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# A CLASS OF SATURATED ROW-COLUMN DESIGNS

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**Abstract** A new class of saturated, binary row-column designs is proposed. It is shown that these designs are treatment connected in the sense that these permit the estimability of all contrasts among treatment effects.

**Key words** Row-column designs, saturated, connectedness.

## 1. Introduction

Row-column designs are used in various fields of applications including agriculture and industry, for eliminating heterogeneity in two directions. Suppose it is desired to compare $v$ treatments using $n = rc$ experimental units, arranged in $r$ rows and $c$ columns. Any allocation of the $v$ treatments over the $n$ experimental units is called a row-column design.

A row-column design is said to be *connected* (or, treatment-connected) if it allows the estimability of all contrasts among treatment effects. The issue of connectedness of row-column designs has been examined by several authors; see e.g., Shah and Khatri [6], Raghavarao and Federer [5], Eccleston and Russell [2], Baksalary and Kala [1], Godolphin and Godolphin [3] and Qu and Ogunyemi [4]. Since a row-column design is not necessarily connected even if the component block designs, obtained by treating the rows as blocks and columns as blocks, are each individually connected, there is no simple way to check the connectedness of a row-column design. In this communication, we propose a class of row-column designs with $r = c$. These designs are binary, i.e., a treatment appears at most once in each row and each column and are saturated in the sense that no degrees of freedom are available for the estimation of the error variance. We show that these designs are connected. Some comments on the efficiency of the proposed designs are also made.

## 2. Preliminaries

Throughout, for a positive integer $s$, $\mathbf{1}_s$ and $I_s$ will denote respectively, an $s \times 1$ vector of all ones and an identity matrix of order $s$. Also, for positive integers $a, b$, $\mathbf{0}_{ab}$ will denote an $a \times b$ null matrix. The null column vector $\mathbf{0}_{a1}$ will be simply written as $\mathbf{0}_a$. A prime over a matrix or vector will denote its transpose. Finally, for a pair of matrices $E = (e_{ij})$ and $F$, the Kronecker (tensor) product of $E$ and $F$ will be denoted by $E \otimes F$, i.e., $E \otimes F = (e_{ij}F)$.

Consider a row-column design $d$ involving $v$ treatments arranged in $r$ rows and $c$ columns. Let $Y_{ijk}$ denote the observable random variable corresponding to the observation pertaining to the $k$th treatment in the $i$th row and $j$th column, $1 \le i \le r, 1 \le j \le c, 1 \le k \le v$, and $\boldsymbol{Y}$ be the $n \times 1$ vector of the quantities $\{Y_{ijk}\}$. We postulate the following model for the observations collected via $d$:

$$\begin{aligned} \mathbb{E}(\boldsymbol{Y}) &= \mu \mathbf{1}_n + D_{1d}\boldsymbol{\alpha} + D_{2d}\boldsymbol{\beta} + D_{3d}\boldsymbol{\gamma}, \\ \mathbb{D}(\boldsymbol{Y}) &= \sigma^2 I_n, \end{aligned} \tag{1}$$

where

$$\begin{aligned} \mu &= \text{a general mean}, \\ \boldsymbol{\alpha} &= \text{the } r \times 1 \text{ vector of row effects}, \\ \boldsymbol{\beta} &= \text{the } c \times 1 \text{ vector of column effects}, \\ \boldsymbol{\gamma} &= \text{the } v \times 1 \text{ vector of treatment effects}, \\ D_{1d} &= \text{the } n \times r \text{ observations versus rows incidence matrix}, \\ D_{2d} &= \text{the } n \times c \text{ observations versus columns incidence matrix}, \\ D_{3d} &= \text{the } n \times v \text{ observations versus treatments incidence matrix}, \\ n &= \text{the total number of observations in } d, \end{aligned}$$

and $\mathbb{E}(\cdot)$ and $\mathbb{D}(\cdot)$ are respectively, the expectation and dispersion operators.

The incidence matrices $D_{id},\ 1 \le i \le 3$, are defined in the usual manner. For instance, if $D_{1d} = (d_{ui}^{(1)})$, then

$$\begin{aligned} d_{ui}^{(1)} &= 1, \quad \text{if the } u\text{th observation corresponds to the } i\text{th row} \\ &= 0, \quad \text{otherwise.} \end{aligned}$$

The matrices $D_{2d}$ and $D_{3d}$ are defined similarly.

