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CONVOLUTION ROOTS AND EMBEDDINGS OF PROBABILITY MEASURES ON LOCALLY COMPACT GROUPS

S. G. Dani

School of Mathematics, Tata Institute of Fundamental Research,
Homi Bhabha Road, Colaba, Mumbai 400 005, India
e-mail: dani@math.tifr.res.in

Abstract We discuss some properties of nilpotent Lie groups and their application in proving the embedding theorem for infinitely divisible probability measures on locally compact groups.

Key words Nilpotent Lie group, infinitely divisible measure.

1. Introduction

Let G be a locally compact second countable group. A probability measure μ on G is said to be *infinitely divisible* if for every $n \in \mathbb{N}$, μ admits a n th convolution root, namely there exists a probability measure μ_n on G such that $\mu_n^n = \mu$ (the exponent stands for the n th convolution power), and μ is said to be *embeddable* if there exists a continuous one-parameter convolution semigroup $\{\mu_t\}_{t \geq 0}$ such that $\mu_1 = \mu$. Evidently an embeddable probability measure is infinitely divisible. Under what conditions, say on the group or on the measure, the converse holds has been a topic of much investigation in literature. It is conjectured in particular that if G is connected Lie group then every infinitely divisible probability measure on G is embeddable, and a similar statement should also be true for other classes of groups, though indeed it does not hold for all groups (see [7, 8] for various details). The above mentioned conjecture for connected Lie groups was established in [5] for a large class of Lie groups, called class $\mathcal{C}$ groups, including all (connected) linear Lie groups, all simply connected Lie groups and all semisimple Lie groups. A more transparent approach to the problem was introduced in [6]. The focus of the proof in [6] nevertheless remained on algebraic groups, from which the results were deduced by extension to the class $\mathcal{C}$ groups. Here we isolate some of the underlying ideas, depending essentially only on properties of nilpotent Lie groups, without reference to algebraic groups, and present a generalisation which technically has a broader scope for application. Theorem 2.2 below, which is stated after recalling some preliminaries, is the main technical result in this respect and Corollary 2.4 describes a general application of it to the problem of embeddability of infinitely divisible probability measures. We shall indicate how together with some special features of the almost algebraic and more generally the class $\mathcal{C}$ groups Corollary 2.4 implies the embedding theorem of [6].

2. Preliminaries

Let G be a locally compact second countable group. We denote by $P(G)$ the space of all probability measures on G equipped with the usual weak* topology and the convolution product (see [7] for various generalities). For any closed subset S of G, $P(S)$ will denote the subspace consisting of probability measures on G with the support contained in S. For $\mu \in P(G)$ we denote by $\operatorname{supp}\mu$ the support of μ, and by $G(\mu)$ the smallest closed subgroup containing $\operatorname{supp}\mu$. We denote by $N(\mu)$ the normaliser of $G(\mu)$ in G and by $Z(\mu)$ the centraliser of $G(\mu)$ in G. We note that $Z(\mu)$ is a normal subgroup of $N(\mu)$. For any closed subgroup H of G we denote by H^0 the connected component of the identity in H.

For $n \in \mathbb{N}$ we denote by $R_n(\mu)$ the set of all nth roots of μ on G; namely $R_n(\mu) = \{\rho \in P(G) \mid \rho^n = \mu\}$. We recall that for any root ρ of μ the support of ρ is contained in $N(\mu)$; this holds more generally for all two-sided factors of μ (see [3], Proposition 1.1, or [1], Lemma 5.1). Furthermore, if $\rho \in R_n(\mu)$ for $n \in \mathbb{N}$, then there exists $x \in N(\mu)$ such that $\rho \in P(G(\mu)x)$, and for any such x we have $x^n \in G(\mu)$ (this is straightforward to deduce from the fact that the image of μ in the quotient $N(\mu)/G(\mu)$ is the point mass at identity; for details see [5], Proposition 2.7, or [1], Lemma 5.2). For $\rho \in R_n(\mu)$ we define an automorphism α_ρ of $Z(\mu)$ by $\alpha_\rho(g) = xgx^{-1}$ for all $g \in Z(\mu)$, where $x \in N(\mu)$ is an element such that $\rho \in P(G(\mu)x)$; we note that the automorphism α_ρ depends only on ρ and not on the specific choice of x, and that α_ρ^n is the identity automorphism. We recall the following simple fact which plays a crucial role in the arguments in the sequel, as also in much of the earlier work on the embedding theorem; a proof is recalled for convenience of the reader.

