices.

Example 3 — For $n = 3$ we have

$$\begin{aligned}
\mathcal{Q}_3 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & \frac{1-q^2}{1-q} & 1 & 0 \\ 1 & \frac{1-q^3}{1-q} & \frac{1-q^3}{1-q} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & q & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & q & 1 & 0 \\ 0 & 0 & q^2 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & q & 1 & 0 \\ 1 & q & q^2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & q & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \\
\mathcal{Q}_3^{-1} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ q & -\frac{1-q^2}{1-q} & 1 & 0 \\ -q^3 & q\frac{1-q^3}{1-q} & -\frac{1-q^3}{1-q} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -q^2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -q & 1 & 0 \\ 0 & 0 & -q & 1 \end{pmatrix} \\
&\cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ q & -q & 1 & 0 \\ -q^3 & q^3 & -q^2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & q & -q & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}.
\end{aligned}$$

The q -Stirling matrices (see [11]) are obtained by substituting x_i with $x_0 = 0$, and $x_i = [i] = 1 + q + \cdots q^{i-1}$ in the elementary symmetric polynomial matrix $\mathcal{E}_n[x_0, x_1, \dots, x_n]$ and complete symmetric polynomial matrix $\mathcal{H}_n[x_0, x_1, \dots, x_n]$, i.e., $\mathcal{H}_q^{(n)} = \mathcal{H}_n[0, [1], [2], \dots, [n]]$ and $\mathcal{E}_q^{(n)} = \mathcal{E}_n[0, [1], [2], \dots, [n]]$. From Theorem 2 and Theorem 4, $\mathcal{H}_q^{(n)}$ and $\mathcal{E}_q^{(n)}$ can be factorized into products of 1-banded and lower triangular matrices.

3. FACTORIZATIONS OF VANDERMONDE AND ITS INVERSE

The $(n+1) \times (n+1)$ Vandermonde matrix $\mathcal{V}_n$ is of the form

$$\mathcal{V}_n = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_0 & x_1 & x_2 & \cdots & x_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_0^n & x_1^n & x_2^n & \cdots & x_n^n \end{pmatrix},$$

where it is assumed that $x_0, x_1, \dots, x_n$ are pairwise distinct real numbers. Let $R_n[x]$ be the vector space of polynomial in x over the real number field R of degree at most n , then the sets $B_1 = \{1, x, x^2, \dots, x^n\}$ and $B_2 = \{[x]_0, [x]_1, \dots, [x]_n\}$ are both bases for $R_n[x]$, where $[x]_r = (x - x_0)(x - x_1) \cdots (x - x_{r-1})$ for $1 \leq r \leq n$, and $[x]_0 = 1$. Considering the relation between two bases, we can easily get the following lemma [17].

Lemma 3 — For $m = 0, 1, \dots, n$,

$$[x]_m = \sum_{k=0}^m (-1)^{m-k} e_{m-k}(x_0, x_1, \dots, x_{m-1}) x^k, \quad (8)$$

$$x^m = \sum_{k=0}^m h_{m-k}(x_0, x_1, \dots, x_k) [x]_k. \quad (9)$$

This lemma means that the transition matrix from the basis $B_1 = \{1, x, x^2, \dots, x^n\}$ to the basis $B_2 = \{[x]_0, [x]_1, \dots, [x]_n\}$ is the elementary symmetric polynomial matrix $\mathcal{E}_n[x_0, x_1, \dots, x_n]$, and the transition matrix from the basis B_2 back to the basis B_1 is the complete symmetric polynomial matrix $\mathcal{H}_n[x_0, x_1, \dots, x_n]$, i.e.,

$$\begin{pmatrix} [x]_0 \\ [x]_1 \\ \vdots \\ [x]_n \end{pmatrix} = \mathcal{E}_n \begin{pmatrix} 1 \\ x \\ \vdots \\ x^n \end{pmatrix}, \quad (10)$$

$$\begin{pmatrix} 1 \\ x \\ \vdots \\ x^n \end{pmatrix} = \mathcal{H}_n \begin{pmatrix} [x]_0 \\ [x]_1 \\ \vdots \\ [x]_n \end{pmatrix}. \quad (11)$$

From equation (9) we have $x_j^i = \sum_{k=0}^i h_{i-k}(x_0, x_1, \dots, x_k) [x]_k$, $i, j = 0, 1, 2, \dots, n$. Thus we have the following theorem (see [10, 17]).

