Indian J. Pure Appl. Math., **41**(1): 199-224, February 2010

NEUMANN BOUNDARY CONDITION FOR A NON-AUTONOMOUS HAMILTON-JACOBI EQUATION IN A QUARTER PLANE

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Dedicated to the memory of Professor Shyam Lal Yadava

Abstract We consider Hamilton-Jacobi equation $u_t + H(u,\, u_x) = 0$ in the quarter plane and study initial boundary value problems with Neumann boundary condition on the line $x = 0$. We assume that $p \to H(u,p)$ is convex, positively homogeneous of degree one. In general, this problem need not admit a continuous viscosity solution when $s \to H(s,p)$ is non increasing. In this paper, explicit formula for a viscosity solution of the initial boundary value problem is given for the cases $s \to H(s,p)$ is non decreasing as well as $s \to H(s,p)$ is non increasing.

Key words Hamilton-Jacobi equation, viscosity solution, Neumann boundary condition.

1. Introduction

Let $I \subset I\!R$ be an open interval and let $\Omega = I \times (0,\infty)$. Let $H \in C(I\!R \times I\!R)$, $u_0 \in W^{1,\infty}(I)$ and $\lambda \in C(\partial I \times (0,\infty))$. Consider the following initial boundary value problem for the Hamilton Jacobi equation:

$$\begin{array}{rcll} u_t + H(u,\, u_x) &=& 0 & \text{in} \quad \Omega \\ u(x,0) &=& u_0(x) & \text{for} \quad x \in I \\ \frac{\partial u}{\partial x} &=& \lambda & \text{on} \quad \partial I \times (0,\infty). \end{array} \tag{1.1}$$

In the case of a pure initial value problem i.e., $I = I\!R$, when $\dfrac{\partial H}{\partial u} \geq 0$, then the existence and uniqueness of viscosity solution of (1.1) is well studied by Crandall and Lions [4, 7, 13]. In the case of a initial boundary value problem (1.1), then it is not always possible to prescribe boundary condition on the boundary and hope to have a solution and hence these boundary conditions has to be understood in a relaxed sense. Lions [13, 14] has discussed this problem in detail and showed the existence and uniqueness of the solution.

When $H(s,p) = H(p)$, H is super linear growth and convex in p, in fact one can derive the explicit formula for the solution of (1.1). In the case of a pure initial value problem i.e., $I = I\!R$, Hopf [8] has obtained an explicit formula for a viscosity solution as a minimization over controlled paths of a certain functional with controlled paths being straight lines. In the case of a quarter plane i.e., $I = I\!R_+$, problem was studied by Joseph and Gowda [10] and obtained an explicit formula for a viscosity solution of (1.1) similar to Hopf's formula. Here the controlled paths being certain piecewise linear curves having atmost three linear curves. Using explicit formula, Lax-Olenik in the case of pure initialvalue problem and Joseph-Gowda [10] in the case of a quarter plane, entropy solution $\vartheta = u_x$ of the following scalar conservation law has been derived.

$$\begin{aligned} \vartheta_t + H(\vartheta)_x &= 0 && \text{in} \quad \Omega \\ \vartheta(x,0) &= \frac{\partial u_0}{\partial x}(x) && \text{for} \quad x \in I \\ \vartheta &= \lambda && \text{on} \quad \partial I \times (0,\infty). \end{aligned} \tag{1.2}$$

The problem (1.2) has also been studied by Lefloch [12] where a formula was derived which contains a solution of a variational inequality which may not be solvable explicitly. Furthermore using the numerical scheme, Joseph and Gowda [11] obtained a similar formula for a solution of (1.1) in the quarter plane when $H(p) = |p|$.

Now what happens when H depends on "u" and $p \to H(u,p)$ is convex?. In general, obtaining an explicit formula for a solution is quite difficult. For the pure initiavalue problem, under the assumption $\dfrac{\partial H}{\partial u} \geq 0, p \to H(u,p)$ is convex, positively homogeneous of degree one, Barron, Jensen and Liu [6] have obtained an explicit formula for a viscosity solution.

If $\dfrac{\partial H}{\partial u}$ is non positive, this problem is not well studied in the literature. In general this problem admits discontinuous solution. In [1, 2, 3] explicit formulas for viscosity solution is given in the case of pure initial value problem under the assumptions either $p \to H(u,p)$ is convex and positively homogeneous of degree greater than one or $p \to H(u,p)$ is convex, positively homogeneous of degree one and finitely many oscillations in u.

Now the question is, under the same assumptions as above, does there exists an explicit formula for a solution of initial boundary value problem (1.1)?. In this paper, for the case $p \to H(u,p)$ is convex, positively homogeneous of degree one and either $\dfrac{\partial H}{\partial u} \geq 0$ or ≤ 0 explicit formula for a viscosity solution is obtained for initial boundary value problem. In section 2, we state our main results. In section 3, the necessary preliminaries are given. In section 4, we prove the result for the case $u \to H(u,p)$ is non decreasing. In section 5, proof of the result for the case $u \to H(u,p)$ is non increasing is given.

2. Main Results

Hereafter we assume that $u_0 \in W^{1,\infty}(\mathbb{R}+), \lambda \in C(\mathbb{R}^+)$ and H satisfies:

(H_0) : $p \to H(u,p)$ is convex, positively homogeneous of degree one and $H(u,p) > 0$ for all $p \neq 0$. Assume that η is a diffeomorphism from $\mathbb{R}$ onto $\mathbb{R}$ where

$$\eta(u) = \int_0^u \frac{d\theta}{H(\theta,1)}. \tag{2.1}$$

Difinition 2.1. (Admissible curves). Let $0 \leq s < t$ and $\beta \in C([s,t], \overline{\mathbb{R}}_+)$. Then β is called an admissible curve if the following holds.

1. β consists of atmost three linear curves (Fig. 1a., Fig. 1b.).
2. Let $s = t_3 \leq t_2 \leq t_1 \leq t_0 = t$ be such that for $i = 1,2,3, \quad \beta_i = \beta|_{[t_i,t_{i-1}]}$ be the linear parts of β. Then $\beta_2 = 0$.

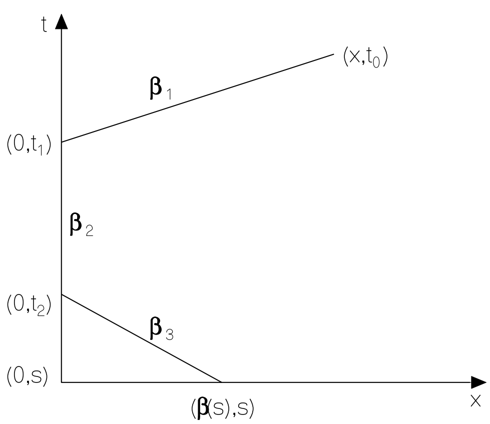

Fig. 1a. With β_2 is nonempty

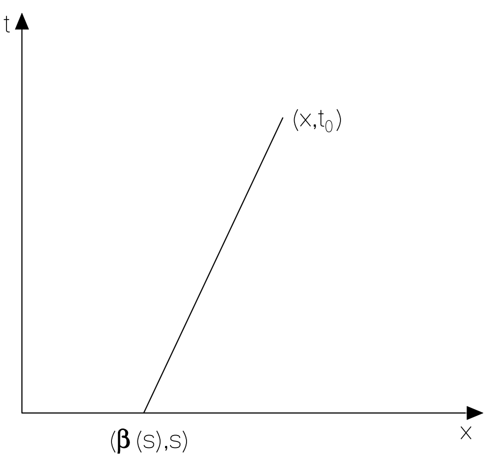

Fig. 1b. With β_2 is empty

Represent an admisible curve $\beta = \{\beta_1, \beta_2, \beta_3\}$ and $\dot\beta = (\dot\beta_1, \dot\beta_2, \dot\beta_3)$, where $\dot\beta_i = \dfrac{d\beta_i}{d\theta}$ and $|\dot\beta| = \max_{1\leq i\leq 3}\left\{|\dot\beta_i|\right\}$. Let $M > 0,\ x \in \overline{\mathbb{R}}_+,\ 0 \leq s < t$ and define

$$c(x,s,t) \ = \ \left\{\beta \in C([s,t], \overline{\mathbb{R}}_+); \beta(t) = x, \beta \text{ is an admissible curve}\right\} \tag{2.2}$$

$$c_M(x,s,t) = \left\{\beta \in c(x,s,t); \quad |\dot\beta| \leq M\right\}, \tag{2.3}$$

$$c(x,t) = c(x,0,t),\ c_M(x,t) = c_M(x,0,t). \tag{2.4}$$

For $a\,, b \ \in \mathbb{R}$, denote $a \bigvee b \ = \ \max(a,b)$. The main results are divided into two parts.

Part 2.1. Assume that H satisfies the following assumption.
$(H_1)\quad u \mapsto H(u,p)$ is non decreasing for each p.
Associated to H, let h be its quasi convex dual defined by

$$h(q) = \inf\left\{\gamma; H(\gamma,p) \geq pq \quad \forall |p| \leq 1\right\} . \tag{2.5}$$

For $(x,t) \in \overline{I\!R}_+ \times I\!R_+$ and $0 \leq s < t,\ \beta = (\beta_1,\beta_2,\beta_3) \in c(x,s,t),$ define

$$\rho_3(\beta) \;=\; \eta(h(\dot{\beta}_3)), \tag{2.6}$$

$$\rho_1(\beta) \;=\; \eta(h(\dot{\beta}_1)), \tag{2.7}$$

$$\int_\beta \lambda^+ \;=\; \int_{\{\beta=0\}} \lambda^+(\theta)d\theta, \tag{2.8}$$

with $\rho_i(\beta) = -\infty$ if $\beta_i = \phi$. Then we have the following :

Theorem 2.1. *Let $(x,t) \in \overline{I\!R}_+ \times I\!R_+$ and define*

$$\eta(u(x,t)) = \inf_{\beta\in c(x,t)} \left\{\eta(u_0(\beta(0))) \bigvee \rho_3(\beta) - \int_\beta \lambda^+\right\} \bigvee \rho_1(\beta) . \tag{2.9}$$

Then for any $T > 0,\ u \in W^{1,\infty}(\overline{I\!R}_+ \times [0,T])$ and u is a viscosity solution of (1.1) with $I = I\!R_+$. Furthermore minimizer in (2.9) exist.

Part 2.2. Assume that H satisfies the following assumption.
$(H_2)\quad u \mapsto H(u,p)$ is non increasing for each fixed p.
Associated to H, let h be the quasi concave dual defined by

$$h(q) = \sup\left\{\nu; H(\nu,p) \geq pq \,\forall\, |p| \leq 1\right\} . \tag{2.10}$$

Let $M > 0,\ 0 \leq s < t$ and $(x,t) \in \overline{I\!R}_+ \times I\!R_+$. For a function ϑ on $\overline{I\!R_+ \times I\!R_+}$, define

$$\begin{aligned}\underline{A}(\vartheta,x,s,t) \;&=\; \Big\{\beta \in c(x,s,t); \vartheta(\beta(s),s) \leq h(\dot{\beta}_3),\\ &\qquad \eta(\vartheta(\beta(s),s)) - \textstyle\int_\beta \lambda^+ \leq \eta(h(\dot{\beta}_1))\Big\}\end{aligned} \tag{2.11}$$

$$\begin{aligned}\overline{A}(\vartheta,x,s,t) \;&=\; \Big\{\beta \in c(x,s,t); \vartheta(\beta(s),s) < h(\dot{\beta}_3),\\ &\qquad \eta(\vartheta(\beta(s),s)) - \textstyle\int_\beta \lambda^+ < \eta(h(\dot{\beta}_1))\Big\}\end{aligned} \tag{2.12}$$

$$\begin{aligned}\underline{A}_M(\vartheta,x,s,t) \;&=\; \Big\{\beta \in \underline{A}(\vartheta,x,s,t) : |\dot{\beta}| \leq M\Big\}\\ \overline{A}_M(\vartheta,x,s,t) \;&=\; \Big\{\beta \in \overline{A}(\vartheta,x,s,t) : |\dot{\beta}| \leq M\Big\},\end{aligned}$$

$$\left\{\begin{aligned}\underline{A}(\vartheta,x,t) \;&=\; \underline{A}(\vartheta,x,0,t),\ \overline{A}(\vartheta,x,t) = \overline{A}(\vartheta,x,0,t),\\ \underline{A}_M(\vartheta,x,t) \;&=\; \underline{A}_M(\vartheta,x,0,t),\ \overline{A}_M(\vartheta,x,t) = \overline{A}_M(\vartheta,x,0,t),\end{aligned}\right. \tag{2.13}$$

with $h(\dot{\beta}_i) = \infty$ if $\beta_i = \phi$. Then we have the following.

Theorem 2.2. *Define*

$$\underline{u}(x,t) = \inf \eta^{-1}\left\{\eta(u_0(\beta(0))) - \int_\beta \lambda^+ ; \beta \in \underline{A}(u_0,x,t)\right\}, \qquad (2.14)$$

$$\overline{u}(x,t) = \inf \eta^{-1}\left\{\eta(u_0(\beta(0))) - \int_\beta \lambda^+ , \beta \in \overline{A}(u_0,x,t)\right\}. \qquad (2.15)$$

Let $I = I\!R_+$. Then $\underline{u}$ is a lower semicontinuous function, $\overline{u}$ is an upper semicontinuous function, $\underline{u}^* = \overline{u}$ and $\overline{u}_* = \underline{u}$. Furthermore $\underline{u}$ and $\overline{u}$ are viscosity solutions of (1.1).

Remark 2.1. The assumption on η being diffeomorphism from $I\!R$ onto $I\!R$ is not required. With a slight modification, all theorems are still valid except for the fact that, in general, minimiser may not exist.

3. Preliminaries

In this section we recall some definitions and some known facts from [5, 6, 9 and 14].

Definition 3.1. *Let $\Omega \subset I\!R^n$ be a domain and V be a locally bounded function. For $x \in \Omega$ define*

$$\begin{aligned} V^*(x) &= \lim_{r\to 0} \sup\{V(z) : |x-z| \le r\} \\ V_*(x) &= \lim_{r\to 0} \inf\{V(z) : |x-z| \le r\}. \end{aligned}$$

Then it follows easily that V^* and V_* are upper and lower semicontinuous functions respectively. Also $V_* \le V^*$.