Lemma 2.1. *Let the notation be as above. Let $\rho \in R_n(\mu)$, $n \in \mathbb{N}$. Then, for all $z \in Z(\mu)$, $\rho z = \alpha_\rho(z)\rho$ and $(\rho z)^n = \alpha_\rho(z)\alpha_\rho^2(z)\cdots\alpha_\rho^n(z)\mu$.*

Proof : Let $x \in N(\mu)$ be such that $\rho \in P(G(\mu)x)$. Then $\rho z = (\rho x^{-1})xz = (\rho x^{-1})\alpha_\rho(z)x = \alpha_\rho(z)(\rho x^{-1})x = \alpha_\rho(z)\rho$. By repeated application of this we see that $(\rho z)^n = \alpha_\rho(z)\alpha_\rho^2(z)\cdots\alpha_\rho^n(z)\rho^n = \alpha_\rho(z)\alpha_\rho^2(z)\cdots\alpha_\rho^n(z)\mu$. □

Now let N be a compactly generated nilpotent Lie subgroup of $Z(\mu)$. We say that $\rho \in R_n(\mu)$ is *compatible* with N if N is α_ρ-invariant and moreover $\alpha_\rho(gN^0) = gN^0$ for all $g \in N$, namely the factor of α_ρ on N/N^0 is trivial.

Let T denote the unique maximal torus in N^0. Then T is contained in the center of N^0 and N^0/T is simply connected. For ρ which is compatible with N we denote by $M(\rho, N)$ the subgroup $\{g \in N^0 \mid \alpha_\rho(g) \in gT\}$; equivalently $M(\rho, N)$ is the closed subgroup of N^0 containing T and such that $M(\rho, N)/T$ is the set of fixed points of the factor action of α_ρ on N^0/T. Since N^0/T is a simply connected nilpotent Lie group, the subgroup $M(\rho, N)$ is a closed connected Lie subgroup.

We now state our main results:

Theorem 2.2. *Let G be a locally compact second countable group and $\mu \in P(G)$. Let N be a closed compactly generated nilpotent Lie subgroup of $Z(\mu)$. Let $n \in \mathbb{N}$ and $\{\nu_i\}$ be a sequence in $R_n(\mu)$ such that each ν_i is compatible with N and there exists a sequence $\{x_i\}$ in N such that $\{\nu_i x_i\}$ is relatively compact. Then there exists a sequence $\{z_i\}$ in N^0 such that $\{z_i \nu_i z_i^{-1}\}$ is relatively compact.*

Theorem 2.3. *Let G, μ and N be as in Theorem 2.2. Let $r, n \in \mathbb{N}$, and let $\{\lambda_i\}$ be a sequence in $R_{rn}(\mu)$. Suppose that there exist sequences $\{y_i\}$ and $\{z_i\}$ in N^0 such that $\{y_i \lambda_i y_i^{-1}\}$ and $\{z_i \lambda_i^r z_i^{-1}\}$ are relatively compact, and $M(\lambda_i, N) = M(\lambda_i^r, N)$ for all i. Then $\{z_i \lambda_i z_i^{-1}\}$ is relatively compact.*

From these general results we shall deduce the following embedding theorem. Given a sequence $\{\lambda_i\}$ in $P(G)$ we say that it is *relatively compact modulo N*, where N is a closed subgroup of G, if there exists a sequence $\{x_i\}$ in N such that $\{\lambda_i x_i\}$ is relatively compact.

We recall that $\mu \in P(G)$ is said to be *rationally embeddable* if there exists a homomorphism $\iota : \mathbb{Q}_+ \to P(G)$, where $\mathbb{Q}_+$ is the semigroup of positive rational numbers, such that $\iota(1) = \mu$.

Corollary 2.4. *Let G be a locally compact second countable group and $\mu \in P(G)$ be infinitely divisible. Let N be a compactly generated nilpotent Lie subgroup of $Z(\mu)$. For each $k \in \mathbb{N}$ let ν_k be a $k!$th root of μ which is compatible with N. Suppose that there exist an infinite subset Σ of $\mathbb{N}$ such that for all $n \in \Sigma$ the sequence $\{\nu_k^{k!/n!}\}$ of $n!$th roots is relatively compact modulo N. Then there exists sequences $\{k_j\}$ in $\mathbb{N}$ and $\{\zeta_j\}$ in N^0, and $r \in \mathbb{N}$, such that $k_j \to \infty$ as $j \to \infty$ and for $n \in \Sigma$ with $n \geq r$ the sequence $\{\zeta_j \nu_{k_j}^{k_j!/n!} \zeta_j^{-1} \mid j \in \mathbb{N}, k_j \geq n!\}$ is relatively compact. Consequently, μ is rationally embeddable in $P(G)$, and if G is a Lie group then μ is embeddable in a continuous one-parameter convolution semigroup contained in $P(G)$.*

3. Some Results on Nilpotent Lie Groups

In this section we prove some properties of nilpotent Lie groups needed in the proof of Theorem 2.2. Under certain further restrictions the results were proved earlier (see below for references). We begin by noting the following simple fact; it may be considered standard, but a short proof is included for the convenience of the reader.