Theorem 5 — The $(n+1) \times (n+1)$ Vandermonde matrix $\mathcal{V}_n = (x_j^i)$ can be factorized as $\mathcal{V}_n = \mathcal{L}_n \mathcal{U}_n$, where $\mathcal{L}_n$ is the complete symmetric polynomial matrix $\mathcal{H}_n[x_0, x_1, \dots, x_n]$, and $\mathcal{U}_n$ is an upper triangular matrix with $\mathcal{U}_n(i, j) = [x]_j^i = (x_j - x_0)(x_j - x_1) \cdots (x_j - x_{i-1})$, $i \leq j$.

It is well known that every $f(x) \in R_n[x]$ can be expressed as

$$f(x) = \sum_{j=0}^n f(x_j) g_j(x), \quad (12)$$

where

$$g_j(x) = \prod_{\substack{k=0 \\ k \neq j}}^n \frac{x - x_k}{x_j - x_k}, \quad j = 0, 1, \dots, n, \quad (13)$$

is the Lagrange basis [21] of $R_n[x]$.

Putting $f(x) = x^i$ in (12) for $i = 0, 1, \dots, n$, respectively, we obtain

$$\begin{pmatrix} 1 \\ x \\ \vdots \\ x^n \end{pmatrix} = \mathcal{V}_n \begin{pmatrix} g_0(x) \\ g_1(x) \\ \vdots \\ g_n(x) \end{pmatrix} = \mathcal{L}_n \mathcal{U}_n \begin{pmatrix} g_0(x) \\ g_1(x) \\ \vdots \\ g_n(x) \end{pmatrix}, \quad (14)$$

$$\begin{pmatrix} g_0(x) \\ g_1(x) \\ \vdots \\ g_n(x) \end{pmatrix} = \mathcal{V}_n^{-1} \begin{pmatrix} 1 \\ x \\ \vdots \\ x^n \end{pmatrix} = \mathcal{U}_n^{-1} \mathcal{E}_n \begin{pmatrix} 1 \\ x \\ \vdots \\ x^n \end{pmatrix}. \quad (15)$$

Combining (10), (11), (15) and (15), we get

$$\begin{pmatrix} [x]_0 \\ [x]_1 \\ \vdots \\ [x]_n \end{pmatrix} = \mathcal{U}_n \begin{pmatrix} g_0(x) \\ g_1(x) \\ \vdots \\ g_n(x) \end{pmatrix}, \quad (16)$$

$$\begin{pmatrix} g_0(x) \\ g_1(x) \\ \vdots \\ g_n(x) \end{pmatrix} = \mathcal{U}_n^{-1} \begin{pmatrix} [x]_0 \\ [x]_1 \\ \vdots \\ [x]_n \end{pmatrix}, \quad (17)$$

i.e., the transition matrices between the basis B_2 and the Lagrange basis are $\mathcal{U}_n$ and $\mathcal{U}_n^{-1}$.

Recall that the interpolating polynomial $p_n(x)$ for a function $f(x)$ at distinct points $x_0, x_1, \dots, x_n$ in Newton form is (see [6, 13]).

$$p_n(x) = f(x_0) + f[x_0, x_1][x]_1 + f[x_0, x_1, x_2][x]_2 + \dots + f[x_0, x_1, x_2, \dots, x_n][x]_n, \quad (18)$$

where $[x]_j = (x - x_0)(x - x_1) \cdots (x - x_{j-1})$ and

$$f[x_0, x_1, x_2, \dots, x_j] = \sum_{l=0}^j \frac{f(x_l)}{\prod_{\substack{k=0 \\ k \neq l}}^j (x_l - x_k)} \quad (19)$$

is the j th divided difference of $f(x)$ with respect to points $x_0, x_1, \dots, x_n$. When $f(x)$ is a polynomial of degree at most n , then $p_n(x) = f(x)$ (see [13]).