Under model (1), the information matrix for estimating linear functions of treatment effects under a row-column design $d$ is given by

$$C_d = R_d - c^{-1} N_{1d} N_{1d}' - r^{-1} N_{2d} N_{2d}' + (rc)^{-1} \boldsymbol{r}_d \boldsymbol{r}_d', \tag{2}$$

where

$$\begin{aligned} R_d &= \text{diag}(r_{d1}, \ldots, r_{dv}), \\ \boldsymbol{r}_d &= (r_{d1}, \ldots, r_{dv})', \\ N_{1d} &= \text{the } v \times r \text{ treatments versus rows incidence matrix}, \\ N_{2d} &= \text{the } v \times c \text{ treatments versus columns incidence matrix}, \end{aligned}$$

and for $1 \leq i \leq v$, $r_{di}$ denotes the number of times the $i$th treatment appears in $d$.

A row-column design is said to be *connected* (or, treatment-connected) if it allows the estimability of all contrasts among treatment effects. A necessary condition for a row-column design with $v$ treatments, $r$ rows and $c$ columns to be connected is that

$$rc \geq v + r + c - 2. \tag{3}$$

It is well-known that a row-column design $d$ is connected if and only if

$$\mathrm{Rank}(C_d) = v - 1. \tag{4}$$

Equivalently, a row-column design $d$ is connected if and only if

$$\mathrm{Rank}(X_d) = v + r + c - 2, \tag{5}$$

where

$$X_d = [\mathbf{1}_n \; D_{1d} \; D_{2d} \; D_{3d}]. \tag{6}$$

One can write (2) as

$$C_d = C_{1d} + C_{2d} - C_{0d}, \tag{7}$$

where

$$\begin{aligned} C_{1d} &= R_d - c^{-1} N_{1d} N_{1d}', \\ C_{2d} &= R_d - r^{-1} N_{2d} N_{2d}', , \\ C_{0d} &= R_d - (rc)^{-1} \boldsymbol{r}_d \boldsymbol{r}_d'. \end{aligned} \tag{8}$$

Note that $C_{1d}$ (respectively, $C_{2d}$) is the information matrix under a sub-model of (1) obtained by ignoring the column (respectively, row) effects and $C_{0d}$ is the information matrix under a model with both row and column effects ignored. A row-column design $d$ is row-connected (respectively, column connected) if and only if $\mathrm{Rank}(C_{1d}) = v - 1$ (respectively, $\mathrm{Rank}(C_{2d}) = v - 1$). This means that a row-column design is row-connected (respectively, column-connected) if and only if the block design obtained by treating the rows (respectively, columns) is connected when viewed as a block design. It is known (see e.g., Raghavarao and Federer [5]) that a connected row-column design is both row- and column-connected. However, as noted by Shah and Khatri [6], the converse is not true, i.e., a row-column design is not necessarily connected even if it is both row- and column-connected.

A row-column design is said to be *saturated* if equality holds in (3). In this communication, we propose a new class of saturated row-column designs involving $s$ rows and $s$ columns, where $s \geq 3$ is an integer. Clearly, then from (3), for a saturated row-column design, we have $v = s^2 - 2s + 2$. We show that each member of this class of designs is connected.

## 3. A Class of Saturated Row-Column Designs

We propose a new class of saturated row-column designs, say $d_0$, involving $v = s^2 - 2s + 2$ treatments, $s$ rows and $s$ columns, where $s \geq 3$ is an integer. The design $d_0$ with treatments labelled as $1, 2, \ldots, s^2 - 2s + 2$, is given by

$$d_0 = \begin{matrix} 1 & 2 & \cdots & s-1 & s \\ s & s+1 & \cdots & 2s-2 & 2s-1 \\ 2s-1 & 2s & \cdots & 3s-3 & 3s-2 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ s^2-2s+2 & 1 & \cdots & s-2 & s-1 \end{matrix}.$$