Definition 3.2. Let U be a locally bounded function in $\overline{I\!R_+ \times I\!R_+}$.
1. U is said to be a *subsolution* of (1.1) if for any $(x_0,t_0) \in \overline{I\!R}_+ \times I\!R_+, \varphi \in C^1(\overline{I\!R}_+ \times I\!R_+)$ such that (x_0,t_0) is a local maximum for $U^* - \varphi$ with $U^*(x_0,t_0) = \varphi(x_0,t_0)$. Then at (x_0,t_0) either $\varphi_t + H(\varphi,\varphi_x) \le 0$ or if $x_0 = 0$ and $\varphi_t + H(\varphi,\varphi_x) > 0$ then $\varphi_x(x_0,t_0) \ge \lambda(t_0)$. Further more $\lim_{t\to 0} U^*(x,t) \le u_0(x)$.
2. U is said to be a *super solution* of (1.1) if for any $(x_0,t_0) \in \overline{I\!R}_+ \times I\!R_+, \varphi \in C^1(\overline{I\!R}_+ \times I\!R_+)$ such that (x_0,t_0) is a local minimum for $U_* - \varphi$ with $U_*(x_0,t_0) = \varphi(x_0,t_0)$. Then at (x_0,t_0), either $\varphi_t + H(\varphi,\varphi_x) \ge 0$ or if $x_0 = 0$ and $\varphi_t + H(\varphi,\varphi_x) < 0$, then $\varphi_x(x_0,t_0) \le \lambda(t_0)$. Further more $\lim_{t\to 0} U_*(x,t) \ge u_0(x)$.
3. U is said to be a *viscosity solution* of (1.1) if U is both sub and super solution of (1.1).

Now recall some properties of quasiconvex (concave) dual of H.

Lemma 3.2. *Let H satiefies (H_0) of section 2. Then*
(a) Suppose $u \mapsto H(u,p)$ is non decreasing and h be its quasi convex dual defined by (2.5). Then h satisfies
(A_1) *h is a lower semicontinuous function. For any $q_1, q_2 \in I\!R$, $t \in [0,1], h(tq_1 + (1-t)q_2) \le \max\{h(q_1), h(q_2)\}$,*
(A_2) $h(0) = -\infty, \lim_{|q|\to\infty} h(q) = \infty$,
(A_3) $H(s,p) = \sup\{pq; h(q) \le s\}$.

(b) Suppose $u \mapsto H(u,p)$ *is non increasing and* h *be its quasi concave dual defined by (2.10). Then*
(A_4) h *is an upper semicontinuous function. For any* $q_1, q_2 \in I\!R,\ t \in [0,1],\ h(tq_1 + (1-t)q_2) \geq \min\{h(q_1), h(q_2)\}$,
(A_5) $h(0) = +\infty, \lim_{|q|\to\infty} h(q) = -\infty$,
(A_6) $H(s,p) = \sup\{pq;\ s \leq h(q)\}$.

Proofs of (A_1) to (A_3) follow from Theorem 2.1 and Lemma 2.2 of [6]. (A_4) to (A_6) follow from (A_1) to (A_3) applied to the Hamiltonian $\tilde{H}(s,p) = H(-s,p)$.

4. Proofs of Theorems

In this section we prove the Theorem 2.1 which proves explicit formula for the solution given by (2.9) is a viscosity solution when $s \to H(s,p)$ is non decreasing. In order to prove Theorem 2.1, first we prove the following Lemmas.

Lemma 4.1. *(Existence of a minimizer) Let* $T > 0,\ 0 \leq s < t \leq T,\ x \in \overline{I\!R}_+$. *Let* $\vartheta : \overline{I\!R_+ \times I\!R_+} \to I\!R$ *be a function such that* $|\vartheta|_T = \sup_{\overline{I\!R}_+ \times [0,T]} |\vartheta(x,t)| < \infty$. *Define*

$$\eta(V(x,t)) = \inf_{\beta \in c(x,s,t)} \left\{ \eta(\vartheta(\beta(s),s)) \bigvee \rho_3(\beta) - \int_\beta \lambda^+ \right\} \bigvee \rho_1(\beta). \tag{4.1}$$

Then
(1) $|V|_T < \infty$ *and there exist* $M = M(\vartheta, \lambda, T) > 0$ *such that*

$$\eta(V(x,t)) = \inf_{C_M(x,s,t)} \left\{ \eta(\vartheta(\beta(s),s)) \bigvee \rho_3(\beta) - \int_\beta \lambda^+ \right\} \bigvee \rho_1(\beta). \tag{4.2}$$

(2) Suppose $x \mapsto \vartheta(x,\theta)$ *is lower semicontinuous for each* θ, *then there exist a minimizer* β *of (4.2).*

Proof : Since $\eta(\pm\infty) = \pm\infty$ and $h(0) = -\infty$, by taking $\beta(\theta) = x$ for $\theta \in [s,t]$ to obtain $\eta(V(x,t)) \leq \eta(\vartheta(x,s)) - \int_\beta \lambda^+ \leq \eta(\vartheta(x,s))$. Hence $V(x,t) \leq |\vartheta|_T$. Also for any $\beta \in c(x,s,t)$

$$\begin{aligned} \left\{ \eta(\vartheta(\beta(s),s)) \bigvee \rho_3(\beta) - \int_\beta \lambda^+ \right\} \bigvee \rho_1(\beta) \quad &\geq \quad \eta(\vartheta(\beta(s),s)) \\ &- \quad \int_\beta \lambda^+ \geq \eta(-|\vartheta|_T) - T|\lambda^+|_T, \end{aligned}$$

and hence $|V|_T < \infty$. Since $h(p) \to \infty$ as $|p| \to \infty$ and hence can choose a $M > 0$ such that whenever $|\dot{\beta}| \geq M$ then either $\rho_1(\beta) > \eta(|V|_T)$ or $\rho_3(\beta) > \eta(|\vartheta|_T) + \eta(|V|_T) + T|\lambda^+|_T$. In this case

$$(\eta(\vartheta(\beta(s),s)) \bigvee \rho_3(\beta) - \int_\beta \lambda^+) \bigvee \rho_1(\beta) > \eta(|V|_T)$$

and hence there exist $M > 0$ such that $|\dot{\beta}| \leq M$ and (4.2) holds. This proves (1).

Let $\{\beta_k\}$ be a minimizing sequence for (4.2). Since $|\dot{\beta}_k| \leq M$ and hence there exist a subsequence still denoted by $\{\beta_k\}$ converging to β in $W^{1,\infty}$. Since $x \mapsto \vartheta(x,\theta)$ and $p \mapsto h(p)$ are lower semicontinuous functions, it follows that

$$\begin{aligned}(\eta(\vartheta(\beta(s),s)) &\bigvee \rho_3(\beta) - \textstyle\int_\beta \lambda^+)\bigvee\rho_1(\beta) \geq \eta(V(x,t)) \\ &= \lim_{k\to\infty}\left\{\eta(\vartheta(\beta_k(s),s))\bigvee\rho_3(\beta_k) - \textstyle\int_{\beta_k}\lambda^+\right\}\bigvee\rho_1(\beta_k) \\ &\geq (\eta(\vartheta(\beta(s),s))\bigvee\rho_3(\beta) - \textstyle\int_\beta\lambda^+)\bigvee\rho_1(\beta),\end{aligned}$$

and hence β is a minimizer. This proves the Lemma. □

Lemma 4.2. *(Dynamic programming principle) Let u be as in (2.9). Let $x \in \overline{I\!R}_+$ and $0 \leq s < t$, then*

$$\eta(u(x,t)) = \inf_{\beta\in c(x,s,t)}(\eta(u(\beta(s),s))\bigvee\rho_3(\beta) - \int_\beta \lambda^+)\bigvee\rho_1(\beta). \tag{4.3}$$

Proof : Let $\eta(\vartheta(x,t))$ denote the right hand side of (4.3). Let $\beta = (\beta_1,\beta_2,\beta_3) \in c(x,t)$ defined on the partition $0 = t_3 \leq t_2 \leq t_1 \leq t_0 = t$. Let $\beta^{(1)} = \beta|_{[0,s]} \in c(\beta(s),s)$ and $\beta^{(2)} = \beta|_{[s,t]} \in c(x,s,t)$.

Suppose $t_1 < s < t$, then $\rho_3(\beta^{(2)}) = -\infty,\ \rho_1(\beta^{(2)}) = \rho_1(\beta),\ \rho_3(\beta^{(1)}) = \rho_3(\beta)$ and $\rho_1(\beta^{(1)}) = \rho_1(\beta)$. Hence

$$\begin{aligned}\eta(\vartheta(x,t)) &\leq \eta(u(\beta^{(2)}(s),s))\bigvee\rho_1(\beta^{(2)}) \\ &\leq (\eta(u_0(\beta^{(1)}(0)))\bigvee\rho_3(\beta^{(1)}) - \textstyle\int_{\beta^{(1)}}\lambda^+)\bigvee\rho_1(\beta^{(2)}) \\ &= (\eta(u_0(\beta(0)))\bigvee\rho_3(\beta) - \textstyle\int_\beta\lambda^+)\bigvee\rho_1(\beta).\end{aligned}$$

If $t_2 \leq s \leq t_1$, then $\rho_3(\beta^{(2)}) = -\infty, \rho_1(\beta^{(2)}) = \rho_1(\beta), \rho_1(\beta^{(1)}) = -\infty,\ \rho_3(\beta^{(1)}) = \rho_3(\beta), \int_{\beta^{(1)}}\lambda^+ + \int_{\beta^{(2)}}\lambda^+ = \int_\beta\lambda^+$. Hence

$$\begin{aligned}\eta(\vartheta(x,t)) &\leq (\eta(u(\beta^{(2)}(s),s)) - \textstyle\int_{\beta^{(2)}}\lambda^+)\bigvee\rho_1(\beta^{(2)}) \\ &\leq (\eta(u_0(\beta^{(1)}(0)))\bigvee\rho_3(\beta^{(1)}) - \textstyle\int_{\beta^{(1)}}\lambda^+ - \textstyle\int_{\beta^{(2)}}\lambda^+)\bigvee\rho_1(\beta^{(2)}) \\ &= (\eta(u_0(\beta(0)))\bigvee\rho_3(\beta) - \textstyle\int_\beta\lambda^+)\bigvee\rho_1(\beta).\end{aligned}$$

If $0 \leq s < t_2$, then $\rho_1(\beta^{(2)}) = \rho_1(\beta), \quad \rho_3(\beta^{(1)}) = \rho_3(\beta), \rho_1(\beta^{(1)}) = -\infty, \rho_3(\beta^{(2)}) = \rho_3(\beta)$. Hence

$$\begin{aligned}\eta(\vartheta(x,t)) &\leq (\eta(u(\beta^{(2)}(s),s))\bigvee\rho_3(\beta^{(2)}) - \textstyle\int_{\beta^{(2)}}\lambda^+)\bigvee\rho_1(\beta^{(2)}) \\ &\leq (\eta(u_0(\beta^{(1)}(0)))\bigvee\rho_3(\beta^{(1)})\bigvee\rho_3(\beta^{(2)}) - \textstyle\int_{\beta^{(2)}}\lambda^+)\bigvee\rho_1(\beta^{(2)}) \\ &= (\eta(u_0(\beta(0)))\bigvee\rho_3(\beta) - \textstyle\int_\beta\lambda^+)\bigvee\rho_1(\beta).\end{aligned}$$

Now taking infimum over β to obtain $\eta(\vartheta(x,t)) \leq \eta(u(x,t))$ and hence $\vartheta(x,t) \leq u(x,t)$. From Lemma 4.1, $|\vartheta|_T < \infty$ for any $T > 0$. Hence for every $\epsilon > 0$, there exist $\beta \in c(x,s,t)$ and $\alpha \in c(\beta(s),s)$ such that

$$\begin{aligned}\eta(\vartheta(x,t)) &\geq (\eta(u(\beta(s),s))\bigvee\rho_3(\beta) - \textstyle\int_\beta\lambda^+)\bigvee\rho_1(\beta) - \epsilon \\ \eta(u(\beta(s),s)) &= (\eta(u_0(\alpha(0)))\bigvee\rho_3(\alpha) - \textstyle\int_\alpha\lambda^+)\bigvee\rho_1(\alpha).\end{aligned}$$

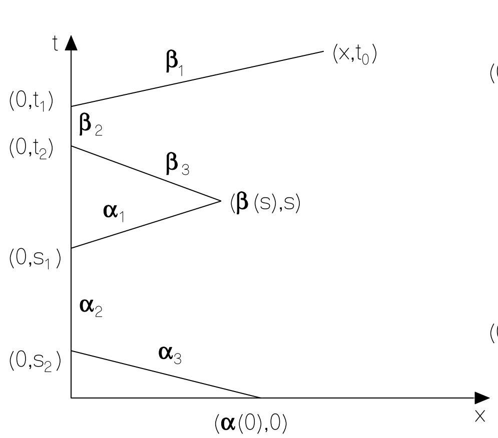

Fig. 2a.