Lemma 3.1. *Let N be a compactly generated nilpotent Lie group. Then N has a unique maximal compact subgroup.*

Proof : Passing to quotient modulo the maximal central torus of N^0 in proving the lemma we may assume that N^0 is simply connected. Then all compact subgroups of N are finite and we have to show that there is a unique maximal one. Let Z be the center of N. Then $N^0/N^0 \cap Z$ is also simply connected, and by an inductive hypothesis we may assume that N/Z has a unique maximal finite subgroup, say M/Z where M is a closed subgroup of N containing Z. Replacing N by M we may assume that N/Z is finite. We now proceed by induction on the order of N/Z. As N/Z is nilpotent, its center is nontrivial. Let A be a subgroup of N containing Z and such that A/Z is a nontrivial cyclic subgroup contained in the center of N/Z. Then A is a closed normal abelian subgroup of N. We note also that Z is compactly generated, and hence so is A. Also, A does not contain any torus of positive dimension, and hence it has a unique maximum finite subgroup, say Φ. As A is normal in N the uniqueness property of Φ implies that Φ is normal in N. Let $q : N \to N/\Phi$ be the quotient homomorphism. Now A/Φ is contained in the center of N/Φ and since the order of N/A is less than the order of N/Z by an inductive hypothesis we may assume that N/Φ has a unique maximal finite subgroup, say F. Then $q^{-1}(F)$ is the unique maximal finite subgroup of N. This proves the lemma. □

Remark 3.2. The argument above shows also that if $x \in N$ is such that xN^0 is of finite order in N/N^0 then xN^0 contains an element of finite order in N. Hence the maximal compact subgroup intersects all connected components of N.

Lemma 3.3. *Let N be a compactly generated nilpotent Lie group such that N^0 is simply connected. Let $x \in N$ and N' be the subgroup of N generated by N^0 and x. If N'/N^0 is infinite then N' can be realised as a codimension-one Lie subgroup of a simply connected Lie group M. If N'/N is finite then N' is a direct product of a finite subgroup F of N with N^0.*

Proof : Let $\mathrm{Aut}N^0$ be the group of all Lie automorphisms of N^0, and σ be the automorphism of N^0 induced by the conjugation action of x. Then $\mathrm{Aut}N^0$ is an algebraic group (viewed as the group of Lie automorphisms of the Lie algebra of N^0), and σ is a unipotent element in $\mathrm{Aut}N^0$. Now suppose that N'/N^0 is infinite. If σ is trivial, then the desired assertion is readily seen to hold if we choose M to be $N^0 \times \mathbb{R}$. Now suppose σ is nontrivial. Then there exists a (nontrivial) unipotent one-parameter subgroup $\{\sigma_t\}$ in $\mathrm{Aut}N^0$ such that $\sigma_1 = \sigma$. We define M to be the semidirect product of the groups $\{\sigma_t\}$ and N^0 corresponding to the action of $\{\sigma_t\}$ on N^0 by automorphisms. Then clearly M is a simply connected nilpotent Lie group and the subgroup generated by N^0 and σ is a codimension 1 subgroup Lie isomorphic to N'. This proves the first assertion.

Now suppose that N'/N^0 is finite. In this case σ is of finite order, and since it is unipotent it is trivial. Thus x centralises N^0. Let n be the order of σ. Then x^n is contained in the center of N^0. The latter is a vector group and therefore contains an element z such that $x^n = z^n$. Then xz^{-1} is an element of order n centralising N^0, and we see that N' is the direct product of the subgroup cyclic generated by xz^{-1} with N^0. This proves (ii). □

Through the rest of the section we denote by N a compactly generated nilpotent Lie group. We denote by $\mathrm{Aut}\,N$ the group of all Lie automorphisms of N, equipped with its usual topology as a Lie group, and by $\mathrm{Aut}'\,N$ the subgroup of $\mathrm{Aut}\,N$ consisting of those automorphisms whose factor action on N/N^0 is trivial. We denote by e the identity element of N and by I the identity automorphism of N.

We shall denote by C the unique maximal compact subgroup of N and by T the maximal central torus in N^0. By their uniqueness property C and T are normal subgroups of N.

Proposition 3.4. *Let $n \in \mathbb{N}$ and $\alpha \in \mathrm{Aut}' N$ be such that $\alpha^n = I$. Let $x \in N$ be such that $\alpha(x)\alpha^2(x)\cdots\alpha^n(x) \in C$. Then there exists $y \in N^0$ such that $\alpha(x) \in y^{-1}\alpha(y)C$.*

Proof : In proving the proposition we may without loss of generality assume C to be trivial. Then N^0 is simply connected. Also, by Remark 3.2 N/N^0 has no finite subgroup. As the factor action of α on N/N^0 is trivial, the condition on x implies that $(xN^0)^n$ is the identity element in N/N^0, and since the latter has no finite subgroup it follows that $xN^0 = N^0$ and hence $x \in N^0$. The Proposition then follows from Lemma 5.4 of [8], which is the special case of the proposition for simply connected nilpotent Lie groups. □