Theorem 6 — *The inverse of Vandermonde matrix $\mathcal{V}_n$ can be factorized as $\mathcal{V}_n^{-1} = \mathcal{U}_n^{-1} \mathcal{L}_n^{-1}$, where $\mathcal{L}_n^{-1}$ is the elementary symmetric polynomial matrix $\mathcal{E}_n[x_0, x_1, \dots, x_n]$, and $\mathcal{U}_n^{-1}$ is an upper triangular matrix with*

$$\mathcal{U}_n^{-1}(i, j) = g_i[x_0, x_1, \dots, x_j] = \frac{1}{\prod_{\substack{k=0 \\ k \neq i}}^j (x_i - x_k)}, \quad 0 \leq i \leq j \leq n. \quad (20)$$

PROOF : It follows from (17), (18) and (19) that

$$(\mathcal{U}_n)^{-1}(i, j) = g_i[x_0, x_1, \dots, x_j] = \sum_{l=0}^j \frac{g_i(x_l)}{\prod_{\substack{k=0 \\ k \neq l}}^j (x_l - x_k)} = \begin{cases} 0, & \text{if } j < i, \\ \frac{1}{\prod_{\substack{k=0 \\ k \neq i}}^j (x_i - x_k)}, & \text{if } j \geq i. \end{cases}$$

Where we used the facts that $g_i(x_i) = 1$ and $g_i(x_j) = 0$ for $i \neq j, i, j = 0, 1, \dots, n$.

Example 4 : If $n = 3$, then $\mathcal{V}_3 = \mathcal{L}_3 \mathcal{U}_3$, $\mathcal{V}_3^{-1} = \mathcal{U}_3^{-1} \mathcal{L}_3^{-1}$,

$$\mathcal{V}_3 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ x_0 & x_1 & x_2 & x_3 \\ x_0^2 & x_1^2 & x_2^2 & x_3^2 \\ x_0^3 & x_1^3 & x_2^3 & x_3^3 \end{pmatrix}, \mathcal{L}_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ x_0 & 1 & 0 & 0 \\ x_0^2 & x_0 + x_1 & 1 & 0 \\ x_0^3 & x_0^2 + x_0 x_1 + x_1^2 & x_0 + x_1 + x_2 & 1 \end{pmatrix},$$

$$\mathcal{U}_3 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & x_1 - x_0 & x_2 - x_0 & x_3 - x_0 \\ 0 & 0 & (x_2 - x_0)(x_2 - x_1) & (x_3 - x_0)(x_3 - x_1) \\ 0 & 0 & 0 & (x_3 - x_0)(x_3 - x_1)(x_3 - x_2) \end{pmatrix},$$

$$\mathcal{L}_3^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -x_0 & 1 & 0 & 0 \\ x_0 x_1 & -(x_0 + x_1) & 1 & 0 \\ -x_0 x_1 x_2 & x_0 x_1 + x_0 x_2 + x_1 x_2 & -(x_0 + x_1 + x_2) & 1 \end{pmatrix},$$

$$\mathcal{U}_3^{-1} = \begin{pmatrix} 1 & \frac{1}{x_0 - x_1} & \frac{1}{(x_0 - x_1)(x_0 - x_2)} & \frac{1}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} \\ 0 & \frac{1}{x_1 - x_0} & \frac{1}{(x_1 - x_0)(x_1 - x_2)} & \frac{1}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} \\ 0 & 0 & \frac{1}{(x_2 - x_0)(x_2 - x_1)} & \frac{1}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} \\ 0 & 0 & 0 & \frac{1}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)} \end{pmatrix},$$

$$\mathcal{V}_3^{-1} = \begin{pmatrix} \frac{-x_1 x_2 x_3}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} & \frac{x_1 x_2 + x_1 x_3 + x_2 x_3}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} & \frac{-(x_1 + x_2 + x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} & \frac{1}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} \\ \frac{-x_0 x_2 x_3}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} & \frac{x_0 x_2 + x_0 x_3 + x_2 x_3}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} & \frac{-(x_0 + x_2 + x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} & \frac{1}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} \\ \frac{-x_0 x_1 x_3}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} & \frac{x_0 x_1 + x_0 x_3 + x_1 x_3}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} & \frac{-(x_0 + x_1 + x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} & \frac{1}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} \\ \frac{-x_0 x_1 x_2}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)} & \frac{x_0 x_1 + x_0 x_2 + x_1 x_2}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)} & \frac{-(x_0 + x_1 + x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)} & \frac{1}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)} \end{pmatrix}.$$