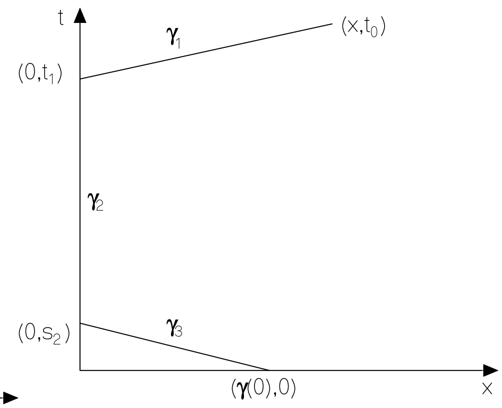

Fig. 2b. With $\gamma_1 = \beta_1 = \alpha_3$ and γ_2 is the line joining the points (0, t_1) and (0, s_2)

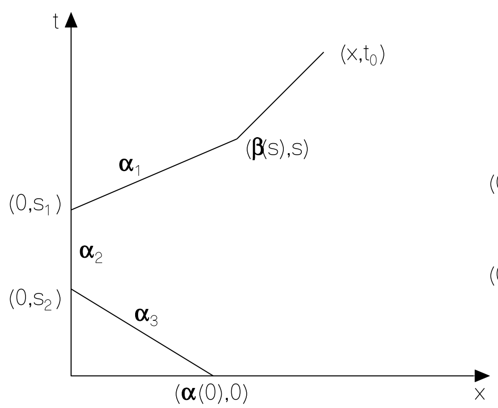

Fig. 3a. With β_2 is empty

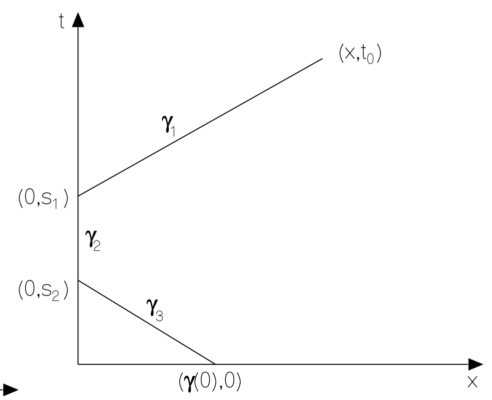

Fig. 3b. With g_1 is the line joining the points (x, t_0) and (0, s_1), $\gamma_2 = \alpha_2$ and $\gamma_3 = \alpha_3$

Suppose $[t_2, t_1] \neq \phi, [s_2, s_1] \neq \phi$ (Fig. 2a.), then define $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ by $\gamma_1 = \beta_1, \gamma_2|_{[s_2,t_1]} = 0, \gamma_3 = \alpha_3$ (Fig. 2b.). Then $\rho_1(\gamma) = \rho_1(\beta), \rho_3(\gamma) = \rho_3(\alpha)$ and $\int_\beta \lambda^+ + \int_\alpha \lambda^+ \leq \int_\gamma \lambda^+$. Hence

$$\begin{aligned}
\eta(\vartheta(x,t)) &\geq (\eta(u(\beta(s),s))\bigvee\rho_3(\beta) - \textstyle\int_\beta \lambda^+)\bigvee\rho_1(\beta) - \epsilon \\
&\geq (\eta(u(\beta(s),s)) - \textstyle\int_\beta \lambda^+)\bigvee\rho_1(\beta) - \epsilon \\
&\geq (\eta(u_0(\alpha(0)))\bigvee\rho_3(\alpha) - \textstyle\int_\alpha \lambda^+ - \int_\beta \lambda^+)\bigvee\rho_1(\beta) - \epsilon \\
&\geq (\eta(u_0(\gamma(0)))\bigvee\rho_3(\gamma) - \textstyle\int_\gamma \lambda^+)\bigvee\rho_1(\gamma) - \epsilon \\
&\geq \eta(u(x,t)) - \epsilon.
\end{aligned}$$

Suppose $[t_2, t_1] = \phi, [s_2, s_1] \neq \phi$ (Fig. 3a.), then define $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ by $\gamma_1(\theta) = \dfrac{x}{t - s_1}(\theta - s_1)$ for $\theta \in [s_1, t], \gamma_2 = \alpha_2, \gamma = \alpha_3$ (Fig. 3b.). Since $\dot{\gamma}_1$ is a convex combination of $\dot{\beta}$ and $\dot{\alpha}_1$ and hence $\rho_1(\gamma) \leq \rho_1(\beta)\bigvee\rho_1(\alpha)$ if $\dot{\beta} \geq 0$. Also

$\rho_1(\gamma) \leq \rho_1(\alpha)$ if $\dot{\beta} < 0$. Therefore

$$\begin{aligned}\eta(\vartheta(x,t)) &\geq \eta(u(\beta(s),s))\bigvee\eta(h(\dot{\beta})) - \epsilon \\ &= (\eta(u_0(\alpha(0)))\bigvee\rho_3(\alpha) - \textstyle\int_\alpha \lambda^+)\bigvee\rho_1(\alpha)\bigvee\eta(h(\dot{\beta})) - \epsilon \\ &\geq (\eta(u_0(\gamma(0)))\bigvee\rho_3(\gamma) - \textstyle\int_\gamma \lambda^+)\bigvee\rho_1(\gamma) - \epsilon \\ &\geq \eta(u(x,t)) - \epsilon.\end{aligned}$$

Similarly, if $[t_2, t_1] \neq \phi, [s_2, s_1] = \phi$ and $[t_2, t_1] = [s_2, s_1] = \phi$ it follows that $\eta(\vartheta(x,t)) \geq \eta(u(x,t)) - \epsilon$. Since ϵ is arbitrary, it follows that $\vartheta(x,t) \geq u(x,t)$ and hence the Lemma. □

Lemma 4.3. *(Lipschitz continuity) Let u be as in (2.9). Then u is a Lipschitz continuous function on $\overline{I\!R}_+ \times [0,T]$ for every $T > 0$.*

Proof : Let $0 \leq x_2 \leq x_1$ and $0 < t \leq T$. From (2) of Lemma 4.1, there exist a $\beta = (\beta_1, \beta_2, \beta_3) \in c(x_1, t)$ defined on the partition $0 = t_3 \leq t_2 \leq t_1 \leq t_0 = t$ such that

$$\eta(u(x_1,t)) = (\eta(u_0(\beta(0)))\bigvee\rho_3(\beta) - \int_\beta \lambda^+)\bigvee\rho_1(\beta).$$

Suppose β is a line segment with $\dot{\beta} \leq 0$. Then define $\gamma(\theta) = x_2 - x_1 + \beta(\theta) \in c(x_2, t)$. Hence

$$\begin{aligned}\eta(u(x_2,t)) - \eta(u(x_1,t)) &\leq (\eta(u_0(\gamma(0)))\bigvee\rho_3(\gamma) - \textstyle\int_\gamma \lambda^+) \\ &\quad -\eta(u_0(\beta(0)))\bigvee\rho_3(\beta) + \textstyle\int_\beta \lambda^+ \\ &\leq |\eta(u_0(\gamma(0))) - \eta(u_0(\beta(0)))| \\ &\leq M_1M_2|\gamma(0) - \beta(0)| = M_1M_2|x_1 - x_2| \ ,\end{aligned}$$

where M_1 and M_2 are Lipschitz constants for η on $[-|u_0|_\infty, |u_0|_\infty]$ and u_0 respectively.

Suppose β is not like above. Let $\tilde{t}_1 = t - (x_2/\dot{\beta}_1)$ and define $\gamma \in c(x_2, t)$ as follows :

(i) Let $\beta_2 \neq \phi$, then

$$\gamma(\theta) = \begin{cases} (\theta - t)\dot{\beta}_1 + x_2 & \theta \in [\tilde{t}_1, t] \\ 0 & \theta \in [t_2, \tilde{t}_1] \\ \beta_3(\theta) & \theta \in [0, t_2]. \end{cases}$$

(ii) Let $\beta_2 = \phi$, then

$$\gamma(\theta) = \begin{cases} (\theta - t)\dot{\beta}_1 + x_2 & \text{if} \quad \theta \in [\tilde{t}_1\bigvee 0, t] \\ 0 & \text{if} \quad \theta \in [0, \tilde{t}_1\bigvee 0]. \end{cases}$$

Clearly $\dot{\gamma}_1 = \dot{\beta}_1,\ \dot{\gamma}_3 = \dot{\beta}_3, \quad \{\beta = 0\} \subset \{\gamma = 0\}$ and $|\gamma(0) - \beta(0)| \leq |x_1 - x_2|$. Hence

$$\begin{aligned}\eta(u(x_1,t)) - \eta(u(x_2,t)) &\leq (\eta(u_0(\gamma(0)))\bigvee\rho_3(\gamma) - \textstyle\int_\gamma \lambda^+)\bigvee\rho_1(\gamma) \\ &\quad -(\eta(u_0(\beta(0)))\bigvee\rho_3(\beta) - \textstyle\int_\gamma \lambda^+)\bigvee\rho_1(\beta) \\ &\leq |\eta(u_0(\gamma(0))) - \eta(u_0(\beta(0)))| \\ &\leq M_1M_2|\gamma(0) - \beta(0)| \leq M_1M_2|x_1 - x_2|.\end{aligned}$$

In order to get the opposite inequality, let $\beta = (\beta_1, \beta_2, \beta_3) \in c_M(x_2, t)$ defined on the partition $0 = t_3 \le t_2 \le t_1 \le t_0 = t$ such that

$$\begin{aligned} \eta(u(x_2,t)) &= (\eta(u_0(\beta(0)))\bigvee \rho_3(\beta) - \int_\beta \lambda^+)\bigvee \rho_1(\beta) \\ &\ge \eta(u_0(\beta(0)))\bigvee \rho_3(\beta) - \int_\beta \lambda^+. \end{aligned}$$

Now choose a $p_0 > 0$ such that

$$\eta(h(p_0)) \le \eta(-|u_0|_\infty) - T|\lambda^+|_T\,. \tag{4.4}$$

Case (i) : $\dot\beta_1 \le p_0$.

Let $\tilde t_1 = t - \dfrac{x_1}{p_0}$ and define $\gamma \in c(x_1, t)$ by

$$\gamma(\theta) = \begin{cases} \dfrac{\theta - \tilde t_1}{t - \tilde t_1} x_1 & \text{if} \quad \tilde t_1 > 0, \quad \theta \in [\tilde t_1, t] \text{ or if } \quad \tilde t_1 \le 0, \theta \in [0,t] \\ 0 & \text{if} \quad \tilde t_1 \ge t_1,\ \theta \in [t_1, \tilde t_1] \\ \beta(\theta) & \text{if} \quad \tilde t_1 \ge t_1,\ \theta \in [0, t_1] \text{ or if } \quad \tilde t_1 \in [t_2, t_1],\ \theta \in [0, \tilde t_1] \\ (\theta - \tilde t_1)\dot\beta_3 & \text{if} \quad \tilde t_1 \in [0, t_2],\ \theta \in [0, \tilde t_1]. \end{cases}$$

Clearly $\dot\gamma_1 = p_0, \dot\gamma_3 = \dot\beta_3$ if $\tilde t_1 > 0$. Hence from (4.4)

$$\begin{aligned} \eta(u_0(\gamma(0)))\bigvee \rho_3(\gamma) - \int_\gamma \lambda^+ &\ge \eta(-|u_0|_\infty) - T|\lambda^+|_T \\ &\ge \eta(h(p_0)) = \rho_1(\gamma)\,, \end{aligned}$$

and hence

$$\begin{aligned} \eta(u(x_1,t)) &\le (\eta(u_0(\gamma(0)))\bigvee \rho_3(\gamma) - \int_\gamma \lambda^+)\bigvee \rho_1(\gamma) \\ &= \eta(u_0(\gamma(0)))\bigvee \rho_3(\gamma) - \int_\gamma \lambda^+\,. \end{aligned} \tag{4.5}$$

Let $\tilde t_1 \ge t_1$, then $\beta(0) = \gamma(0),\ \int_\beta \lambda^+ \le \int_\gamma \lambda^+, \rho_3(\beta) = \rho_3(\gamma)$. Therefore from (4.5),

$$\begin{aligned} \eta(u(x_2,t)) - \eta(u(x_1,t)) &\ge \eta(u_0(\beta(0)))\bigvee \rho_3(\beta) \\ &\quad - \int_\beta \lambda^+ - \eta(u_0(\gamma(0)))\bigvee \rho_3(\gamma) + \int_\gamma \lambda^+ \\ &\ge 0 \ge -|x_1 - x_2|\,. \end{aligned}$$

Let $\tilde t_1 \in [t_2, t_1]$, then $\beta(0) = \gamma(0),\ \rho_3(\beta) = \rho_3(\gamma)$. Now $\dfrac{x_2}{t - t_1} = \dot\beta_1 \le p_0$, hence $\dfrac{x_2}{p_0} \le t - t_1$. Therefore

$$\begin{aligned} \eta(u(x_2,t)) - \eta(u(x_1,t)) &\ge -\int_\beta \lambda^+ + \int_\gamma \lambda^+ = -\int_{\tilde t_1}^{t_1} \lambda^+ \ge -|\lambda^+|_T(t_1 - \tilde t_1) \\ &= -|\lambda^+|_T(t_1 - t + \frac{x_1}{p_0}) \ge -\frac{|\lambda^+|_T(x_1 - x_2)}{p_0}\,. \end{aligned}$$

Let $\tilde{t}_1 \in [0, t_2]$, then $t - t_2 \leq \frac{x_1}{p_0},\ \dot{\beta}_3 = \dot{\gamma}_3,\ \gamma(0) = -\tilde{t}_1\dot{\beta}_3 = \frac{\tilde{t}_1}{t_2}\beta(0)$. Since $\beta \in c_M(x_2, t)$ implies that $\frac{\beta(0)}{t_2} = |\dot{\beta}_3| \leq M$. Hence

$$
\begin{aligned}
|\gamma(0) - \beta(0)| &= \frac{\beta(0)}{t_2}(t_2 - \tilde{t}_1) \leq M(t_2 - t + \frac{x_1}{p_0}) \\
&\leq M(t_1 - t + \frac{x_1}{p_0}) \leq \frac{M}{p_0}(x_1 - x_2),
\end{aligned}
$$

$$
t_1 - t_2 = t_1 - t + t - t_2 \leq -\frac{x_2}{p_0} + t - \tilde{t}_1 = \frac{x_1 - x_2}{p_0}.
$$

$$
\begin{aligned}
\eta(u(x_2,t)) - \eta(u(x_1,t)) &\geq \eta(u_0(\beta(0)))\bigvee\rho_3(\beta) - \textstyle\int_\beta \lambda^+ - \eta(u_0(\gamma(0)))\bigvee\rho_3(\gamma) \\
&\geq -M_1M_2|\beta(0) - \gamma(0)| - |\lambda^+|_T(t_1 - t_2) \\
&\geq -[M_1M_2M + |\lambda^+|_T]\frac{(x_1 - x_2)}{p_0}.
\end{aligned}
$$

Let $\tilde{t}_1 \leq 0$. Then $t \leq \frac{x_1}{p_0}, \beta(0) \leq Mt_2 \leq M(t_2 - \tilde{t}_1) \leq \frac{M}{p_0}(x_1 - x_2), \gamma(0) = \frac{-\tilde{t}_1}{t - \tilde{t}_1}x_1 \leq x_1 - x_2, \quad t_1 - t_2 \leq \frac{M}{p_0}(x_1 - x_2)$. Hence

$$
\begin{aligned}
\eta(u(x_2,t)) - \eta(u(x_1,t)) &\geq \eta(u_0(\beta(0)))\bigvee\rho_3(\beta) - \textstyle\int_\beta \lambda^+ - \eta(u_0(\gamma(0)))\bigvee\rho_3(\gamma) \\
&\geq -|\eta(u_0(\beta(0))) - \eta(u_0(\gamma(0)))| - |\lambda^+|_T(t_1 - t_2) \\
&\geq -M_1M_2(|\beta(0)| + |\gamma(0)|) - |\lambda^+|_T\frac{(x_1 - x_2)}{p_0} \\
&\geq -(MM_1M_2 + |\lambda^+|_T)\frac{x_1 - x_2}{p_0}.
\end{aligned}
$$

Case (ii) : $\dot{\beta}_1 \geq p_0$.