Proposition 3.5. *Let $n \in \mathbb{N}$, and $\{\alpha_i\}$ be a sequence in $\mathrm{Aut}' N$ such that $\alpha_i^n = I$ for all $i \in \mathbb{N}$. Let $\{x_i\}$ be a sequence in N such that $\{\alpha_i(x_i)\alpha_i^2(x_i)\cdots\alpha_i^n(x_i)\}$ is relatively compact. Then there exists a sequence $\{y_i\}$ in N, such that $\{y_i^{-1}x_i\}$ is relatively compact, and $\alpha_i(y_i)\alpha_i^2(y_i)\cdots\alpha_i^n(y_i) \in T$ for all i.*

Proof : In proving the proposition we may without loss of generality assume that T is trivial, namely that N^0 is simply connected. Since $\{\alpha_i\}$ is contained in $\mathrm{Aut}' N$ the condition on x_i's implies that $\{(x_iN^0)^n\}$ is a finite subset of N/N^0. By the root compactness of compactly generated nilpotent groups (see [7], Chapter III) this implies that $\{x_iN^0\}$ is finite. Passing to subsequences we may assume that there exists $x \in N$ such that $x_iN^0 = xN^0$ for all i.

Let N' be the Lie subgroup generated by N^0 and x. From the definition of $\mathrm{Aut}' N$ it follows that N' is invariant under the action of $\mathrm{Aut}' N$. Now suppose first that N'/N^0 is infinite. Let M be the simply connected nilpotent Lie group as in Lemma 3.3, containing N' as a codimension 1 (normal) subgroup; then M/N^0 is isomorphic to $\mathbb{R}$ and M/N' is the circle group. Now for all i let $\omega_i = \alpha_i(x_i)\alpha_i^2(x_i)\cdots\alpha_i^n(x_i)$ and let θ_i be the unique n th root of ω_i in M. We note that $(\theta_iN^0)^n = \omega_iN^0 = (x_iN^0)^n = (xN^0)^n$ and since M/N^0 is isomorphic to $\mathbb{R}$ we have $\theta_i \in xN^0$, and in particular $\theta_i \in N' \subset N$ for all i.

For all i let $y_i = x_i\theta_i^{-1}$; by the above observation $y_i \in N^0$. Clearly, for all i we have $\alpha_i(\omega_i) = \alpha_i(x_i)^{-1}\omega_i\alpha_i(x_i)$, and by the uniqueness of n th roots in M it follows that $\alpha_i(\theta_i) = \alpha_i(x_i)^{-1}\theta_i\alpha_i(x_i)$. Therefore, for all i, $\alpha_i(y_i) = \alpha_i(x_i)\alpha_i(\theta_i)^{-1} = \alpha_i(x_i)(\alpha_i(x_i)^{-1}\theta_i^{-1}\alpha_i(x_i)) = \theta_i^{-1}\alpha_i(x_i)$. Furthermore, for all $j \geq 1$ we get, arguing by induction over j, that $\alpha_i^j(y_i) = \alpha_i^{j-1}(x_i)^{-1}\cdots\alpha_i(x_i)^{-1}\theta_i^{-1}\alpha_i(x_i)\cdots\alpha_i^j(x_i)$. Hence $\alpha_i(y_i)\alpha_i^2(y_i)\cdots\alpha_i^n(y_i) = \theta_i^{-n}\alpha_i(x_i)\alpha_i^2(x_i)\cdots\alpha_i^n(x_i) = e$ for all i. Also $(y_i^{-1}x_i)^n = \theta_i^n = \omega_i$, so by hypothesis we get that $\{(y_i^{-1}x_i)^n\}$ is relatively compact. By the root compactness of compactly generated nilpotent Lie groups it follows that $\{y_i^{-1}x_i\}$ is relatively compact. This proves the proposition in the case at hand.

Now suppose that N'/N^0 is finite. Then by Lemma 3.3 N' is a direct product of N^0 with a finite subgroup, say F. Being the unique finite normal subgroup of N', F is invariant under the action of $\mathrm{Aut}N'$, and considering the action modulo N^0 we see also that F is fixed pointwise by all α_i. Replacing the original x such that $x_iN^0 = xN^0$, within the N^0 coset, we may assume that $x \in F$. Let $\{z_i\}$ be the sequence in N^0 such that $x_i = xz_i$ for all i. Then $\alpha_i(x_i)\alpha_i^2(x_i)\cdots\alpha_i^n(x_i) = x^n\alpha_i(z_i)\alpha_i^2(z_i)\cdots\alpha_i^n(z_i)$ and hence $\{\alpha_i(z_i)\alpha_i^2(z_i)\cdots\alpha_i^n(z_i)\}$ is relatively compact. Since N^0 is simply connected, by the argument as above we deduce from this that there exists a sequence $\{y_i\}$ in N^0 such that $\{y_i^{-1}z_i\}$ is relatively compact, and $\alpha_i(y_i)\alpha_i^2(y_i)\cdots\alpha_i^n(y_i) = e$ for all i. As $y_i^{-1}x_i = y_i^{-1}z_ix$ we see that $\{y_i^{-1}x_i\}$ is relatively compact. This proves the proposition. □