4. A CONNECTION BETWEEN $\mathcal{V}_n^{-1}$ AND $\mathcal{V}_n^T$

In this section we give another factorization for the inverse of Vandermonde matrix $\mathcal{V}_n$. As a result, a connection between $\mathcal{V}_n^{-1}$ and $\mathcal{V}_n^T$ is established. Multiplying the right hand side of (13) out yields

$$g_i(x) = \prod_{\substack{k=0 \\ k \neq i}}^n \frac{x - x_k}{x_i - x_k} = \frac{1}{\prod_{\substack{k=0 \\ k \neq i}}^n (x_i - x_k)} \sum_{j=0}^n (-1)^{n-j} e_{n-j}(x_0, x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) x^j.$$

Thus from the last equation and (15), the (i, j) th entry of $\mathcal{V}_n^{-1}$ is (see also [7, 12, 21]):

$$\mathcal{V}_n^{-1}(i, j) = \frac{(-1)^{n-j} e_{n-j}(x_0, x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)}{\prod_{\substack{k=0 \\ k \neq i}}^n (x_i - x_k)}, \quad i, j = 0, 1, \dots, n. \quad (21)$$

Hence if we let $\mathcal{W}_n$ be the $(n+1) \times (n+1)$ matrix with

$$\mathcal{W}_n(i, j) = (-1)^{n-j} e_{n-j}(x_0, x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n), \quad i, j = 0, 1, \dots, n,$$

then from (21) we have

$$\mathcal{V}_n^{-1} = \mathcal{D}_n^{-1} \mathcal{W}_n, \quad (22)$$

where $\mathcal{D}_n = \text{diag}\left(\prod_{\substack{k=0 \\ k \neq 0}}^n (x_0 - x_k), \prod_{\substack{k=0 \\ k \neq 1}}^n (x_1 - x_k), \dots, \prod_{\substack{k=0 \\ k \neq n}}^n (x_n - x_k)\right)$.

Now we introduce an $(n+1) \times (n+1)$ Hankel matrix $\mathcal{X}_n$ as follows:

$$\mathcal{X}_n = \begin{pmatrix} (-1)^n e_n & (-1)^{n-1} e_{n-1} & (-1)^{n-2} e_{n-2} & \cdots & -e_1 & 1 \\ (-1)^{n-1} e_{n-1} & (-1)^{n-2} e_{n-2} & (-1)^{n-3} e_{n-3} & \cdots & 1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -e_1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

where the (i, j) th entry of $\mathcal{X}_n$ is

$$\mathcal{X}_n(i, j) = \begin{cases} (-1)^{n-i-j} e_{n-i-j}(x_0, x_1, \dots, x_n), & \text{if } 0 \leq i \leq n \text{ and } 0 \leq j \leq n-i, \\ 0, & \text{otherwise,} \end{cases}$$

we can prove the following results, which indicate a connection between $\mathcal{V}_n^{-1}$ and $\mathcal{V}_n^T$.

Theorem 7 — The matrix $\mathcal{X}_n$ links $\mathcal{W}_n$ and the Vandermonde matrix $\mathcal{V}_n$ by

$$\mathcal{W}_n^T = \mathcal{X}_n \mathcal{V}_n, \quad (23)$$

$$\mathcal{V}_n^{-1} = \mathcal{D}_n^{-1} \mathcal{V}_n^T \mathcal{X}_n^T. \quad (24)$$

PROOF : In order to verify (23) we need to show that

$$e_{n-i}(x_0, x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n) = \sum_{k=0}^{n-i} (-1)^k e_{n-i-k}(x_0, x_1, \dots, x_n) x_j^k, \quad i, j = 0, 1, \dots, n. \quad (25)$$