Then $p_0 \leq \dot{\beta}_1 = \frac{x_2}{t - t_1}$ and hence $\frac{t - t_1}{x_2} \leq \frac{1}{p_0}$. Let $\tilde{t}_1 = (1 - \frac{x_1}{x_2})t + \frac{x_1}{x_2}t_1$, then $\tilde{t} < t_1$ and define $\gamma \in c(x_1, t)$ by

$$
\gamma(\theta) = \begin{cases}
\frac{\theta - \tilde{t}_1}{t - \tilde{t}_1}x_1 & \text{if } \theta \in [\tilde{t}_1\bigvee 0, t] \\
0 & \text{if } \theta \in [t_2, \tilde{t}_1] \text{ and } \tilde{t}_1 \geq t_2 \\
\beta(\theta) & \text{if } \theta \in [0, t_2] \text{ and } \tilde{t}_1 \geq t_2 \\
(\theta - \tilde{t}_1)\dot{\beta}_3 & \text{if } 0 \leq \tilde{t}_1 \leq t_2\,.
\end{cases}
$$

Clearly $\dot{\gamma}_1 = \dot{\beta}_1,\ \dot{\gamma}_3 = \dot{\beta}_3$ if $\dot{\gamma}_3 \neq 0$ and $t_1 - \tilde{t}_1 = (\frac{x_1}{x_2} - 1)(t - t_1) = (x_1 - x_2))\frac{t - t_1}{x_2} \leq \frac{x_1 - x_2}{p_0}$. Let $\tilde{t}_1 \in [t_2, t_1]$, then $\beta(0) = \gamma(0), \quad \dot{\beta}_1 = \dot{\gamma}_1, \quad \dot{\beta}_3 = \dot{\gamma}_3$.

Hence

$$\begin{aligned}\eta(u(x_2,t))-\eta(u(x_1,t)) &\geq (\eta(u_0(\beta(0)))\bigvee\rho_3(\beta)-\textstyle\int_\beta\lambda^+)\bigvee\rho_1(\beta)\\ &\qquad -(\eta(u_0(\gamma(0)))\bigvee\rho_3(\gamma)-\textstyle\int_\gamma\lambda^+)\bigvee\rho_1(\gamma)\\ &\geq -\textstyle\int_{\tilde{t}_1}^{t_1}\lambda^+\geq-|\lambda^+|_T(t_1-\tilde{t}_1)\geq\dfrac{-|\lambda^+|_T(x_1-x_2)}{p_0}\,.\end{aligned}$$

Let $0\leq\tilde{t}_1\leq t_2$, then $\dot{\beta}_1=\dot{\gamma}_1,\quad\dot{\beta}_3=\dot{\gamma}_3,\quad\gamma(0)=-\tilde{t}_1\dot{\beta}_3=\dfrac{\tilde{t}_1}{t_2}\beta(0)$. Since $\beta\in c_M(x_2,t)$ and hence $\dfrac{\beta(0)}{t_2}=|\dot{\beta}_3|\leq M$. Hence

$$|\gamma(0)-\beta(0)|=\frac{\beta(0)}{t_2}(t_2-\tilde{t}_1)\leq M(t_1-\tilde{t}_1)=\frac{M}{p_0}(x_1-x_2).$$

Hence

$$\begin{aligned}\eta(u(x_2,t))-\eta(u(x_1,t)) &\geq (\eta(u_0(\beta(0)))\bigvee\rho_3(\beta)-\textstyle\int_\beta\lambda^+)\bigvee\rho_1(\beta)\\ &\quad -(\eta(u_0(\gamma(0)))\bigvee\rho_3(\gamma)-\textstyle\int_\gamma\lambda^+)\bigvee\rho_1(\gamma)\\ &\geq -|\eta(u_0(\beta(0)))-\eta(u_0(\gamma(0)))|-|\lambda^+|_T(t_1-t_2)\\ &\geq -M_1M_2|\beta(0)-\gamma(0)|-\frac{|\lambda^+|_T}{p_0}(x_1-x_2)\\ &\geq -[M_1M_2M+|\lambda^+|_T]\frac{x_1-x_2}{p_0}\,.\end{aligned}$$

Let $\tilde{t}_1\leq 0$, then $\dot{\beta}_1=\dot{\gamma}_1,\quad\gamma(0)=\dfrac{-\tilde{t}_1}{t-\tilde{t}_1}x_1\leq x_1-x_2,\quad\beta(0)\leq Mt_2\leq M(t_1-\tilde{t}_1)\leq\dfrac{M}{p_0}(x_1-x_2)$. Hence

$$\begin{aligned}\eta(u(x_2,t))-\eta(u(x_1,t)) &\geq \eta(u_0(\beta(0)))-\textstyle\int_\beta\lambda^+-\eta(u_0(\gamma(0)))\\ &\geq -M_1M_2(\beta(0)+\gamma(0))-|\lambda^+|_T(t_1-t_2)\\ &\geq -(M_1M_2M+p_0+|\lambda^+|_T)\frac{x_1-x_2}{p_0}.\end{aligned}$$

Combining all the above inequalities to obtain a constant $M(T)$ such that for $x_1,x_2\in\overline{I\!R}_+,\quad 0\leq t\leq T$,

$$|u(x_1,t)-u(x_2,t)|\leq M(T)|x_1-x_2|\,. \tag{4.6}$$

From (4.6), $x\mapsto u(x,t)$ is continuous and hence from (2) of Lemma 4.1, for any $0\leq s<t\leq T$ and $x\in\overline{I\!R}_+$, there exists a $\beta\in c_M(x,s,t)$ such that

$$\eta(u(x,t))=(\eta(u(\beta(s),s))\bigvee\rho_3(\beta)-\int_\beta\lambda^+)\bigvee\rho_1(\beta)\,. \tag{4.7}$$

Since $|\dot{\beta}|\leq M$ and hence $|\beta(s)-x|\leq M(t-s)$. Therefore

$$\begin{aligned}\eta(u(x,t))-\eta(u(x,s)) &\geq \eta(u(\beta(s),s))-\eta(u(x,s))-|\lambda^+|_T(t-s)\\ &\geq -M_1M(T)|\beta(s)-x|-|\lambda^+|_T(t-s)\\ &\geq -[MM_1M(T)+|\lambda^+|_T](t-s)\,.\end{aligned}$$

Now let $\gamma(\theta) = x$ for $\theta \in [s,t]$. Then

$$\eta(u(x,t)) \leq \eta(u(x,s)) - \int_\gamma \lambda^+ \leq \eta(u(x,s))\,.$$

Hence there exists a constant $\tilde{M}(T)$ such that $|u(x,t) - u(x,s)| \leq \tilde{M}(T)(t-s)$. This together with (4.6) proves the Lemma. □

Proof of Theorem 2.1 : From Lemma 4.3, $u \in W^{1,\infty}(\overline{I\!R}_+ \times [0,T])$ for any $T > 0$.

Sub soulution : Suppose u is not a subsolution, then there exist $(x_0,t_0) \in \overline{I\!R}_+ \times I\!R_+,\ \ \mu > 0, \varphi \in C^1(\overline{I\!R_+ \times I\!R_+})$ such that $u - \varphi$ has a local maximum at $(x_0,t_0),\quad u(x_0,t_0) = \varphi(x_0,t_0)$ and at $(x_0,t_0),\ \varphi_t + H(\varphi,\varphi_x) \geq 4\mu$ and $\varphi_x(x_0,t_0) < \lambda(t_0)$ if $x_0 = 0$. By continuity, there exist a $\delta > 0$ such that at $(x_0,t_0),\ \varphi_t + H(\varphi - \delta, \varphi_x) \geq 3\mu$ and $\varphi_x(x_0,t_0) < \lambda(t_0)$ if $x_0 = 0$. Hence from (A_3) there exist a p such that at $(x_0,t_0), \varphi_t + p\varphi_x \geq 2\mu,\ h(p) \leq \varphi(x_0,t_0) - \delta,\ \varphi_x(x_0,t_0) < \lambda(t_0)$ if $x_0 = 0$. Hence there exist a ball B centred at (x_0,t_0) such that in B

$$\varphi_t + p\varphi_x \geq \mu, \quad h(p) \leq \varphi, \quad \varphi_x < \lambda \ \text{ if } x_0 = 0. \tag{4.8}$$

Suppose $x_0 > 0$. By shrinking B if necessary, we can assume that $B \subset I\!R_+ \times I\!R_+$. Now choose s_0 such that for $\theta \in [s_0,t_0], (\beta(\theta),\theta) \in B$ where $\beta(\theta) = p(\theta - t_0) + x_0 \in c(x_0,s_0,t_0)$. Then from (4.3) and (4.8) we have

$$\begin{array}{rcl} \eta(\varphi(x_0,t_0)) = \eta(u(x_0,t_0)) & \leq & \eta(u(\beta(s_0),s_0)) \bigvee \rho_3(\beta) \bigvee \rho_1(\beta) \\ & = & \eta(u(\beta(s_0),s_0) \bigvee h(p)) \\ & \leq & \eta(\varphi(\beta(s_0),s_0) \bigvee h(p)) \\ & \leq & \eta(\varphi(\beta(s_0),s_0))\,. \end{array}$$

This implies that $\varphi(x_0,t_0) \leq \varphi(\beta(s_0),s_0)$. On the other hand, from (4.8)

$$\begin{array}{rcl} \varphi(x_0,t_0) - \varphi(\beta(s_0),s_0) & = & \int_{s_0}^{t_0} \dfrac{d}{d\theta}\varphi(\beta(\theta),\theta)d\theta \\ & = & \int_{s_0}^{t_0} (\varphi_t + p\varphi_x)d\theta \geq \mu(t_0 - s_0), \end{array}$$

which is a contradiction.

Suppose $x_0 = 0$. If $p < 0$, like in the above proof, choose $\beta(\theta) = p(\theta - t_0) \in c(x_0,s_0,t_0)$ such that $\beta(\theta) \in B \cap (\overline{I\!R}_+ \times I\!R_+)$ and the contradiction follows. Therefore assume that $p > 0$. Suppose $\varphi_x(x_0,t_0) \leq 0$, then $2\mu \leq (\varphi_t + p\varphi_x)(x_0,t_0) \leq \varphi_t(x_0,t_0)$ and hence in $B,\quad \mu \leq \varphi_t$. This implies that if $s_0 < t_0$ such that $(x_0,s_0) \in B$, then $\mu(t_0 - s_0) \leq \varphi(x_0,t_0) - \varphi(x_0,s_0)$. But from (4.3), taking $\beta(\theta) = 0$ for $\theta \in [s_0,t_0]$ to obtain $\varphi(x_0,t_0) \leq \varphi(x_0,s_0)$ which is a contradiction. Now let $\varphi_x(x_0,t_0) > 0$ and hence $\lambda(t_0) > 0$. By shrinking B if necessary, we can assume that $\lambda > 0$. Choose s_0,t_0 such that $\{0\} \times [s_0,t_0] \subset B$. Let $\beta(\theta) = x_0$ for $\theta \in [s_0,t_0]$, then from (4.3) $\eta(\varphi(x_0,t_0)) = \eta(u(x_0,t_0)) \leq \eta(\varphi(x_0,s_0)) - \int_{s_0}^{t_0} \lambda^+$ and hence

$$\begin{array}{rcl} (\eta'(\varphi)\varphi_t)(x_0,t_0) & = & \lim_{s_0 \to t_0} \dfrac{\eta(\varphi(x_0,t_0)) - \eta(\varphi(x_0,s_0))}{t_0 - s_0} \\ & \leq & -\lim_{s_0 \to t_0} \dfrac{1}{t_0 - s_0} \displaystyle\int_{s_0}^{t_0} \lambda^+ = -\lambda^+(t_0) = -\lambda(t_0), \end{array}$$

that is, $(\varphi_t+\lambda(t_0)H(\varphi,1))(x_0,t_0)\leq 0$ and hence $\mu\leq(\varphi_t+H(\varphi,\varphi_x))(x_0,t_0)=(\varphi_t+\varphi_x H(\varphi,1))(x_0,t_0)\leq(\varphi_t+\lambda H(\varphi,1))(x_0,t_0)\leq 0$, which is a contradiction. This proves that u is a subsolution.

Super Solution : Suppose u is not a super solution, then there exist $(x_0,t_0)\in\overline{I\!R}_+\times I\!R_+,\mu>0,\varphi\in C^1(\overline{I\!R}_+\times I\!R_+)$ such that $u-\varphi$ has local minimum at $(x_0,t_0), u(x_0,t_0)=\varphi(x_0,t_0)$ and $(\varphi_t+H(\varphi,\varphi_x))(x_0,t_0)\leq -4\mu,\quad \varphi_x(x_0,t_0)>\lambda(t_0)$ if $x_0=0$. Hence in a Ball B centred at (x_0,t_0), $\varphi_t+H(\varphi,\varphi_x)\leq-3\mu,\varphi_x>\lambda$ if $x_0=0$ holds. Hence form (A_6), in B,

$$\left\{\begin{array}{lll} h(p) & \leq & \varphi \text{ implies that } \varphi_t+p\varphi_x\leq-3\mu \\ \varphi_x & > & \lambda \quad \text{if } x_0=0\ . \end{array}\right. \tag{4.9}$$

From (2) of Lemma 4.1 and 4.3, for each $s<t_0$, choose a minimizing curve $\beta_s\in c_M(x_0,s,t_0)$. For $\theta\in[s_0,t_0],|\beta_s(\theta)-x_0|\leq(t_0-s)|\dot\beta_s|\leq M(t_0-s)$, hence choose $s_1<t_0$ such that for any $s_1\leq s<t_0,\quad (\beta_s(\theta),\theta)\in B$ for $\theta\in[s,t_0]$.
Claim : There exist an $s_0\in[s_1,t_0]$ such that for almost all $\theta\in[s_0,t_0]$

$$(\varphi_t+\dot\beta_{s_0}\varphi_x)(\beta_{s_0}(\theta),\theta)\leq-\mu. \tag{4.10}$$

Suppose not, then there exist a sequence $s_k\to t_0,\quad \theta_k\in[s_k,t_0]$ with $\beta_k=\beta_{s_k},p_k=\dot\beta_k(\theta_k)$ such that $-\mu\leq(\varphi_t+p_k\varphi_x)(\beta_k(\theta_k),\theta_k)$ holds. By going to a sub sequence assume that $p_k\to p_0$. Hence by lower semi continuity

$$\begin{array}{rcl} \eta(\varphi(x_0,t_0))=\eta(u(x_0,t_0)) & = & \lim_{k\to\infty}(\eta(u(\beta_k(s_k),s_k))\bigvee\rho_3(\beta_k) \\ & & -\int_{\beta_k}\lambda^+)\bigvee\rho_1(\beta_k) \\ & \geq & \lim_{k\to\infty}[\eta(\varphi(\beta_k(s_k),s_k))\bigvee\rho_3(\beta_k) \\ & & -\int_{\beta_k}\lambda^+)\bigvee\rho_1(\beta_k)] \\ & \geq & \eta(\varphi(x_0,t_0)\bigvee h(p))\ , \end{array}$$

and hence $\varphi(x_0,t_0)\geq h(p)$. Hence from (4.9) we have $-\mu\leq\lim_{k\to\infty}(\varphi_t+p_k\varphi_x)(\beta_k(\theta_k),\theta_k)=(\varphi_t+p\varphi_x)(x_0,t_0)\leq-3\mu$, which is a contradiction and hence the claim.