Proposition 3.6. *Let $n \in \mathbb{N}$ and $\{\alpha_i\}$ be a relatively compact sequence in $\mathrm{Aut}'N$ such that $\alpha_i^n = I$ for all i. Let $\{x_i\}$ be a sequence in N such that $\{\alpha_i(x_i)x_i^{-1}\}$ is relatively compact. Then there exist sequences $\{c_i\}$ and $\{y_i\}$ in N such that $x_i = c_iy_i$ for all i, $\{c_i\}$ is relatively compact, and $\alpha_i(y_i) \in y_iT$ for all i.*

Proof : In proving the proposition without loss of generality we may assume that T is trivial or equivalently that N^0 is a simply connected nilpotent Lie group. For the case of (connected) simply connected N the proposition is same as Proposition 3.3 in [6]. In [6] the assertion is first proved for the case of vector groups, using which it is then deduced for all connected simply connected nilpotent groups. Here we shall assume the case of vector groups and deduce the general result as above. We proceed by induction on the dimension of N^0. If the dimension is 0 then the N is discrete and $\mathrm{Aut}'N$ consists only of the identity automorphism and the assertion follows trivially. Now suppose the assertion holds in all lower dimensional cases than the one under consideration. Let Z be the subgroup consisting of all elements of N^0 contained in the center of N. Since N is nilpotent Z is a vector subgroup of positive dimension. Using an obvious inductive hypothesis and considering the sequence of factors of α_i on N/Z we may assume that there exist sequences $\{g_i\}$ and $\{h_i\}$ in N such that $x_i = g_ih_i$ for all i, $\{g_i\}$ is relatively compact, and $\alpha_i(h_i) \in h_iZ$ for all i. Then $\alpha_i(h_i)h_i^{-1} = \alpha_i(g_i)^{-1}\alpha_i(x_i)x_i^{-1}g_i^{-1}$ for all i, and as $\{g_i\}$, $\{\alpha_i(x_i)x_i^{-1}\}$ and $\{\alpha_i\}$ are relatively compact this shows that $\{\alpha_i(h_i)h_i^{-1}\}$ is a relatively compact sequence in Z. Passing to subsequences we may assume that either $\{h_i\}$ is contained in Z, or consists entirely of elements outside Z. If $\{h_i\}$ is contained in Z then the assertion follows from the special case of the proposition for vector groups (which we recalled from [6] above). Now suppose that $\{h_i\}$ consists of elements outside Z. For each i let Z_i be the subgroup generated by Z and h_i; then, as $\alpha_i(h_i) \in h_iZ$, we see that Z_i is α_i-invariant. Furthermore, since Z is contained in the center of N, each Z_i is abelian and hence a direct product of the vector subgroup Z and the discrete subgroup generated by h_i. We shall deduce from this that for each i there exists a $z_i \in Z$ such that h_iz_i is fixed by α_i: If h_i is of finite order then clearly $\alpha_i(h_i) = h_i$. Suppose h_i is of infinite order. Then there exists a vector group V_i with Z_i as a subgroup of V_i of codimension 1, and α_i is the restriction of a linear transformation of V_i, of finite order. Since Z is

α_i-invariant, and of codimension 1, it follows that there exists a one-dimensional subspace complementing Z in V_i and fixed pointwise by α_i. This shows that there exists $z_i \in Z$ such that $\alpha_i(h_i z_i) = h_i z_i$ (in the multiplicative notation of the group). Then $\alpha_i(h_i)h_i^{-1} = \alpha_i(z_i^{-1})z_i$ for all i, and we get that $\{\alpha_i(z_i^{-1})z_i\}$ is relatively compact. Again from the special case of vector groups it follows that there exist sequences $\{a_i\}$ and $\{b_i\}$ in Z such that $z_i^{-1} = a_i b_i$ for all i, $\{a_i\}$ is relatively compact and $\alpha(b_i) = b_i$ for all i. Then $x_i = g_i h_i = g_i(h_i z_i)z_i^{-1} = g_i(h_i z_i)a_i b_i = (g_i a_i)(h_i z_i b_i)$, for all i, so the desired assertion holds with $c_i = g_i a_i$ and $y_i = h_i z_i b_i$ for all i. This completes the proof of the proposition. □

The following is an extension of Theorem 3.2 of [6].