For the design $d_0$, it is easy to see that

$$D_{1d_0} = \begin{pmatrix} \mathbf{1}_s & \mathbf{0}_s & \cdots & \mathbf{0}_s \\ \mathbf{0}_s & \mathbf{1}_s & \cdots & \mathbf{0}_s \\ \vdots & \vdots & \cdots & \vdots \\ \mathbf{0}_s & \mathbf{0}_s & \cdots & \mathbf{1}_s \end{pmatrix} = I_s \otimes \mathbf{1}_s \tag{9}$$

$$D_{2d_0} = \begin{pmatrix} I_s \\ I_s \\ \vdots \\ I_s \end{pmatrix} = \mathbf{1}_s \otimes I_s. \tag{10}$$

The matrix $D_{3d_0}$ can be written as

$$D_{3d_0} = [\boldsymbol{d}_1\ \boldsymbol{d}_2\ \ldots \boldsymbol{d}_v], \tag{11}$$

where, as before, $v = s^2 - 2s + 2$ and for $u = 1, 2, \ldots, s, 2s-1, 3s-2, \ldots, s^2 - 2s + 2$, $\boldsymbol{d}_u$ has exactly two unities, the remaining entries being zero and the remaining columns in $D_{3d_0}$ have exactly one unity, other entries in these columns being zero. By (5), $d_0$ is connected if and only if

$$\text{Rank}(X) = \text{Rank}[\mathbf{1}_{s^2}\ D_{1d_0}\ D_{2d_0}\ D_{3d_0}] = s^2.$$

Through a sequence of elementary column transformations, one can see that

$$\text{Rank}(X) = 1 + \text{Rank}(Y), \tag{12}$$

where $Y = [E_1\ E_2\ E_3]$ is a square matrix of order $(s^2 - 1)$ with

$$E_1 = \begin{bmatrix} \mathbf{0}_{s-1,s-1} \\ I_{s-1} \otimes \mathbf{1}_s \end{bmatrix}, \qquad E_2 = \begin{bmatrix} I_{s-1} \\ \mathbf{0}'_{s-1} \\ I_{s-1} \\ \mathbf{0}'_{s-1} \\ \vdots \\ \mathbf{0}'_{s-1} \\ I_{s-1} \end{bmatrix}, \tag{13}$$

and

$$E_3 = [\boldsymbol{e}_1\ \boldsymbol{e}_2\ \ldots \boldsymbol{e}_{v-1}],$$

where for $u = 1, 2, \ldots, s-1, 2s-2, 3s-3, \ldots, s^2-2s+1$, $\boldsymbol{e}_u$ has exactly two unities and the remaining columns have exactly one unity, the other entries in these columns being zero. Since $Y$ is a square matrix of order $s^2-1$, in order to show that the design is connected, it suffices to show that $Y$ is nonsingular. Since all the columns except the columns with indices $1, 2, \ldots, s-1, 2s-2, 3s-3, \ldots, s^2-2s+1$ have a single nonzero entry, namely unity, the determinant of $Y$ can be obtained by expanding along these elements to obtain $\det(Y) = \pm\det(Z)$, where for a square matrix, $\det(\cdot)$ denotes its determinant and $Z$ is a square matrix of order $(4s-5)$, given by

$$Z = [Z_1'\ Z_2'\ Z_3']'.$$

The matrices $Z_i,\ 1 \le i \le 3$, are given by

$$Z_1 = \begin{bmatrix} \mathbf{0}_{s-1}' & \mathbf{1}_2' & \mathbf{0}_2' & \cdots & \mathbf{0}_2' & \mathbf{0}_s' \\ \mathbf{0}_{s-1}' & \mathbf{0}_2' & \mathbf{1}_2' & \cdots & \mathbf{0}_2' & \mathbf{0}_s' \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0}_{s-1}' & \mathbf{0}_2' & \mathbf{0}_2' & \cdots & \mathbf{1}_2' & \mathbf{0}_s' \\ \mathbf{0}_{s-1}' & \mathbf{0}_2' & \mathbf{0}_2' & \cdots & \mathbf{0}_2' & \mathbf{1}_s' \end{bmatrix}, \tag{14}$$

$$Z_2 = [I_{s-1}\ \mathbf{0}_{s-1}\ \boldsymbol{g}\ \ldots\ \boldsymbol{g}\ I_{s-1}], \tag{15}$$