Let $x_0>0$. Then by shrinking B if necessary such that $B\subset I\!R_+\times I\!R_+$. Hence β_{s_0} is a line segment. Hence

$$\begin{array}{rcl} \eta(\varphi(x_0,t_0))=\eta(u(x_0,t_0)) & = & \eta(u(\beta_{s_0}(s_0),s_0)\bigvee h(\dot\beta_{s_0})) \\ & \geq & \eta(\varphi(\beta_{s_0}(s_0),s_0)\bigvee h(\dot\beta_{s_0})), \end{array}$$

hence $\varphi(x_0,t_0)\geq\varphi(\beta_{s_0}(s_0),s_0)$. Integrating (4.10) from s_0 to t_0 to obtain $\varphi(x_0,t_0)-\varphi(\beta_{s_0}(s_0),s_0)\leq-\mu(t_0-s_0)$ which is a contradiction.

Let $x_0=0$ and $\lambda(t_0)<0$. Shrinking B if necessary, assume that $\lambda<0$ in B and hence $\lambda^+=0$ in B. Hence by (4.3) it follows that $\varphi(x_0,t_0)\geq\varphi(\beta_{s_0}(s_0),s_0)$ and a contradiction is obtained from integrating (4.10) from s_0 to t_0.

Let $x_0=0$ and $\lambda(t_0)\geq 0$. Then $\varphi_x(x_0,t_0)>\lambda(t_0)\geq 0$. Shrinking B if necessary, can assume that $\varphi_x>0$ in $B\cap(\overline{I\!R}_+\times I\!R_+)$. Hence $\varphi(x,s)\geq\varphi(0,s)$.

Therefore

$$\begin{aligned}\eta(\varphi(x_0,t_0)) = \eta(u(x_0,t_0)) &= \eta(u(\beta_s(s),s))\bigvee\rho_3(\beta_s) - \textstyle\int_{\beta_s}\lambda^+\\ &\geq \eta(\varphi(\beta_s(s),s)) - \textstyle\int_{\beta_s}\lambda^+\\ &\geq \eta(\varphi(0,s)) - \textstyle\int_s^{t_0}\lambda^+ .\end{aligned}$$

Hence

$$\begin{aligned}(\eta'(\varphi)\varphi_t)(x_0,t_0) &= \lim_{s\to t}\frac{\eta(\varphi(x_0,t_0)) - \eta(\varphi(0,s))}{t_0 - s}\\ &\geq -\lambda^+(t_0) = -\lambda(t_0)\,.\end{aligned}$$

That is $0 \leq (\varphi_t + \lambda(t_0)H(\varphi,1))(x_0,t_0) \leq (\varphi_t + \varphi_x H(\varphi,1))(x_0,t_0) = (\varphi_t + H(\varphi,\varphi_x))(x_0,t_0) \leq -4\mu$ which is a contradiction. This proves the theorem. □

5. Proof of Theorem 2.2.

In order to prove Theorem 2.2, we need the following Lemmas.

Lemma 5.1. *Let $T > 0$ and ϑ be a real valued function defined on $\overline{I\!R_+ \times I\!R_+}$ such that $|\vartheta|_T = \sup\{|\vartheta(x,t)|; x \in I\!R, 0 \leq t \leq T\} < \infty$. Let $x \in \overline{I\!R}_+$ and $0 \leq s < t \leq T$. Define*

$$\eta(\underline{V}(x,t)) = \inf \left\{\eta(\vartheta(\beta(s),s)) - \int_\beta \lambda^+; \beta \in \underline{A}(\vartheta,x,s,t)\right\}, \quad (5.1)$$

$$\eta(\overline{V}(x,t)) = \inf \left\{\eta(\vartheta(\beta(s),s)) - \int_\beta \lambda^+; \beta \in \overline{A}(\vartheta,x,s,t)\right\}. \quad (5.2)$$

Then there exist a constant $M = M(T,|\vartheta|_T,|\lambda|_\infty)$ such that $|\underline{V}_T| + |\overline{V}_T| < \infty$,

$$\eta(\underline{V}(x,t)) = \inf\left\{\eta(\vartheta(\beta(s),s)) - \int_\beta \lambda^+; \beta \in \underline{A}_M(\vartheta,x,s,t)\right\}, \quad (5.3)$$

$$\eta\overline{V}(x,t)) = \inf\left\{\eta(\vartheta(\beta(s),s)) - \int_\beta \lambda^+; \beta \in \overline{A}_M(\vartheta,x,s,t)\right\}. \quad (5.4)$$

(1) *If $x \mapsto \vartheta(x,\theta)$ is lower semicontinous for each θ, then a minimizer $\beta \in \underline{A}_M(\vartheta,x,s,t)$ exist in (5.3).*

(2) *If $x \mapsto \vartheta(x,\theta)$ is continuous, then there exist a sequence $\beta_k \in \overline{A}_M(\vartheta,x,s,t)$ converging to $\beta \in \underline{A}_M(\vartheta,x,s,t)$ in $W^{1,\infty}$ such that*

$$\eta(\overline{V}(x,t)) = \eta(\vartheta(\beta(s),s)) - \int_\beta \lambda^+. \quad (5.5)$$

Proof : From $(A_5), h(0) = +\infty, h(q) \to -\infty$, as $|q| \to \infty$. Let $\beta(\theta) = x\ \forall\theta \in [s,t]$, then $h(\dot\beta) = h(0) = \infty$. Hence $\beta \in \underline{A}(\vartheta,x,s,t) \cap \overline{A}(\vartheta,x,s,t)$. Therefore $\eta(\underline{V}(x,t)) \leq \eta(\vartheta(\beta(s),s)) - \int_\beta \lambda^+ \leq \eta(|\vartheta|_T)$ and similarly $\eta(\overline{V}(x,t)) \leq \eta(|\vartheta|_T)$. Now for any $\beta \in \underline{A}(\vartheta,x,s,t), \eta(\vartheta(\beta(s),s)) - \int_\beta \lambda^+ \geq \eta(-|\vartheta|_T) - T|\lambda^+|$ and hence by taking infimum over β, $\eta(\underline{V}(x,t)) \geq \eta(-|\vartheta|_T) - T|\lambda^+|_T$. This proves $|\underline{V}|_T < \infty$ and similarly $|\overline{V}|_T < \infty$. Since $h(q) \to -\infty$ as $|q| \to \infty$, we can choose a $M > 0$ such that on $|q| \geq M,\ h(q) < -|\vartheta|_T$. Hence if $\beta \in c(x, s, t)$ such that $|\dot{\beta}| \geq M$, then either $h(\dot{\beta}_1) < -|\vartheta|_T$ or $h(\dot{\beta}_3) < -|\vartheta|_T$. In either case $\beta \notin \underline{A}(\vartheta, x, s, t)$ and $\beta \notin \overline{A}(\vartheta, x, s, t)$. This proves (5.3) and (5.4).

Let $\{\beta_k\} \subset \underline{A}_M(\vartheta, x, s, t)$ be a minimizing sequence in (5.3). Extract a subsequence converging to $\beta \in c_M(\vartheta, x, s, t)$ in $W^{1,\infty}$. From (A_4) h is an upper semicontinuous and $x \mapsto \vartheta(x, \theta)$ is lower semicontinous, it follows that $\beta \in \underline{A}_M(\vartheta, x, s, t)$ and $\eta(\underline{V}(x, t)) = \eta(\vartheta(\beta(s), s)) - \int_\beta \lambda^+$. This proves (1). Apply the similar argument for $\overline{V}(x, t)$ to conclude (5.5). The only difference is that in this case β need not be in $\overline{A}_M(\vartheta, x, s, t)$. This proves the Lemma. □

Lemma 5.2 : *Let $\underline{u}$ be as in (2.14). Let $T > 0,\ x \in \overline{I\!R}_+,\ 0 \leq s < t \leq T$. Then $|\underline{u}|_T < \infty$, $x \mapsto \underline{u}(x, \theta)$ is lower semicontinuous and there exist an $M = M(|u|_T, |\lambda^+|_T, T)$ such that*

$$\underline{u}(x, t) \leq \inf \eta^{-1} \left\{ \eta(\underline{u}(\beta(s), s)) - \int_\beta \lambda^+; \beta \in \underline{A}_M(\underline{u}^*, x, s, t) \right\},$$

$$\eta(\underline{u}^*(x, t)) \leq \inf \left\{ \eta(\underline{u}^*(\beta(s), s)) - \int_\beta \lambda^+; \beta \in \overline{A}_M(\underline{u}^*, x, s, t) \right\}. \tag{5.6}$$

Proof : Since $|u_0|_T < \infty$ then from Lemma 5.1, $|\underline{u}_T| < \infty$. Let $(x_m, t_m) \to (x_0, t_0)$ as $m \to \infty$. Since u_0 is continuous, from (1) of Lemma 5.1 there exist a $M_1 > 0$ and a minimizer $\beta_m \in \underline{A}_{M_1}(u_0, x_m, t_m)$ of (2.15). Hence by going to a subsequence, let $\{\beta_m\}$ converges to $\beta \in \underline{A}_{M_1}(u_0, x_0, t_0)$ in $W^{1,\infty}$. Hence

$$\begin{aligned}
\eta(\underline{u}(x_0, t_0)) &\leq \eta(u_0(\beta(0))) - \textstyle\int_\beta \lambda^+ \\
&= \underline{\lim}_{m\to\infty} \left\{ \eta(u_0(\beta_m(0))) - \textstyle\int_{\beta_m} \lambda^+ \right\} \\
&= \underline{\lim}_{m\to\infty} \eta(\underline{u}(x_m, t_m)),
\end{aligned}$$

and this proves $\underline{u}$ is lower semicontinuous. Next define ϑ_1 by

$$\eta(\vartheta_1(x, t)) = \inf \left\{ \eta(\underline{u}(\beta(s), s)) - \int_\beta \lambda^+; \beta \in \underline{A}(\underline{u}, x, s, t) \right\} \tag{5.7}$$

Step 1 : $\vartheta_1(x, t) \geq \underline{u}(x, t)$.

Since $|\underline{u}|_T < \infty$ and $x \mapsto \underline{u}(x, \theta)$ is lower semicontinuous, hence from (1) of Lemma 5.1, there exist $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \underline{A}(\underline{u}, x, s, t)$ defined on the partition $s = t_3 \leq t_2 \leq t_1 \leq t_0 = t$ and $\beta = (\beta_1, \beta_2, \beta_3) \in \underline{A}(u_0, \alpha(s), s)$ defined on the partition $0 = s_3 \leq s_2 \leq s_1 \leq s_0 = s$ such that $\eta(\vartheta_1(x, t)) = \eta(\underline{u}(\alpha(s), s)) - \int_\alpha \lambda^+, \quad \eta(\underline{u}(\alpha(s), s)) = \eta(u_0(\beta(0))) - \int_\beta \lambda^+$. Hence

$$\eta(\vartheta_1(x, t)) = \eta(u_0(\beta(0))) - \int_\alpha \lambda^+ - \int_\beta \lambda^+). \tag{5.8}$$

Suppose $[t_2, t_1] \neq \phi, [s_2, s_1] \neq \phi$. Define $\gamma = (\gamma_1, \gamma_2, \gamma_3) \in c(x,t)$ by $\gamma_1 = \alpha_1, \quad \gamma_2|_{[s_2,t_1]} = 0, \quad \gamma_3 = \beta_3$. Then $u_0(\gamma(0)) = u_0(\beta(0)) \leq h(\dot{\beta}_3) = h(\dot{\gamma}_3), \eta(u_0(\gamma(0))) - \int_\gamma \lambda^+ = \eta(u_0(\beta(0))) - \int_\gamma \lambda^+ \leq \eta(u_0(\beta(0))) - \int_\beta \lambda^+ - \int_\alpha \lambda^+ = \eta(\underline{u}(\alpha(s), s)) - \int_\alpha \lambda^+ \leq \eta(h(\dot{\alpha}_1)) = \eta(h(\dot{\gamma}_1))$. Hence $\gamma \in \underline{A}(u_0, x, t)$ and from (4.18) we have

$$\begin{aligned}\eta(\vartheta_1(x,t)) = \eta(u_0(\gamma(0))) - \int_\alpha \lambda^+ \quad &- \quad \int_\beta \lambda^+ \geq \eta(u_0(\gamma(0))) \\ &- \quad \int_\gamma \lambda^+ \geq \eta(\underline{u}(x,t)).\end{aligned}$$

Hence $\vartheta_1 \geq \underline{u}$.

Suppose $[t_2, t_1] \neq \phi, [s_2, s_1] = \phi$. Then β is a line segment and $\int_\beta \lambda^+ = 0$. Hence $\underline{u}(\alpha(s), s) = u_0(\beta(0))$ and $u_0(\beta(0)) \leq h(\dot{\beta})$. Now define $\gamma = (\gamma_1, \gamma_2, \gamma_3) \in c(x,t)$ by $\gamma_1 = \alpha_1, \gamma_2 = \alpha_2, \gamma_3(\theta) = \dfrac{t_2 - \theta}{t_2}\beta(0)$ for $\theta \in [0, t_2]$. Suppose $\dot{\alpha_3} \leq \dot{\beta}$, then by construction $h(\dot{\alpha_3}) \leq h(\dot{\gamma_3})$ and hence $u_0(\gamma(0)) = u_0(\beta(0)) = \underline{u}(\alpha(s), s) \leq h(\dot{\alpha_3}) \leq h(\dot{\gamma_3})$ and $\eta(u_0(\gamma(0))) - \int_\gamma \lambda^+ = \eta(\underline{u}(\alpha(s), s)) - \int_\alpha \lambda^+ \leq \eta(h(\dot{\alpha_1})) = \eta(h(\dot{\gamma_1}))$. Hence $\gamma \in \underline{A}(u_0, x, t)$. If $\dot{\beta} \leq \dot{\alpha}_3 \leq 0$, then clearly $h(\dot{\beta}) \leq h(\dot{\gamma}_3)$ and hence $\eta(\gamma(0)) - \int_\gamma \lambda^+ = \eta(\underline{u}(\alpha(s), s)) - \int_\alpha \lambda^+ \leq \eta(h(\dot{\alpha}_1)) = \eta(h(\dot{\gamma}_1))$. Hence $\gamma \in \underline{A}(u_0, x, t)$. This implies that $\eta(\vartheta_1(x,t)) \ = \ \eta(u_0(\beta(0))) - \int_\alpha \lambda^+ \ = \ \eta(u_0(\gamma(0))) - \int_\gamma \lambda^+ \geq \eta(\underline{u}(x,t))$. Hence $\vartheta_1(x,t) \geq \underline{u}(x,t)$.