Theorem 3.7. *Let G, μ and N be as in the statement of Theorem 2.2. Let $n \in \mathbb{N}$, $\rho \in P(G)$ be a nth root of μ compatible with N. Let $z \in N$ be such that $(\rho z)^n = c\mu$ for some $c \in C$. Then there exist $y \in N^0$, $h \in C$ such that $\rho z = y\rho y^{-1}h$.*

Proof : By Lemma 2.1, we have $(\rho z)^n = \alpha_\rho(z)\alpha_\rho^2(z)\cdots\alpha_\rho^n(z)\mu$. Since $(\rho z)^n = c\mu$ we get that $c^{-1}\alpha_\rho(z)\alpha_\rho^2(z)\cdots\alpha_\rho^n(z)\mu = \mu$. We recall that $\{g \in N \mid g\mu = \mu\}$ is a compact subgroup of N and hence it is contained in C. Hence we get that $\alpha_\rho(z)\alpha_\rho^2(z)\cdots\alpha_\rho^n(z) \in C$. Therefore by Proposition 3.4, there exist $y \in N^0$ and $h' \in C$ such that $\alpha_\rho(z) = y\alpha_\rho(y^{-1})h'$. Let $x \in N(\mu)$ be such that $\rho \in P(xG(\mu)) = P(G(\mu)x)$. Since ρ is compatible with N, x normalises N, and hence also C. Then we have $xzx^{-1} = yxy^{-1}x^{-1}h'$ and so $xz = yxy^{-1}h$, where $h = x^{-1}h'x \in C$. Therefore $\rho z = (\rho x^{-1})xz = (\rho x^{-1})(yxy^{-1})h = y(\rho x^{-1})xy^{-1}h = y\rho y^{-1}h$, which proves the theorem. □

4. Proofs of the Main Results

We now prove the theorems using the results proved in the preceding sections.

Proof of Theorem 2.2: Let $n \in \mathbb{N}$, $\{\nu_i\}$ and $\{x_i\}$ be as in the hypothesis. Since $\{\nu_i x_i\}$ is relatively compact, so is the sequence $\{(\nu_i x_i)^n\}$. For each i let α_i denote the restriction of the automorphism α_{ν_i} to N, in the notation as above. By Lemma 2.1 we have $(\nu_i x_i)^n = \alpha_i(x_i)\alpha_i^2(x_i)\cdots\alpha_i^n(x_i)\mu$, for all i. As $\{(\nu_i x_i)^n\}$ is relatively compact, this shows that $\{\alpha_i(x_i)\alpha_i^2(x_i)\cdots\alpha_i^n(x_i)\}$ is relatively compact. Hence by Proposition 3.5 there exists sequence $\{x_i'\}$ in N and $\{t_i\}$ in T such that $\alpha_i(x_i')\alpha_i^2(x_i')\cdots\alpha_i^n(x_i') = t_i$ for all i, and $\{x_i^{-1}x_i'\}$ is relatively compact. Then $(\nu_i x_i')^n = \alpha_i(x_i')\alpha_i^2(x_i')\cdots\alpha_i^n(x_i')\mu = t_i\mu$ for all i. Also, as $\nu_i x_i' = \nu_i x_i(x_i^{-1}x_i')$ for all i and $\{\nu_i x_i\}$ and $\{x_i^{-1}x_i'\}$ are relatively compact, we get that $\{\nu_i x_i'\}$ is relatively compact. As $(\nu_i x_i')^n = t_i\mu$, by Theorem 3.7 there exist sequences $\{z_i\}$ in N^0 and $\{c_i\}$ in C such that $\nu_i x_i' = z_i\nu_i z_i^{-1}c_i$, for all i. Since $\{\nu_i x_i'\}$ and $\{c_i\}$ are relatively compact, we also get that $\{z_i\nu_i z_i^{-1}\}$ is relatively compact. This proves the theorem. □

Proof of Theorem 2.3 : Let $r, n \in \mathbb{N}$ and $\{\lambda_i\}$ be a sequence in $R_{rn}(\mu)$, as in the hypothesis, such that $M(\lambda_i, N) = M(\lambda_i^r, N)$ for all i, and $\{y_i\}$ and $\{z_i\}$ be sequences in N^0 such that $\{y_i\lambda_i y_i^{-1}\}$ and $\{z_i\lambda_i^r z_i^{-1}\}$ are relatively compact. For all i let $\nu_i = \lambda^r$. Then $\{y_i\nu_i y_i^{-1}\}$ and $\{z_i\nu_i z_i^{-1}\}$ are relatively compact. Therefore there exists a compact subset K such that, for all i, $\nu_i(y_i^{-1}Ky_i) = y_i\nu_i y_i^{-1}(K) >$

$\frac{1}{2}$, and also $\nu_i(z_i^{-1}Kz_i) > \frac{1}{2}$. (see [9], Proposition 7.2.3). In particular, for all i the set $y_i^{-1}Ky_i \cap z_i^{-1}Kz_i \cap \operatorname{supp}\nu_i$ is nonempty; let x_i be an element from the subset. Then $x_i \in \operatorname{supp}\nu_i$, for all i, and $\{y_ix_iy_i^{-1}\}$ and $\{z_ix_iz_i^{-1}\}$ are relatively compact.