$$\boldsymbol{g} = \begin{pmatrix} \mathbf{0}_{s-2} & \mathbf{0}_{s-2} \\ 1 & 0 \end{pmatrix}, \tag{16}$$

and

$$Z_3 = \begin{pmatrix} I_{s-2} & \mathbf{0}_{s-2,2} & \mathbf{0}_{s-2,2} & \cdots & \mathbf{0}_{s-2,2} & \mathbf{0}_{s-2} & I_{s-2} \\ \mathbf{0}_{s-1,s-2} & G_{(s-1)1} & G_{(s-1)2} & \cdots & G_{(s-1)(s-1)} & \mathbf{0}_{s-1} & \mathbf{0}_{s-1,s-2} \end{pmatrix}. \tag{17}$$

The matrix $Z_1$ given by (14) is of order $(s-1) \times (4s-5)$. In the matrix $Z_2$ given by (15), the matrix $\boldsymbol{g}$ appears $(s-2)$ times and the matrix $G_{mu},\ 1 \le u \le m \le s-1$ appearing in (17) is an $m \times 2$ matrix whose $u$th row has all elements equal to 1 and the remaining elements are all zero.

Again through elementary row and column transformations, it can be seen that $\det(Z) = \pm\det(W)$, where $W = [W_1', W_2', W_3']'$ is a square matrix of order $2s-2$ and the matrices $W_i,\ 1 \le i \le 3$, are given by

$$\begin{aligned} W_1 &= [W_{11}, G_{(s-1)2}, G_{(s-1)3}, \ldots, G_{(s-1)(s-2)}, W_{12}], \\ W_2 &= [0, 1, 0, 1, \ldots, 0, 1], \\ W_3 &= [\boldsymbol{h}, G_{(s-2)1}, G_{(s-2)2}, \ldots, G_{(s-2)(s-2)}, \mathbf{0}_{s-2}], \end{aligned}$$

where $W_{11}$ is a matrix of order $(s-1) \times 2$ with its first row equal to $(2, 1)$ and the remaining elements zero, $W_{12}$ is an $(s-1) \times 2$ matrix with its last row as $(1, s-1)$ and rest of the elements zero and $\boldsymbol{h} = (1, 0, 0, \ldots, 0)'$ is a $(s-2) \times 1$ vector.

Finally, for integers $k \geq 2$ and $t \geq 1$, let $A_{kt}$ be a square matrix of order $2k$ given by $A_{kt} = [B_1', B_2', B_3']'$ where

$$\begin{aligned} B_1 &= [U_{11}, G_{k2}, G_{k3}, \ldots, G_{k(k-1)}, U_{12}], \\ B_2 &= [0, 1, 0, 1, \ldots, 0, 1], \\ B_3 &= [\boldsymbol{f}, G_{(k-1)1}, G_{(k-1)2}, \ldots, G_{(k-1)(k-1)}, \mathbf{0}_{k-1}], \end{aligned}$$

$U_{11}$ is a $k \times 2$ matrix with its first row equal to $(2, 1)$ and all other elements equal to zero, $U_{12}$ is a $k \times 2$ matrix with its last row equal to $(1,\ t)$, rest of the elements being equal to zero and $\boldsymbol{f} = (1, 0, 0, \ldots, 0)'$ is a $(k-1) \times 1$ vector. We then have the following result.

**Lemma 1.** *With $A_{kt}$ as defined above,*

$$\begin{aligned} \det(A_{kt}) &= (-1)^k 2\det(A_{(k-1)t}) + \det(A_{(k-2)t}),\ \text{if } k > 2, \\ &= 2\det(A_{1t}) - 1,\ \text{if } k = 2, \end{aligned}$$

*where* $A_{1t} = \begin{pmatrix} 0 & 1 & t \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.