Suppose $[t_2, t_1] = \phi, [s_2, s_1] = \phi$, then α and β are line segments with $\int_\alpha \lambda^+ = \int_\beta \lambda^+ = 0, \quad \vartheta_1(x,t) = \underline{u}(\alpha(s), s) = u_0(\beta(0)) \leq \min(h(\dot{\alpha}), h(\dot{\beta}))$. Let $\gamma(\theta) = \dfrac{\theta}{t}x + (1 - \dfrac{\theta}{t})\beta(0)$ for $\theta \in [0,t]$, clearly $\gamma \in c(x,t)$ and by quasi concavity of $h, \ \min(h(\dot{\alpha}), h(\dot{\beta})) \leq h(\dot{\gamma})$ and hence $\gamma \in \underline{A}(u_0, x, t)$ with $\int_\gamma \lambda^+ = 0$. Hence $\eta(\vartheta_1(x,t)) = \eta(u_0(\beta(0))) = \eta(u_0(\gamma(0))) \geq \eta(\underline{u}(x,t))$. This implies that $\vartheta_1(x,t) \geq \underline{u}(x,t)$. Suppose $[t_2, t_1] = \phi, \ [s_2, s_1] \neq \phi$. Then $\vartheta_1(x,t) = \underline{u}(\alpha(s), s)$. Define $\gamma = (\gamma_1, \gamma_2, \gamma_3) \in c(x,t)$ by $\gamma_1(\theta) = \dfrac{\theta - s_1}{t - s_1}x$ if $\theta \in [s_1, t], \gamma_2 = \beta_2, \gamma_3 = \beta_3$. Hence $u_0(\gamma(0)) = u_0(\beta(0)) \leq h(\dot{\beta}_3) = h(\dot{\gamma}_3)$. Now $\underline{u}(\alpha(s), s) \leq h(\dot{\alpha})$ and $\eta(\underline{u}(\alpha(s), s) - \int_\beta \lambda^+ \leq \eta(h(\beta_1))$. Hence $\eta(\underline{u}(\alpha(s), s)) - \int_\beta \lambda^+ \leq \min(\eta(h(\dot{\beta}_1)), \eta(h(\dot{\alpha}))) \leq \eta(h(\dot{\gamma}))$ since $\dot{\gamma}$ is a convex combination of $\dot{\alpha}$ and $\dot{\beta}_1$. Hence $\gamma \in \underline{A}(u_0, x, t)$ and $\eta(\vartheta_1(x,t)) = \eta(u_0(\beta(0))) - \int_\beta \lambda^+ = \eta(u_0(\gamma(0))) - \int_\gamma \lambda^+ \geq \eta(\underline{u}(x,t))$. Hence $\vartheta_1(x,t) \geq \underline{u}(x,t)$ and this proves the step-1.

Step 2. Let $(x_m, t_m) \to (x,t)$ such that $\underline{u}^*(x,t) = \lim_{m\to\infty} \underline{u}(x_m, t_m)$. Let $\beta = (\beta_1, \beta_2, \beta_3) \in \overline{A}(\underline{u}^*, x, t, s)$ defined on the partition $s = t_3 \leq t_2 \leq t_1 \leq t_0 = t$. Now for each m , define $\beta^{(m)} = (\beta_1^{(m)}, \beta_2^{(m)}, \beta_3^{(m)})$ by $\beta_1^{(m)}(\theta) = \dfrac{\theta - t_1}{t_m - t_1}x_m$ if $\theta \in [t_1, t]$ and $\beta^{(m)}|_{[0,t_1]} = \beta$. Then $\beta^{(m)} \in c(x_m, s, t_m)$ and $\beta^{(m)} \to \beta$ in $W^{1,\infty}$. Hence by upper semicontinuity of $\underline{u}^*$ and h it follows that there exist an $m_0 > 0$

such that for all $m \ge m_0$, $\quad \beta_m \in \overline{A}(\underline{u}, x_m, t_m, s)$. Hence by the Step 1,

$$\begin{aligned}
\underline{u}^*(x,t) &= \lim_{m\to\infty} \underline{u}(x_m,t_m) \le \lim_{m\to\infty} \vartheta_1(x_m,t_m) \\
&\le \lim_{m\to\infty} \eta^{-1}\left\{\eta(\underline{u}(\beta^m(s),s)) - \int_{\beta^m}\lambda^+\right\} \\
&\le \lim_{m\to\infty} \eta^{-1}\left\{\eta(\underline{u}^*(\beta^m(s),s)) - \int_{\beta^m}\lambda^+\right\} \\
&\le \eta^{-1}\left\{\eta(\underline{u}^*(\beta(s),s)) - \int_{\beta}\lambda^+\right\}.
\end{aligned}$$

Hence by taking infimum over all β and from (2) of Lemma 5.1 to obtain (5.6). This proves the Lemma. □

Lemma 5.3. *Let $\overline{u}$ be as in (2.15). Let $T > 0, x \in \overline{I\!R}_+, 0 \le s < t \le T$. Then $|\overline{u}|_T < \infty, x \mapsto \overline{u}(x,\theta)$ is upper semicontinuous and there exist an $M = M(|\overline{u}|_T, |\lambda^+|_T, T)$ such that*

$$\eta(\overline{u}(x,t)) \ge \inf\left\{\eta(\overline{u}(\beta(s),s)) - \int_\beta \lambda^+; \beta \in \overline{A}_M(\overline{u}, x, s, t)\right\},$$

$$\eta(\overline{u}_*(x,t)) \ge \inf\left\{\eta(\overline{u}_*(\beta(s),s)) - \int_\beta \lambda^+; \beta \in \underline{A}_M(\overline{u}_*, x, s, t)\right\}.$$

Proof : Since $|u_0|_T < \infty$ and hence Lemma 5.1 implies that $|\overline{u}|_T < \infty$. Let $(x_m, t_m) \to (x,t)$ as $m \to \infty$. Since u_0 is continuous, from (2) of Lemma 5.1, there exist $M_1 > 0$ and a sequence $\beta^{(k)} \in \overline{A}_{M_1}(u_0, x, t)$ such that $\beta^{(k)} \to \beta$ in $W^{1,\infty}$ and $\eta(\overline{u}(x,t)) = \eta(u_0(\beta(0))) - \int_\beta \lambda^+$. Let $\beta^{(k)} = (\beta_1^{(k)}, \beta_2^{(k)}, \beta_3^{(k)})$ be defined on the partition $s = t_{3k} \le t_{2k} \le t_{1k} \le t_{0k} = t$. For m large, define $\beta^{m,k}$ as follows :

Case (i) : $x > 0$. If $[t_{2k}, t_{1k}] = \phi$, then $\beta^{(k)}$ is a line segment. Define $\beta^{m,k} \in c(x_m, t_m)$ the line segment parallel to $\beta^{(k)}$. In this case $\beta^{m,k} \to \beta^{(k)}$ as $m \to \infty$ in $W^{1,\infty}$ and $\dot{\beta}^{m,k} = \dot{\beta}^{(k)}$ for all large m. If $[t_{2k}, t_{1k}] \ne \phi$, let $\beta^{m,k} = (\beta_1^{m,k}, \beta_2^{m,k}, \beta_3^{m,k})$ such that $\beta_1^{m,k}$ is the line segment starting at (x_m, t_m) and parallel to $\beta_1^{(k)}$. Let $(0, t_{1,m,k})$ be the end point of $\beta_1^{m,k}$. Then $\beta^{m,k} \to \beta^{(k)}$ in $W^{1,\infty}, \dot{\beta}_i^{m,k} = \dot{\beta}_i^{(k)}$ for $i = 1,2,3$ and $\int_{\beta^{m,k}} \lambda^+ \to \int_{\beta^k} \lambda^+$ as $m \to \infty$. $\beta_2^{m,k} = 0$ in $[t_{2k}, t_{1,m,k}]$ if $t_{1,m,k} \ge t_{2k}$ and $\beta^{m.k} = \beta^{(k)}$ in $[0, t_{2k}]$. If $t_{1,m,k} \in [t_{2k}, t_{1k}]$, then $\beta^{m,k} = \beta^{(k)}$ in $[0, t_{1,m,k}]$. If $t_{1,m,k} < t_{2k}$ then $\beta_3^{m,k}$ is the line segment parallel to $\beta_3^{(k)}$ in $[0, t_{1,m,k}]$.

Case (ii) : $x = 0$. Choose $p_0 > 0$ such that $h(p_0) > \eta(|u_0|_T)$. If $x_m = 0$, define $\beta^{m,k} \in c(x_m, t_m)$ as follows. If $\beta^{(k)}$ is a line segment, then $\beta^{m,k}$ is the line segment parallel to β^k. If $\beta^{(k)}$ is not a line segment, then if $x_m > 0, \beta_1^{m,k}(\theta) = \dfrac{\theta - t_{2k}}{t_m - t_{2k}} x_m$ for $\theta \in [t_{2k}, t_m]$ and $\beta^{m,k} = \beta^{(k)}$ in $[0, t_{2k}]$. If $x_m = 0$, then $\beta_2^{m,k}(\theta) = 0$ in $[t_{2k}, t_m]$ and $\beta^{m,k} = \beta^{(k)}$ in $[0, t_{2k}]$. Now for m large and $x_m > 0, \dot{\beta}_1^{m,k} = \dfrac{x_m}{t_m - t_{2k}} < p_0$ and hence $h(\dot{\beta}_1^{m,k}) \ge h(p_0) > \eta(|u_0|_T) > \eta(u_0(\beta^{m,k}(0)))$. Also $\beta^{m,k} \to \beta^{(k)}$ in $W^{1,\infty}$.

Now by construction, upper semicontinuity of h and by continuity of u_0, it follows easily that there exist $m_k > 0$ such that for all $m \geq m_k$, $\beta^{m,k} \in \overline{A}(u_0, x_m, t_m)$. Hence

$$\begin{aligned}\overline{\lim}_{m\to\infty}\eta(\overline{u}(x_m,t_m)) &\leq \lim_{k\to\infty}\overline{\lim}_{m\to\infty}\left\{\eta(u_0(\beta^{m,k}(0))) - \int_{\beta^{m,k}}\lambda^+\right\}\\ &= \lim_{k\to\infty}\eta(u_0(\beta^{(k)}(0))) - \int_{\beta^k}\lambda^+\\ &= \eta(u_0(\beta(0))) - \int_\beta \lambda^+\\ &= \eta(\overline{u}(x,t)).\end{aligned}$$

This proves that $x \mapsto \overline{u}(x,t)$ is upper semicontinuous.

Claim 1 : Let $\gamma = (\gamma_1, \gamma_2, \gamma_3) \in \overline{A}(u_0, x, t)$ defined on the partition $0 = t_3 \leq t_2 \leq t_1 \leq t_0 = t$ and $\alpha = \gamma|_{[s,t]}$ and $\beta = \gamma|_{[0,s]}$. Then $\alpha \in \overline{A}(\overline{u}, x, s, t)$ and $\beta \in \overline{A}(u_0, \alpha(s), s)$.

Since $\gamma \in \overline{A}(u_0, x, t)$, hence

$$u_0(\gamma(0)) < h(\dot{\gamma}_3), \quad \eta(u_0(\gamma(0))) - \int_\gamma \lambda^+ < \eta(h(\dot{\gamma}_1)). \tag{5.9}$$

Case (i) : $[t_2, t_1] \neq \phi$.

Let $s \in (t_1, t)$ then $\dot{\gamma}_1 = \dot{\alpha} = \dot{\beta}_1,\ \dot{\gamma}_3 = \dot{\beta}_3,\quad \int_\gamma \lambda^+ = \int_\beta \lambda^+$. Hence from (4.19) $\eta(u_0(\beta(0))) - \int_\beta \lambda^+ < \eta(h(\dot{\beta}_1)),\quad u_0(\beta(0)) < h(\dot{\beta}_3)$ and hence $\beta \in \overline{A}(u_0, \alpha(s), s)$. $\eta(\overline{u}(\beta(s), s)) \leq \eta(u_0(\beta(0))) - \int_\beta \lambda^+ < \eta(h(\dot{\beta}_1)) = \eta(h(\dot{\alpha}))$, implies that $\overline{u}(\alpha(s), s) = \overline{u}(\beta(s), s) < h(\dot{\alpha}_1)$. Hence $\alpha \in \overline{A}(\overline{u}, x, s, t)$. Let $s \in [t_2, t_1]$, then $\dot{\gamma}_1 = \dot{\alpha}_1, \dot{\gamma}_3 = \dot{\beta}_3, \int_\alpha \lambda^+ + \int_\beta \lambda^+ = \int_\gamma \lambda^+,\ \beta_1 = \phi, \alpha_3 = \phi$. Hence from (4.19) $u_0(\beta(0)) < h(\dot{\beta}_3)$ implies that $\beta \in \overline{A}(u_0, \alpha(s), s)$. Hence $\eta(\overline{u}(\alpha(s), s)) - \int_\alpha \lambda^+ \leq \eta(u_0(\beta(0))) - \int_\beta \lambda^+ - \int_\alpha \lambda^+ = \eta(u_0(\gamma(0))) - \int_\gamma \lambda^+ < h(\dot{\alpha}_1)$. Hence $\alpha \in \overline{A}(\overline{u}, x, s, t)$. Let $s \in [0, t_2)$. Then $\dot{\gamma}_1 = \dot{\alpha}_1,\ \dot{\gamma}_3 = \dot{\alpha}_3 = \dot{\beta},\ \beta_1 = \beta_2 = \phi$. Hence from (4.19) $\beta \in \overline{A}(u_0, \alpha(s), s)$ and hence $\eta(\overline{u}(\alpha(s), s)) \leq \eta(u_0(\beta(0))$. This implies that $\eta(\overline{u}(\alpha(s), s)) - \int_\alpha \lambda^+ \leq \eta(u_0(\beta(0))) - \int_\alpha \lambda^+ = \eta(u_0(\gamma(0))) - \int_\gamma \lambda^+ < \eta(h(\dot{\gamma}_1)) = \eta(h(\dot{\alpha}_1))$. Hence $\alpha \in \overline{A}(\overline{u}, x, s, t)$.