Firstly, we show it for $j = n$, i.e.,

$$e_{n-i}(x_0, x_1, \dots, x_{n-1}) = \sum_{k=0}^{n-i} (-1)^k e_{n-i-k}(x_0, x_1, \dots, x_n) x_n^k, \quad i = 0, 1, \dots, n. \quad (26)$$

We use induction on i . By routine computation, we have

$$(x_0 - t)(x_1 - t) \cdots (x_n - t) = \sum_{k=0}^{n+1} (-1)^k e_{n+1-k}(x_0, x_1, \dots, x_n) t^k,$$

Setting $t = x_n$ in the last equation, we get

$$\sum_{k=0}^{n+1} (-1)^k e_{n+1-k}(x_0, x_1, \dots, x_n) x_n^k = 0.$$

Thus,

$$\sum_{k=1}^{n+1} (-1)^k e_{n+1-k}(x_0, x_1, \dots, x_n) x_n^k = -e_{n+1}(x_0, x_1, \dots, x_n) = -e_n(x_0, x_1, \dots, x_{n-1}) x_n,$$

which yields

$$\sum_{k=1}^{n+1} (-1)^k e_{n+1-k}(x_0, x_1, \dots, x_n) x_n^{k-1} = -e_n(x_0, x_1, \dots, x_{n-1}),$$

or equivalently

$$e_n(x_0, x_1, \dots, x_{n-1}) = \sum_{k=0}^n (-1)^k e_{n-k}(x_0, x_1, \dots, x_n) x_n^k, \quad (27)$$

i.e., (26) holds for $i = 0$.

We now assume that (26) is true for $i = m$, namely we have

$$e_{n-m}(x_0, x_1, \dots, x_{n-1}) = \sum_{k=0}^{n-m} (-1)^k e_{n-m-k}(x_0, x_1, \dots, x_n) x_n^k, \quad (28)$$

and we proceed to check it for $i = m + 1$.

Using (28), we have

$$e_{n-m}(x_0, x_1, \dots, x_{n-1}) = \sum_{k=1}^{n-m} (-1)^k e_{n-m-k}(x_0, x_1, \dots, x_n) x_n^k + e_{n-m}(x_0, x_1, \dots, x_n),$$

and substituting

$$e_{n-m}(x_0, x_1, \dots, x_{n-1}) - e_{n-m}(x_0, x_1, \dots, x_n) = -x_n e_{n-m-1}(x_0, x_1, \dots, x_{n-1}) \text{ yields}$$

$$-x_n e_{n-m-1}(x_0, x_1, \dots, x_{n-1}) = \sum_{k=1}^{n-m} (-1)^k e_{n-m-k}(x_0, x_1, \dots, x_n) x_n^k.$$

Dividing the last equation by x_n we obtain

$$-e_{n-m-1}(x_0, x_1, \dots, x_{n-1}) = \sum_{k=1}^{n-m} (-1)^k e_{n-m-k}(x_0, x_1, \dots, x_n) x_n^{k-1},$$

which is equivalent to

$$e_{n-m-1}(x_0, x_1, \dots, x_{n-1}) = \sum_{k=0}^{n-m-1} (-1)^k e_{n-m-1-k}(x_0, x_1, \dots, x_n) x_n^k. \quad (29)$$

This shows that (26) holds for $i = m + 1$. By induction, (26) holds for all $i = 0, 1, \dots, n$. Considering the symmetric structures of columns of the matrices $\mathcal{W}_n$ and $\mathcal{V}_n$, we see that the proof of (25) for $0 \leq j \leq n - 1$ is similar. Substituting (23) into (22) we get (24). This completes the proof.

Theorem 8 — Let $\mathcal{X}_n$ be the Hankel matrix defined by elementary symmetric functions as before, then

$$\mathcal{X}_n^{-1} = \mathcal{V}_n \mathcal{D}_n^{-1} \mathcal{V}_n^T, \quad (30)$$

$$\mathcal{X}_n^{-1}(i, j) = h_{i+j-n}(x_0, x_1, \dots, x_n). \quad (31)$$

Hence $\mathcal{X}_n^{-1}$ is a Hankel matrix whose entries are complete symmetric functions.