*Proof* : First, let $k > 3$. Then, from the definition of $A_{kt}$, it is observed that only the first two elements of the first row of $A_{kt}$ are non-zero, these being equal to 2 and 1, respectively. Expanding $\det(A_{kt})$ along the elements of the first row gives

$$\det(A_{kt}) = 2\det(H) - \det(V),\ \text{say}$$

where $H = [H_1', H_2', H_3']'$ with

$$\begin{aligned} H_1 &= [\mathbf{0}_{k-1}, G_{(k-1)1}, G_{(k-1)2}, \ldots, G_{(k-1)(k-2)}, T_{12}], \\ H_2 &= [1, 0, 1, 0, 1, \ldots, 0, 1],\ \text{and} \\ H_3 &= [G_{(k-1)1}, G_{(k-1)2}, \ldots, G_{(k-1)(k-1)}, \mathbf{0}_{k-1}], \end{aligned}$$

$T_{12}$ is a $(k-1) \times 2$ matrix whose last row is $(1,\ t)$ and rest of the elements are zero and $V = [V_1', V_2', V_3']'$ with

$$\begin{aligned} V_1 &= H_1, \\ V_2 &= [0, 0, 1, 0, 1, \ldots, 0, 1],\ \text{and} \\ V_3 &= H_3. \end{aligned}$$

Adding the third column of $H$ to its second column and then subtracting the first column of $H$ from the second column and finally rearranging the rows, one gets

$$\det(H) = (-1)^k \det \begin{pmatrix} 1 & \mathbf{0}' \\ * & A_{(k-1)t} \end{pmatrix} = (-1)^k \det(A_{(k-1)t}),$$

where the elements in the column denoted by $*$ are not required in the evaluation of the determinant. Again, it can be shown that

$$\det(V) = (-1)^k \det \begin{pmatrix} V_1^* \\ V_2^* \\ V_3^* \end{pmatrix} = \det(V^*), \text{ say,}$$

where

$$\begin{aligned} V_1^* &= [G_{(k-1)1}, G_{(k-1)2}, \ldots, G_{(k-1)(k-2)}, T_{12}], \\ V_2^* &= [0, 1, 0, 1, \ldots, 0, 1], \text{ and} \\ V_3^* &= [\mathbf{0}_{k-2}, G_{(k-2)1}, \ldots, G_{(k-2)(k-2)}, \mathbf{0}_{k-2}], \end{aligned}$$

and the string $(0, 1)$ appears $(k-1)$ times in $V_2^*$. One can now show that

$$\det(V^*) = (-1)^k \det \begin{pmatrix} V_1^{**} \\ V_2^{**} \\ V_3^{**} \end{pmatrix} = \det(V^{**}), \text{ say,}$$

where

$$\begin{aligned} V_1^{**} &= [\mathbf{0}_{k-2}, G_{(k-2)1}, G_{(k-2)2}, \ldots, G_{(k-2)(k-3)}, T_{12}^*], \\ V_2^{**} &= [1, 0, 1, 0, 1, \ldots, 0, 1], \text{ and} \\ V_3^{**} &= [G_{(k-2)1}, \ldots, G_{(k-2)(k-2)}, \mathbf{0}_{k-2}], \end{aligned}$$

and $T_{12}^*$ is a $(k-2) \times 2$ matrix whose last row is $(1, t)$ and all other elements are equal to zero. Using the operations used for evaluating the determinant of $H$ above, one gets

$$\det(V^{**}) = (-1)^{k-1} \det(A_{(k-2)t}).$$

Therefore,

$$\begin{aligned} \det(A_{kt}) &= (-1)^k 2\det(A_{(k-1)t}) + (-1)^{2k} \det(A_{(k-2)t}) \\ &= (-1)^k 2\det(A_{(k-1)t}) + \det(A_{(k-2)t}). \end{aligned}$$

For $k = 3$, one can easily see that

$$\begin{aligned} \det(A_{3t}) &= \det \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & t \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix} \\ &= -2\det(A_{2t}) + \det(A_{1t}) \\ &= (-1)^3 2\det(A_{2t}) + \det(A_{1t}), \end{aligned}$$

where

$$A_{2t} = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & t \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

and $A_{1t}$ is as defined earlier. Finally, for $k = 2$,

$$\begin{aligned} \det(A_{2t}) &= \det \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & t \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \\ &= 2\det \begin{pmatrix} 0 & 1 & t \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} - \det \begin{pmatrix} 0 & 1 & t \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \\ &= 2\det(A_{1t}) - 1. \end{aligned}$$

□

**Lemma 2.** *For integers $k \geq 1,\ t \geq 1$,*

$$\det(A_{kt}) = \begin{cases} kt + 1, & \text{if } k = 2u - 1 \text{ or } 2u, \text{ for any odd integer } u \geq 1, \\ -kt - 1, & \text{otherwise}. \end{cases}$$