Case (ii) : $[t_2, t_1] = \phi$.

Then α, β, γ are line segments. If $\dot{\gamma} = 0$ then $\dot{\alpha} = \dot{\beta} = 0$. Hence $\alpha \in \overline{A}(u_0, x, s, t)$, $\beta \in \overline{A}(\overline{u}, \alpha(s), s)$ since $h(0) = \infty$. Suppose $\dot{\gamma} < 0$, then $\int_\alpha \lambda^+ = 0, \dot{\alpha} = \dot{\beta} = \dot{\gamma} < 0$ and $u_0(\beta(0)) = u_0(\gamma(0)) < h(\dot{\gamma}) = h(\dot{\beta})$. Hence $\beta \in \overline{A}(u_0, \alpha(s), s)$ and this implies that $\eta(\overline{u}(\alpha(s), s)) \leq \eta(u_0(\beta(0)))$. Therefore $\overline{u}(\alpha(s), s) \leq u_0(\beta(0)) < h(\dot{\beta}) = h(\dot{\alpha})$. Hence $\alpha \in \overline{A}(\overline{u}, x, s, t)$.

Suppose $\dot{\gamma} > 0$, then $\dot{\alpha} = \dot{\beta} = \dot{\gamma}, \int_\gamma \lambda^+ = 0$ and $\eta(u_0(\beta(0))) < \eta(h(\dot{\gamma})) = \eta(h(\dot{\beta}))$. Hence $\beta \in \overline{A}(u_0, \alpha(s), s)$ and $\eta(\overline{u}(\alpha(s), s)) \leq \eta(u_0(\beta(0))) < \eta(h(\dot{\alpha}))$. Hence $\alpha \in \overline{A}(\overline{u}, x, s, t)$. This proves the claim.

From the claim it follows that the set

$$\begin{aligned}D &= \left\{(\alpha, \beta); \alpha = \gamma|_{[s,t]}, \beta = \gamma|_{[0,s]}, \gamma \in \overline{A}(u_0, x, t)\right\}\\ &\subset \left\{(\alpha, \beta); \alpha \in \overline{A}(\overline{u}, x, s, t), \beta \in \overline{A}(u_0, \alpha(s), s)\right\}.\end{aligned}$$

Hence if we define $\vartheta_1(x,t)$ as follows, then from (5.4)

$$\begin{aligned}
\eta(\vartheta_1(x,t)) &= \inf\left\{\eta(\overline{u}(\alpha(s),s)) - \int_\alpha \lambda^+; \alpha \in \overline{A}(\overline{u},x,s,t)\right\} \\
&= \inf_\alpha \left\{\inf_\beta \left\{\eta(u_0(\beta(0))) - \int_\beta \lambda^+; \beta \in \overline{A}(u_0,\alpha(s),s)\right\}\right. \\
&\qquad \left. - \int_\alpha \lambda^+; \alpha \in \overline{A}(\overline{u},x,s,t)\ \right\} \\
&\leq \inf\left\{\eta(u_0(\beta(0))) - \int_\alpha \lambda^+ - \int_\beta \lambda^+; (\alpha,\beta) \in D\right\} \\
&= \inf\left\{\eta(u_0(\gamma(0))) - \int_\gamma \lambda^+; \gamma \in \overline{A}(u_0,x,t)\right\} \\
&= \eta(\overline{u}(x,t)).
\end{aligned} \tag{5.10}$$

Hence the first inequality of the Lemma.

Let $(x_k,t_k) \to (x,t)$ such that $\overline{u}_*(x,t) = \lim_{k\to\infty} \overline{u}(x_k,t_k)$. Let $\epsilon > 0$, from (5.10) choose $\beta^{(k)} \in \overline{A}_M(\overline{u},x_k,s,t_k)$ and $k_0 > 0$ such that for all $k \geq k_0$

$$\eta(\overline{u}_*(x,t)) \geq \eta(\overline{u}(x_k,t_k)) - \epsilon/2 \tag{5.11}$$

$$\eta(\overline{u}(x_k,t_k)) > \eta(\overline{u}(\beta^{(k)}(s),s)) - \int_{\beta^{(k)}} \lambda^+ - \epsilon/2. \tag{5.12}$$

Extract a subsequence still denoted by $\beta^{(k)}$ converging to β in $W^{1,\infty}$. Then

$$\begin{aligned}
\overline{u}_*(\beta(s),s) &\leq \underline{\lim}_{k\to\infty} \overline{u}_*(\beta^{(k)}(s),s) \\
&\leq \lim_{k\to\infty} \overline{u}(\beta^{(k)}(s),s) \\
&\leq \overline{\lim}_{k\to\infty} h(\dot{\beta}_3^{(k)}). \\
&\leq h(\dot{\beta}_3), \\
\eta(\overline{u}_*(\beta(s),s)) - \int_\beta \lambda^+ &\leq \underline{\lim}_{k\to\infty} \left\{\eta(\overline{u}_*(\beta^{(k)}(s),s) - \int_{\beta^{(k)}} \lambda^+\right\} \\
&\leq \underline{\lim}_{k\to\infty} \{\eta(\overline{u}(\beta^k(s),s)) - \int_{\beta^{(k)}} \lambda^+\} \\
&\leq \overline{\lim}_{k\to\infty} \eta(h(\dot{\beta_1}^{(k)})) \\
&\leq \eta(h(\dot{\beta}_1)).
\end{aligned}$$

Hence $\beta \in \underline{A}(\overline{u}_*,x,s,t)$ and from (5.11) and (5.12)

$$\begin{aligned}
\eta(\overline{u}_*(x,t)) &\geq \lim_{k\to\infty} \eta(\overline{u}(x_k,t_k)) - \epsilon/2 \\
&\geq \lim_{k\to\infty} (\eta(\overline{u}(\beta^{(k)}(s),s)) - \int_{\beta^k} \lambda^+) - \epsilon \\
&\geq \lim_{k\to\infty} (\eta(\overline{u}_*(\beta^{(k)}(s),s)) - \int_{\beta^k} \lambda^+) - \epsilon \\
&\geq (\eta(\overline{u}_*(\beta(s),s)) - \int_\beta \lambda^+) - \epsilon \\
&\geq \inf\left\{\eta(\overline{u}_*(\gamma(s),s)) - \int_\gamma \lambda^+; \gamma \in \underline{A}_M(\overline{u}_*,x,s,t)\right\} - \epsilon.
\end{aligned}$$

Since ϵ is arbitrary and hence the Lemma follows. $\square$

Lemma 5.4. *Let* $\underline{u}$ *and* $\overline{u}$ *be as in (2.14) and (2.15) respectivly. Then* $\underline{u}^* = \overline{u}$ *and* $\overline{u}_* = \underline{u}$.

Proof : Proof is divided into three steps.

Step 1 : Let $\alpha > 1$ and $\{q_k\}$ be a bounded sequence. Then $\underline{\lim}_{k\to\infty}\, h(\alpha q_k) < \underline{\lim}_{k\to\infty} h(q_k)$.

Suppose not, then let for a subsequence still denoted by $\{q_k\}$ such that

$$q_k \to q_0, \quad \lim_{k\to\infty} h(\alpha q_k) = \lim_{k\to\infty} h(q_k) = \eta.$$

Choose $|p_k| = |\tilde{p}_k| = 1$ such that for all $|p| = 1$

$$\begin{array}{lrcl} (i) & H(h(q_k), p_k) & = & q_k p_k, \quad H(h(q_k), p) \geq q_k p \\ (ii) & H(h(\alpha q_k), \tilde{p}_k) & = & \alpha q_k \tilde{p}_k, \quad H(h(\alpha q_k), p) \geq \alpha q_k p \end{array}$$

Again going to a subsequence, can assume that $p_k \to p_0,\ \tilde{p}_k \to \tilde{p}_0$ as $k \to \infty$. Then by continuity of $H, q_0 p_0 = H(\eta, p_0) = \lim_{k\to\infty} H(h(\alpha q_k), p_0) \geq \alpha q_0 p_0$. Since $H(\eta, p_0) > 0$, it follows that $\alpha \leq 1$ which is a contradiction. This proves step 1.

Step 2 : Let $\tau > t$. Then $\underline{u}(x,t) \geq \overline{u}(x,\tau)$.

Let $\beta = (\beta_1, \beta_2, \beta_3) \in \underline{A}(u_0, x, t)$ defined on the partition $0 = t_3 \leq t_2 \leq t_1 \leq t_0 = t$ be a minimiser for $\underline{u}$. Then $\eta(\underline{u}(x,t)) = \eta(u_0(\beta(0))) - \int_\beta \lambda^+,\ \ \eta(u_0(\beta(0))) - \int_\beta \lambda^+ \leq \eta(h(\dot{\beta}_1))$ and $u_0(\beta(0)) \leq h(\dot{\beta}_3)$. Let $\gamma = (\gamma_1, \gamma_2, \gamma_3) \in \overline{A}(u_0, x, \tau)$ be defined as follows.

Case (i) : Suppose $u_0(\beta(0)) < h(\dot{\beta}_3)$. Then $\gamma \in \overline{A}(u_0, x, \tau)$ is given by

$$\begin{array}{rcl} \gamma_1(\theta) & = & x + (\theta - \tau)\dfrac{x}{\tau - t_1} \quad \text{if} \quad \theta \in [t_1, \tau] \\ \gamma_2(\theta) & = & \beta_2(\theta) \quad \text{if} \quad \theta \in [t_2, t_1] \\ \gamma_3(\theta) & = & \beta_3(\theta) \quad \text{if} \quad \theta \in [0, t_2]. \end{array}$$

This implies $u_0(\gamma(0)) = u_0(\beta(0)) < h(\dot{\beta}_3) = h(\dot{\gamma}_3),\ \eta(u_0(\gamma(0))) - \int_\gamma \lambda^+ = \eta(u_0(\beta(0)) - \int_\beta \lambda^+ \leq \eta(h(\dot{\beta}_1)) = \eta\left(h\left(\dfrac{x}{t - t_1}\right)\right) = \eta\left(h\left(\dfrac{x}{\tau - t_1}\dfrac{\tau - t_1}{t - t_1}\right)\right) < \eta\left(h\left(\dfrac{x}{\tau - t_1}\right)\right)$, by Step 1. Hence $\gamma \in \overline{A}(u_0, x, \tau)$ and $\underline{u}(x,t) \geq \overline{u}(x,\tau)$.

Case (ii) : Suppose $u_0((\beta(0)) = h(\dot{\beta}_3)$. Then

$$\begin{array}{rcl} \eta\left(h\left(\dfrac{-\beta(0)}{t_2}\right)\right) - \int_{t_2}^{t_1} \lambda^+ & = & \eta(u_0(\beta(0)) - \int_\beta \lambda^+ \\ & \leq & \eta(h(\dot{\beta}_1)) \\ & < & \eta\left(h\left(\dfrac{x}{\tau - t_1}\right)\right). \end{array}$$

Let $A = \left\{\mu; \mu < t_1, \eta\left(h\left(\dfrac{-\beta(0)}{\mu}\right)\right) - \int_\mu^{t_1} \lambda^+ \ < \ \eta\left(h\left(\dfrac{x}{\tau - t_1}\right)\right)\right\}$. h is an upper semicontinuous function implies A is open and hence there exists ε_0 such that $\forall\mu \in [t_2, t_2+\varepsilon_0)$, $\eta\left(h\left(-\frac{\beta(0)}{\mu}\right)\right) - \int_\mu^{t_1}\lambda^+ \;<\; \eta\left(h\left(\frac{x}{\tau - t_1}\right)\right)$. Let $\mu \in (t_2, t_2+\varepsilon_0)$. Define now γ by

$$\begin{aligned}
\gamma_1(\theta) &= x + (\theta-\tau)\frac{x}{\tau-t_1} \quad \text{if} \quad \theta \in [t_1,\tau] \\
\gamma_2(\theta) &= 0 \quad \text{if} \quad \theta \in [\mu, t_1] \\
\gamma_3(\theta) &= \beta(0) + \frac{-\beta(0)}{\mu}\theta \quad \text{if} \quad \theta \in [0,\mu].
\end{aligned}$$

Then $u_0(\gamma(0)) = u_0(\beta(0)) = h\left(-\frac{\beta(0)}{t_2}\right) = h\left(-\frac{\beta(0)}{\mu}\frac{\mu}{t_2}\right) \;<\; h\left(-\frac{\beta(0)}{\mu}\right) = h(\dot\gamma_3)$ by Step 1. Hence

$$\begin{aligned}
\eta(u_0(\gamma(0)) - \int_\gamma \lambda^+ &= \eta(u_0(\beta(0)) - \int_\mu^{t_1}\lambda^+ \\
&< h\left(-\frac{\beta(0)}{\mu}\right) - \int_\mu^{t_1}\lambda^+ \;<\; h(\dot\gamma_3).
\end{aligned}$$

This implies $\gamma \in \overline{A}(u_0,x,\tau)$. Hence $\underline{u}(x,t) \geq \overline{u}(x,\tau)$.

Step 3 : Let $B_r(x,t)$ be a ball centered at (x,t) with radius r. Let $t>0$ and let $t_k < t$ and $t_k \to t$. Then from step (2)

$$\begin{aligned}
\underline{u}^*(x,t) &= \lim_{r\to 0}\sup_{B_r}\underline{u}(z) \\
&\geq \lim_{k\to\infty}\underline{u}(x,t_k) \geq \overline{u}(x,t).
\end{aligned}$$

On the other hand $\underline{u} \leq \overline{u}$ and hence $\underline{u}^*(x,t) \leq \overline{u}(x,t)$, implies that $\underline{u}^*(x,t) = \overline{u}(x,t)$. Similarly $\overline{u}_* = \underline{u}$. This proves the Lemma. □

Lemma 5.5. *(Dynamic programming principle) Let $\underline{u}$ and $\overline{u}$ be defined as in (2.16) and (2.17) respectively. Then*

$$\begin{aligned}
\underline{u}(x,t) &= \inf \eta^{-1}\left\{\eta(\underline{u}(\beta(s),s)) - \int_\beta \lambda^+ ; \beta \in \underline{A}_M(\underline{u},x,s,t)\right\}, \\
\overline{u}(x,t) &= \inf \eta^{-1}\left\{\eta(\overline{u}(\beta(s),s)) - \int_\beta \lambda^+ ; \beta \in \overline{A}_M(\overline{u},x,s,t)\right\}.
\end{aligned}$$

Proof : Proof follows from Lemma 5.2, Lemma 5.3 and Lemma 5.4. □

Proof of Theorem 2.2 : Let $0 < t \leq T$ and $x \in I\!\!R_+$. From (1) and (2) of Lemma 4.4, there exist $M(T) > 0, \quad \alpha^{(t)}, \beta^{(t)} \in C_{M(T)}(x,t)$ such that $\eta(\underline{u}(x,t)) = \eta(u_0(\beta^{(t)}(0))) - \int_{\beta^{(t)}}\lambda^+,\ \eta(\overline{u}(x,t)) = \eta(u_0(\alpha^{(t)}(0))) - \int_{\alpha^{(t)}}\lambda^+$. Now $|(x,x) - (\alpha^{(t)},\beta^{(t)})| = |\int_0^t(\dot\alpha^{(t)}(\theta), \dot\beta^{(t)}(\theta)) \; d\theta| \leq M(T)t$. This implies that $\lim_{t\to 0}(\overline{u}(x,t), \underline{u}(x,t)) = (u_0(x), u_0(x))$.