Now, for each i let $g_i = y_iz_i^{-1}$, and α_i be the automorphism of N^0 defined by $\alpha_i(\xi) = (z_ix_iz_i^{-1})^{-1}\xi(z_ix_iz_i^{-1})$, for all $\xi \in N^0$. As $\{z_ix_iz_i^{-1}\}$ is relatively compact, it follows that $\{\alpha_i\}$ is a relatively compact sequence of automorphisms of N. Furthermore, we see that $\alpha_i^n = I$, the identity automorphism, for every i. Also, for all i, we have $\alpha_i(g_i)g_i^{-1} = (z_ix_iz_i^{-1})^{-1}y_iz_i^{-1}(z_ix_iz_i^{-1})(y_iz_i^{-1})^{-1} = (z_ix_iz_i^{-1})^{-1}(y_ix_iy_i^{-1})$. As the latter form a relatively compact sequence, $\{\alpha_i(g_i)g_i^{-1}\}$ is relatively compact. Therefore by Proposition 3.6, applied to N^0 in place of N, each g_i can be written as a_ib_i, with $a_i, b_i \in N^0$, $\{a_i\}$ relatively compact, and $\alpha_i(b_i)b_i^{-1} \in T$ for all i. Thus $(z_ix_iz_i^{-1})^{-1}b_i(z_ix_iz_i^{-1})b_i^{-1} \in T$, and by conjugating by $x_iz_i^{-1}$ we get that $(z_i^{-1}b_iz_i)x_i(z_i^{-1}b_i^{-1}z_i)x_i^{-1} \in T$. Therefore $z_i^{-1}b_iz_i \in M(\nu_i, N)$, which by hypothesis coincides with $M(\lambda_i, N)$. Now, for any i, $y_i\lambda_iy_i^{-1} = g_iz_i\lambda_iz_i^{-1}g_i^{-1} = a_ib_iz_i\lambda_iz_i^{-1}b_i^{-1}a_i^{-1}$ and since $z_i^{-1}b_iz_i \in M(\lambda_i, N)$ there exists a sequence $\{t_i\}$ in T such that $z_i^{-1}b_iz_i\lambda_iz_i^{-1}b_i^{-1}z_i = \lambda_it_i$, and hence $a_ib_iz_i\lambda_iz_i^{-1}b_i^{-1}a_i^{-1} = a_iz_i\lambda_iz_i^{-1}a_i^{-1}t_i$. Since $\{y_i\lambda_iy_i^{-1}\}$, $\{a_i\}$ and $\{t_i\}$ are relatively compact, the preceding conclusion shows that $\{z_i\lambda_iz_i^{-1}\}$ is relatively compact; this proves the theorem. □

Proof of Corollary 2.4 : Let the notation be as in the hypothesis. For any $n \in \mathbb{N}$ and $\rho \in R_n(\mu)$ we denote by $d(\rho)$ the dimension of $M(\rho, N)$. For $k, n \in \mathbb{N}$, $k \geq n$ let $d(k,n) = d(\nu_k^{k!/n!})$. We define inductively a sequence $\{d_n\}$ in $\mathbb{N}$ and a decreasing sequence $\{E_n\}$ of subsets of $\mathbb{N}$ as follows: let d_1 be the dimension of N and $E_1 = \mathbb{N}$. Consider the set $\{d(k,2) \mid k \geq 2\}$. Since it involves only finitely many integers, we can choose d_2 and such that $d(k,2) = d_2$ for infinitely many k in E_1, and define $E_2 = \{k \in E_1 \mid k \geq 2, d(k,2) = d_2\}$. After defining $d_1, \dots, d_{n-1}$ and $E_1, \dots, E_{n-1}$ for some $n \in \mathbb{N}$ we consider $\{d(k,n) \mid k \geq n!, k \in E_{n-1}\}$ and choose d_n to be such that $\{k \in E_{n-1} \mid d(k,n) = d_n\}$ is infinite and define $E_n = \{k \in E_{n-1} \mid k \geq n!, d(k,n) = d_n\}$.

Since for every $k \geq n$, $\nu_k^{k!/n!}$ is a n th root of $\nu_k^{k!/(n-1)!}$ it follows that $\{d_n\}$ is a decreasing sequence in $\mathbb{N}$. Hence there exists $d, l \in \mathbb{N}$ such that $d_n = d$ for all $n \geq l$. Let Σ be the subset of $\mathbb{N}$ as in the hypothesis and let $\Sigma' = \{j \in \Sigma \mid j \geq l\}$.

For each $n \in \Sigma'$ consider the sequence of $n!$ th roots $\left\{\nu_k^{k!/n!}\right\}_{k\in E_n}$. By the condition in the hypothesis for each $n \in \Sigma'$ there exist $x_{k;n}$, $k \in E_n$, in N such that $\left\{\nu_k^{k!/n!}x_{k;n} \mid k \in E_n\right\}$ is relatively compact and in turn by Theorem 2.2 there exist $z_{k;n}$, $k \in E_n$, in N^0 such that $\left\{z_{k;n}\nu_k^{k!/n!}{z_{k;n}}^{-1}\right\}$ is relatively compact. For all $n \in \Sigma'$ we have $d(\nu_k^{k!/n!}) = d(\nu_k^{k!/l!})$ and hence it follows that $M(\nu_k^{k!/n!}, N) = M(\nu_k^{k!/l!}, N)$. Therefore by Theorem 2.3, with $z_k = z_{k;l}$ we get that $\left\{z_k\nu_k^{k!/n!}z_k^{-1} \mid k \in E_n\right\}$ is relatively compact for all $n \in \Sigma'$. For all $j \in \mathbb{N}$ let