PROOF : Eq. (30) follows directly from (24). From (30), $\mathcal{X}_n^{-1}(i, j) = \sum_{k=0}^n x_k^i \cdot \frac{x_k^j}{\prod_{\substack{l=0 \\ l \neq k}}^n (x_k - x_l)} = \sum_{k=0}^n \frac{x_k^{i+j}}{\prod_{\substack{l=0 \\ l \neq k}}^n (x_k - x_l)} = f[x_0, x_1, \dots, x_n]$ for $f(x) = x^{i+j}$ by (19). Thus if $i + j < n$, then $\mathcal{X}_n^{-1}(i, j) = 0$ and if $i + j \geq n$ then $\mathcal{X}_n^{-1}(i, j) = h_{i+j-n}(x_0, x_1, \dots, x_n)$ by [p. 450, Exercise 7.4].

Example 5 — If $n = 3$, then $\mathcal{V}_3 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ x_0 & x_1 & x_2 & x_3 \\ x_0^2 & x_1^2 & x_2^2 & x_3^2 \\ x_0^3 & x_1^3 & x_2^3 & x_3^3 \end{pmatrix},$

$$\mathcal{W}_3 = \begin{pmatrix} -x_1x_2x_3 & x_1x_2 + x_1x_3 + x_2x_3 & -(x_1 + x_2 + x_3) & 1 \\ -x_0x_2x_3 & x_0x_2 + x_0x_3 + x_2x_3 & -(x_0 + x_2 + x_3) & 1 \\ -x_0x_1x_3 & x_0x_1 + x_0x_3 + x_1x_3 & -(x_0 + x_1 + x_3) & 1 \\ -x_0x_1x_2 & x_0x_1 + x_0x_2 + x_1x_2 & -(x_0 + x_1 + x_2) & 1 \end{pmatrix},$$

$$\mathcal{X}_3 = \begin{pmatrix} -(x_0x_1x_2 + x_0x_1x_3 + x_0x_2x_3 + x_1x_2x_3) & x_0x_1 + x_0x_2 + x_0x_3 + x_1x_2 + x_1x_3 + x_2x_3 & -(x_0 + x_1 + x_2 + x_3) & 1 \\ x_0x_1 + x_0x_2 + x_0x_3 + x_1x_2 + x_1x_3 + x_2x_3 & -(x_0 + x_1 + x_2 + x_3) & 1 & 0 \\ -(x_0 + x_1 + x_2 + x_3) & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathcal{D}_3 = \begin{pmatrix} (x_0-x_1)(x_0-x_2)(x_0-x_3) & 0 & 0 & 0 \\ 0 & (x_1-x_0)(x_1-x_2)(x_1-x_3) & 0 & 0 \\ 0 & 0 & (x_2-x_0)(x_2-x_1)(x_2-x_3) & 0 \\ 0 & 0 & 0 & (x_3-x_0)(x_3-x_1)(x_3-x_2) \end{pmatrix},$$

$$\text{and } \mathcal{X}_3^{-1} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & h_1 \\ 0 & 1 & h_1 & h_2 \\ 1 & h_1 & h_2 & h_3 \end{pmatrix}, \text{ where } h_1 = x_0 + x_1 + x_2 + x_3, h_2 = x_0^2 + x_1^2 + x_2^2 + x_3^2 +$$

$$x_0x_1 + x_0x_2 + x_0x_3 + x_1x_2 + x_1x_3 + x_2x_3, h_3 = x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_0^2x_1 + x_0^2x_2 + x_0^2x_3 + x_1^2x_0 + x_1^2x_2 + x_1^2x_3 + x_2^2x_0 + x_2^2x_1 + x_2^2x_3 + x_3^2x_0 + x_3^2x_1 + x_3^2x_2 + x_0x_1x_2 + x_0x_1x_3 + x_0x_2x_3 + x_1x_2x_3.$$

It is easy to check that $\mathcal{W}_3^T = \mathcal{X}_3 \mathcal{V}_3$ and $\mathcal{X}_3^{-1} = \mathcal{V}_3 \mathcal{D}_3^{-1} \mathcal{V}_3^T$.

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