Sub solution : Suppose $\underline{u}$ is not a sub solution. Then there exist $(x_0, t_0) \in (\overline{I\!R}_+ \times I\!R_+), \mu > 0, \varphi \in C^1(\overline{I\!R_+ \times I\!R_+})$ such that $\underline{u}^*(x_0, t_0) = \varphi(x_0, t_0),\ \underline{u}^* - \varphi$ has local maximum at (x_0, t_0) and at $(x_0, t_0),\ \varphi_t + H(\varphi, \varphi_x) \geq 4\mu$ and $\varphi_x(x_0, t_0) < \lambda(t_0)$ if $x_0 = 0$. By continuity, there exist $\delta > 0$ such that at $(x_0, t_0),\ \varphi_t + H(\varphi + \delta, \varphi_x) \geq 3\mu, \varphi_x(x_0, t_0) < \lambda(t_0)$ if $x_0 = 0$. Hence from (A_6) there exist a p such that at $(x_0, t_0), \varphi_t + p\varphi_x \geq 2\mu, \varphi(x_0, t_0) + \delta \leq h(p)$ and $\varphi_x(x_0, t_0) < \lambda(t_0)$ if $x_0 = 0$. Hence there exist a ball B around (x_0, t_0) such that for all $(x, t) \in B \cap (\overline{I\!R}_+ \times I\!R_+)$,

$$\varphi_t + p\varphi_x \geq \mu,\ \varphi \leq h(p),\ \varphi_x < \lambda \text{ if } x_0 = 0. \tag{5.13}$$

Suppose $x_0 > 0$. Then by shrinking B if necessary, can assume that $B \subset I\!R_+ \times I\!R_+$. Let $\beta(\theta) = p(\theta - t_0) + x_0$ and choose $s_0 < t_0$ such that $(\beta(\theta), \theta) \in B$ for $\theta \in [s_0, t_0]$. Since $\underline{u}^* \leq \varphi \leq h(p) = h(\dot{\beta})$ and $\int_\beta \lambda^+ = 0$, it follows from (5.6) that $\varphi(x_0, t_0) = \underline{u}^*(x_0, t_0) \leq \underline{u}^*(\beta(s_0), s_0) \leq \varphi(\beta(s_0), s_0)$. On the other hand from (5.13)

$$\begin{aligned} \varphi(x_0, t_0) - \varphi(\beta(s_0), s_0) &= \int_{s_0}^{t_0} \frac{d}{d\theta}\varphi(\beta(\theta), \theta)\ d\theta \\ &= \int_{s_0}^{t_0} (\varphi_t + p\varphi_x)(\beta(\theta), \theta)\ d\theta \\ &\geq \mu(t_0 - s_0) > 0, \end{aligned}$$

which is a contradiction.

Suppose $x_0 = 0$. If $p < 0$, let $\beta(\theta) = p(\theta - t_0) + x_0$ and choose $s_0 < t_0$ such that $(\beta(\theta), \theta) \in B \cap (\overline{I\!R}_+ \times I\!R_+)$ for $\theta \in [s_0, t_0]$. Since $\{\beta = 0\} = \{x_0\}$ and hence $\int_\beta \lambda^+ = 0$. Since $\underline{u}^* < \varphi \leq h(p) = h(\dot{\beta})$ and hence as in earlier situation we obtain a contradiction.

Let $p \geq 0$. Suppose $\varphi_x(x_0, t_0) \leq 0$, then $2\mu \leq (\varphi_t + p\varphi_x)(x_0, t_0) \leq \varphi_t(x_0, t_0)$. By shrinking B if necessary it follows that $\varphi_t \geq \mu$ in $B \cap (\overline{I\!R}_+ \times I\!R_+)$ and hence $\varphi(x_0, t_0) \geq \varphi(x_0, s_0) + \mu(t_0 - s_0)$ for $s_0 < t_0$ sufficiently close to t_0. Let $p_1 < 0$ with $\varphi < h(p_1)$ in $B \cap (\overline{I\!R}_+ \times I\!R_+)$ and $\beta(\theta) = p_1(\theta - t_0) + x_0$, then $\int_\beta \lambda^+ = 0$ and $\underline{u}^* \leq \varphi \leq h(p_1)$. Hence from (4.16), $\phi(x_0, t_0) = \underline{u}^*(x_0, t_0) \leq \underline{u}^*(\beta(s_0), s_0) \leq \varphi(\beta(s_0), s_0)$. Letting $p_1 \to 0$ to obtain $\varphi(x_0, t_0) \leq \varphi(x_0, s_0)$ which is a contradiction. Suppose $\varphi_x(x_0, t_0) > 0$ then $\lambda(t_0) > 0$. Let $\beta(\theta) = x_0$ for $\theta \in [s_0, t_0]$. Since $h(\dot{\beta}) = h(0) = \infty$, hence $\beta \in \overline{A}(\underline{u}^*, x_0, s_0, t_0)$ and hence from (4.16)

$$\begin{aligned} \eta(\varphi(x_0, t_0)) &= \eta(\underline{u}^*(x_0, t_0)) \leq \eta(\underline{u}^*(x_0, s_0)) - \int_{s_0}^{t_0} \lambda^+ \\ &\leq \eta(\varphi(x_0, s_0)) - \int_{s_0}^{t_0} \lambda^+. \end{aligned}$$

This implies that

$$\begin{aligned} \eta'(\varphi(x_0, t_0))\varphi_t(x_0, t_0) &= \lim_{s_0 \to t_0} \frac{\eta(\varphi(x_0, t_0)) - \eta(\varphi(x_0, s_0))}{t_0 - s_0} \\ &\leq -\lambda^+(t_0) = -\lambda(t_0). \end{aligned}$$

Since $\eta'(u) = \dfrac{1}{H(u,1)}$ it follows from the above inequality that at $(x_0, t_0), 4\mu \leq \phi_t + H(\varphi, \varphi_x) = \varphi_t + \varphi_x H(\varphi, 1) \leq (\varphi_t + \lambda H(\varphi, 1))(x_0, t_0) \leq 0$, which is a contradiction. This proves that $\underline{u}$ is a subsolution.

Super solution : Suppose $\overline{u}$ is not a super solution. Then there exist $(x_0, t_0) \in (\overline{I\!R}_+ \times I\!R_+), \mu > 0, \varphi \in C^1(\overline{I\!R_+ \times I\!R_+})$ such that $\overline{u}_*(x_0, t_0) = \varphi(x_0, t_0), \overline{u}_* - \varphi$ has local minima at (x_0, t_0) and at $(x_0, t_0), \varphi_t + H(\varphi, \varphi_x) \leq -4\mu, \varphi_x(x_0, t_0) > \lambda(t_0)$ if $x_0 = 0$. Hence in a ball B around (x_0, t_0), $\varphi_t + H(\varphi, \varphi_x) \leq -3\mu$ and $\varphi_x > \lambda$ if $x_0 = 0$. Hence from (A_6), for $(x, t) \in B \cap (\overline{I\!R}_+ \times I\!R_+)$,

$$\text{if } \varphi(x,t) \leq h(p), \text{ then } \varphi_t + p\varphi_x \leq -3\mu \quad \text{and} \quad \varphi_x > \lambda \quad \text{if} \quad x_0 = 0. \tag{5.14}$$

From (1) of Lemma 5.1, let for each $s < t_0, \beta_s$ be a minimizer for

$$\eta(\overline{u}(x,t)) = \inf\left\{\eta(\overline{u}_*(\beta(s), s)) - \int_\beta \lambda^+; \quad \beta \in \underline{A}_M(\overline{u}_*, x, s, t)\right\}, \tag{5.15}$$

then from Lemma 5.3 we have

$$\begin{aligned} \eta(\varphi(x_0, t_0)) &= \eta(\overline{u}_*(x_0, t_0)) \\ &\geq \eta(\overline{u}_*(\beta_s(s), s)) - \textstyle\int_{\beta_s} \lambda^+ \\ &\geq \eta(\varphi(\beta_s(s), s)) - \textstyle\int_{\beta_s} \lambda^+, \end{aligned}$$

$$\eta(\varphi(x_0, t_0)) - \eta(\varphi(x_0, s)) \geq \eta(\varphi(\beta_s(s), s)) - \eta(\varphi(x_0, s)) - \int_{\beta_s} \lambda^+. \tag{5.16}$$

Let $x_0 > 0$. Without loss of generality assume that $B \subset (I\!R_+ \times I\!R_+)$ and there exist a $s_0 < t_0$ such that $(\beta_s(\theta), \theta) \in B$ for all $s \in [s_0, t_0],\ \theta \in [s, t_0]$. Furthermore $\beta_s(\theta) = p_s(\theta - t_0) + x_0$ and $\{\beta_s = 0\} = \phi$. As $s \to t_0$, let $p_s \to p_0$. Divide $(t_0 - s)$ in (5.16) and letting $s \to t_0$, to obtain $\eta'(\varphi(x_0, t_0))(\varphi_t + p_0\varphi_x)(x_0, t_0) \geq 0$, $\quad \varphi(x_0, t_0) = \lim_{s \to t_0} \varphi(\beta_s(s), s) \leq \lim_{s \to t_0} \overline{u}_*(\beta_s(s), s) \leq \lim_{s \to t_0} h(p_s) \leq h(p_0)$. This contradicts (5.14).

Let $x_0 = 0$. If $\beta_s = x_0 + p_s(\theta - t_0)$, then $p_s \leq 0$. Let $p_s \to p_0$ as $s \to t_0$. Divide by $(t_0 - s)$ and letting $s \to t_0$ in (5.16) to obtain $\eta'(\varphi(x_0, t_0))(\varphi_t + p_0\varphi_x)(x_0, t_0) \geq -\lambda^+(t_0)$ and $\varphi(x_0, t_0) \leq h(p_0)$. If $\lambda(t_0) \leq 0$, then $(\varphi_t + p_0\varphi_x)(x_0, t_0) \geq 0$ which contradicts (5.14). If $\lambda(t_0) \geq 0$, then $\varphi_x(x_0, t_0) > \lambda(t_0) \geq 0$ hence $0 \leq (H(\varphi, 1)\lambda(t_0) + (\varphi_t + p_0\varphi_x))(x_0, t_0) \leq (H(\varphi, \varphi_x) + \varphi_t)(x_0, t_0) \leq -4\mu$ which is a contradiction.

Suppose β_s is not a line segment, let $\beta_s = (\phi, \beta_{2s}, \beta_{3s})$ defined on the partition $s = s_3 < s_2 < s_1 = s_0 = t_0$. Let $\beta_{3s}(\theta) = p_s(\theta - s_1)$ for $\theta \in [s, s_2]$ with $\varphi(\beta_s(s), s) \leq h(\dot{\beta}_{3s}) = h(p_s)$ and $p_s < 0$. Let $p_s \to p_0$ as $s \to t_0$. Suppose $\lambda(t_0) \geq 0$, then $\varphi_x(x_0, t_0) > 0$ and hence in $B \cap (\overline{I\!R}_+ \times I\!R_+),\ \varphi_x \geq 0$. This implies that $\varphi(\beta_s(s), s) \geq \varphi(x_0, s)$. Hence from Lemma 5.3, $\eta(\varphi(x_0, t_0)) = \eta(\overline{u}_*(x_0, t_0)) \geq \eta(\varphi(\beta_s(s), s)) - \int_s^{t_0} \lambda^+ \geq \eta(\varphi(x_0, s)) - \int_s^{t_0} \lambda^+$. Divide by $t_0 - s$ and letting $s \to t_0$ to obtain $\eta'(\varphi(x_0, t_0))\varphi_t(x_0, t_0) \geq -\lambda^+(t_0) = -\lambda(t_0)$. That is $0 \leq (\varphi_t + \lambda(t_0)H(\varphi, 1))(x_0, t_0)$, but $(\varphi_t + \varphi_x H(\varphi, 1))(x_0, t_0) \leq -4\mu$ which is a contradiction. Suppose $\lambda(t_0) < 0$, then $\lambda^+ = 0$ hence from Lemma 5.3, $\varphi(x_0,t_0) = \overline{u}_*(x_0,t_0) \geq \overline{u}_*(\beta_s(s),s) \geq \varphi(\beta_s(s),s)$. Hence $0 \leq \lim_{s\to t_0} \dfrac{\varphi(x_0,t_0)-\varphi(\beta_s(s),s)}{t_0-s} = \varphi_t+\varphi_x$. Suppose $\varphi_x \leq 0$, then $0 \leq (\varphi_t+\varphi_x)(x_0,t_0) \leq \varphi_t(x_0,t_0) \leq (\varphi_t + H(\varphi,\varphi_x))(x_0,t_0) \leq -4\mu$, which is a contradiction. If $\varphi_x > 0$, then $\varphi(x_0,t_0) > \varphi(\beta_s(s),s) \geq \varphi(x_0,s)$ and hence $\varphi_t(x_0,t_0) \geq 0$. But $\varphi_t(x_0,t_0) \leq (\varphi_t + H(\varphi,\varphi_x))(x_0,t_0) \leq -4\mu$ which is a contradiction. This proves that $\overline{u}$ is a super solution. Since $\underline{u}^* = \overline{u}$ and $\overline{u}_* = \underline{u}$ and hence $\underline{u},\overline{u}$ are viscosity solutions of (1.1). This completes the proof of Theorem 2.2. □

Acknowledgement

The authors would like to acknowledge funding from the Indo-French Centre for Promotion of Advanced Research, under Project 3401-2.

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