$k_j \in E_j$ be such that $k_j \to \infty$ and $\zeta_j = z_{k_j}$. Then $\left\{\zeta_j \nu_{k_j}^{k_j!/n!} \zeta_j^{-1} \mid j \in \mathbb{N}, k_j \geq n!\right\}$ is relatively compact for all $n \in \Sigma'$. This proves the first assertion in the Corollary.

Passing to a subsequence, after a process of diagonalisation, we may now assume that for all $n \in \Sigma'$, $\left\{\zeta_j \nu_{k_j}^{k_j!/n!} \zeta_j^{-1}\right\}$ converges as $j \to \infty$, say $\zeta_j \nu_{k_j}^{k_j!/n!} \zeta_j^{-1} \to \rho_n$. Then for each $n \in \Sigma'$, ρ_n is a $n!$ th root of μ and for any $m, n \in \Sigma'$ with $m \geq n$ we have $\rho_m^{m!/n!} = \rho_n$.

We now define a map $\iota : \mathbb{Q}_+ \to P(G)$, where $\mathbb{Q}_+$ is the semigroup of positive rationals, as follows: for $\frac{p}{q} \in \mathbb{Q}_+$ let $n \in \Sigma'$ be such that $q \leq n!$, and define $\iota\left(\frac{p}{q}\right) = \rho_n^{n!p/q}$; we note that on account of the relation between the ρ_n's, $\iota\left(\frac{p}{q}\right)$ is well-defined by this, independent of the choice of the n in question. It is also straightforward to see that ι is a homomorphism of $\mathbb{Q}_+$ into $P(G)$. Thus it is a rational embedding of $\iota(1)$, which is μ. It is known (see [4]) that on a Lie group a probability measure admitting a rational embedding is embeddable in a continuous one-parameter convolution semigroup. This proves the Corollary. □

5. Epilogue

The results presented here suggest a strategy for attacking the embedding problem: Let G be a Lie group and $\mu \in P(G)$ be infinitely divisible. As seen in the proof of Corollary 2.4, by the results of [4], to prove embeddability of μ in a continuous one-parameter semigroup it suffices to show that it is rationally embeddable, namely that there exists a homomorphism $\iota : \mathbb{Q}_+ \to P(G)$ such that $\iota(1) = \mu$. By infinite divisibility for every k there exists a $k!$ th root ν_k, and if we can find a sequence $\{z_k\}$ in $Z(\mu)$ such that $z_k \nu_k^{k!/n!} z_k^{-1}$ converges, as k tends to infinity along a sequence of natural numbers, for infinitely many natural numbers n, then the argument as in the corollary shows that μ is rationally embeddable.

The above sketch suggests looking for such a sequence in a nilpotent Lie subgroup N of $Z(\mu)$, and for this, one would be called upon to ensure that $\left\{\nu_k^{k!/n!} \mid k \geq n!\right\}$ is relatively compact modulo N for infinitely many n. It may be mentioned that something similar is involved in the proofs of the embedding theorems in [6] and less directly in some earlier papers, though the present formulation is slightly new.

Now let G be an almost algebraic group and $\mu \in P(G)$. In this case it has been known [3] that $F(\mu)/Z(\mu)$ is compact. Additionally, it was shown in [5] (see also [6]) that if μ is infinitely divisible in G then there exists an almost algebraic subgroup H of G containing $\operatorname{supp}\mu$ such that μ is infinitely divisible on H and $Z(\mu) \cap H$, which is the analogue of $Z(\mu)$ within H, contains a simply connected nilpotent subgroup N such that $Z(\mu)/N$ is compact. In the light of these facts we get that $\left\{\nu_k^{k!/n!} \mid k \geq n!\right\}$, as above, is indeed relatively compact modulo N for all $n \in \mathbb{N}$ and hence the theorem follows in this case. (This argument is somewhat different from that in [6] where reduction to rational embeddability as above was not used, and the notion of strong root compactness is used instead, but it is similar in spirit; see [1] for a discussion on the alternative approach).

Following certain technical stratagems as in [6] one can also extend the above argument to all class $\mathcal{C}$ groups, namely connected Lie groups which admit a finite dimensional representation with discrete kernel. We will not go into the details of this here. The technique is also used in [2] for a class of groups which are not of class $\mathcal{C}$, together with other various other ideas, to conclude embeddability of infinitely divisible measures.

It would be of interest to prove analogues of Theorems 2.2 and 2.3 for other classes of locally compact groups in the place of compactly generated nilpotent Lie groups. This may provide a unified approach to proving embedding theorems in a wider class of locally compact groups.

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