*Proof* : For simplicity in notation, we denote $\det(A_{kt})$ by $a_{kt}$. Then, the result of Lemma 1 can be written as

$$a_{kt} = (-1)^k 2a_{(k-1)t} + a_{(k-2)t},\ k \geq 2, \tag{18}$$

where we define $a_{0t} \equiv -1$. We need to show that for an integer $m \geq 0$,

$$\begin{aligned} a_{4mt} &= -4mt - 1, \\ a_{(4m+1)t} &= (4m + 1)t + 1, \\ a_{(4m+2)t} &= (4m + 2)t + 1 \text{ and} \\ a_{(4m+3)t} &= -(4m + 3)t - 1. \end{aligned} \tag{19}$$

For $m = 0$, using the matrices $A_{it},\ 1 \leq i \leq 3$, it can be readily seen that

$$\begin{aligned} a_{0t} &= -1, \\ a_{1t} &= t + 1, \\ a_{2t} &= 2t + 1 \text{ and} \\ a_{3t} &= -3t - 1. \end{aligned}$$

We now apply induction to prove (19). Suppose (19) holds for all integers $m = 0, 1, \ldots, l$. We then have by (18),

$$\begin{aligned} a_{4(l+1)t} &= (-1)^{4(l+1)} 2a_{(4l+3)t} + a_{(4l+2)t} \\ &= -2\{(4l + 3)t + 1\} + (4l + 2)t + 1,\ \text{by hypothesis} \\ &= -4(l + 1)t - 1. \end{aligned}$$

On similar lines, one can show that

$$\begin{aligned} a_{(4(l+1)+1)t} &= \{4(l+1)+1\}t+1, \\ a_{(4(l+1)+2)t} &= \{4(l+1)+2\}t+1 \text{ and} \\ a_{(4(l+1)+3)t} &= -\{4(l+1)+3\}t-1. \end{aligned}$$

This completes the proof. □

We can now state the main result of this paper.

**Theorem.** *The row-column design $d_0$ is connected for each integer $s > 2$.*

*Proof* : Observe that the matrix $W$ defined earlier in this section is the same as $A_{(s-1)(s-1)}$ and by Lemma 2, is nonsingular for each $s > 2$. Hence, by the arguments in the proof of Lemma 1, it is clear that $\mathrm{Rank}(X) = s^2$, which in turn implies that $d_0$ is connected for all $s > 2$. □

## 4. Concluding Remarks

In this paper, a class of saturated row-column designs with number of rows equal to the number of columns has been proposed and it has been established that the proposed designs are connected. Row-column designs with the number of rows equal to the number of columns are some times called square designs. Many of the existing square designs, including Latin square designs, accommodate fewer treatments than the number of treatments in the designs proposed here. These designs are thus likely to be useful in situations where a large number of treatments are to be tested and the number of rows/columns is smaller than that of treatments.

While it has not yet been possible to obtain general results regarding the efficiency of the proposed designs, for $s = 4$ our design has an edge over some of the existing designs on the basis of the average variance of the best linear unbiased estimators of all elementary treatment contrasts.

The average variance of the best linear unbiased estimators of all elementary treatment contrasts under the design with $v = 10, r = 4 = c$, i.e., $s = 4$, proposed here is $3.0001\sigma^2$, while the same average variance for the design proposed recently by Qu and Ogunyemi [4] is $3.5810\sigma^2$, where $\sigma^2$ is the variance of an observation. A design with $v = 8$ treatments in 4 rows and 4 columns was given by Eccleston and Russell [2]. This design can be converted to a connected row-column design with $v = 10, r = 4 = c$ (see Qu and Ogunyemi [4]). The average variance of the best linear unbiased estimators of all elementary treatment contrasts for this design is $3.8205\sigma^2$, which is again appreciably higher than that obtained under the design proposed in this paper. Since minimizing the average variance of the best linear unbiased estimators of all elementary treatment contrasts is equivalent to the well-known and commonly used $A$-optimality criterion, the proposed design for $s = 4$ is better than some of the existing designs according to the $A$-criterion. Further investigations are needed to evaluate whether or not the proposed designs for other values of $s$ are better than the existing comparable designs.

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