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NOTES ON EULER'S WORK ON DIVERGENT FACTORIAL SERIES AND THEIR ASSOCIATED CONTINUED FRACTIONS

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Abstract Factorial series which diverge everywhere were first considered by Euler from the point of view of summing divergent series. He discovered a way to sum such series and was led to certain integrals and continued fractions. His method of summation was essentialy what we call Borel summation now. In this paper, we discuss these aspects of Euler's work from the modern perspective.

Key words Divergent series, factorial series, continued fractions, hypergeometric continued fractions, Sturmian sequences.

1. Introductory Remarks

Euler was the first mathematician to develop a systematic theory of divergent series. In his great 1760 paper *De seriebus divergentibus* [1, 2] and in his letters to Bernoulli he championed the view, which was truly revolutionary for his epoch, that one should be able to assign a numerical value to any divergent series, thus allowing the possibility of working systematically with them (see [3]). He anticipated by over a century the methods of summation of divergent series which are known today as the summation methods of Cesaro, Hölder, Abel, Euler, Borel, and so on. Eventually his views would find their proper place in the modern theory of divergent series [4].

But from the beginning Euler realized that almost none of his methods could be applied to the series

$$1 - 1!x + 2!x^2 - 3!x^3 + \ldots = \sum_{n=0}^{\infty} (-1)^n n! x^n \qquad (1)$$

which he called the *divergent series par excellence*. In his paper quoted above he developed a remarkable method of summing this series. His method led him to a class of continued fractions of the form

$$[1, u_1x, u_2x, u_3x, \ldots, u_nx, \ldots] := \frac{1}{1+}\frac{u_1x}{1+}\frac{u_2x}{1+}\ldots\frac{u_nx}{1+}\ldots \tag{2}$$

where u_n is a sequence of positive numbers and x is a real variable, generally positive.

Let us set, with Euler,

$$f = 1 - 1!x + 2!x^2 - 3!x^3 + \ldots, \qquad g = xf. \tag{2a}$$

One views f and g as elements of the formal power series ring $\mathbf{C}[[x]]$ which is a differential ring in the sense that the derivation d/dx acts formally on it. Then g satisfies the differential equation

$$x^2\frac{dg}{dx} + g = x, \qquad g \equiv 0 \pmod x$$

formally. This differential equation makes analytic sense also. If we replace the condition $g \equiv 0 \pmod x$ by its analytic version $g(x) \to 0$ as $x \to 0+$, there is a unique *analytic solution*

$$g(x) = e^{1/x}\int_0^x \frac{e^{-1/t}}{t}dt,$$

so that we can write

$$f \sim \frac{e^{1/x}}{x}\int_0^x \frac{e^{-1/t}}{t}dt.$$

The notation $\sim$ means that the function on the right is *asymptotic* to the series on the left in the classical sense of Poincaré:

$$\frac{e^{1/x}}{x}\int_0^x \frac{e^{-1/t}}{t}dt = 1 - 1!x + 2!x^2 - 3!x^3 + \ldots + (-1)^n n!x^n + O(x^{n+1}),$$

as $x \to 0+$ for every $n = 0, 1, 2, \ldots$. It is quite straightforward to prove this by repeated integration by parts. Euler evaluates the integral for $x = 1$ numerically and gives the resulting value as the “sum”of the divergent factorial series (1). Now, proceeding formally,

$$\begin{aligned}\sum_{n=0}^{\infty}(-1)^n x^n n! &= \sum_{n=0}^{\infty}(-1)^n x^n \int_0^{\infty} e^{-u}u^n du\\ &= \int_0^{\infty}\frac{e^{-u}}{1+xu}du = \frac{e^{1/x}}{x}\int_0^x \frac{e^{-1/t}}{t}dt,\end{aligned}$$

where we use the substitution

$$\frac{1}{t} - \frac{1}{x} = u,$$

in the last step. Taking $x = 1$ we thus obtain formally

$$1 - 1! + 2! - 3! + \ldots = \int_0^\infty \frac{e^{-u}}{1+u} du = \int_0^1 \frac{e^{1-1/t}}{t} dt.$$

We recognize in this method the germ of the idea of what we now know as *Borel summation* [5]. With Borel (see [5], p. 55) we observe that

$$\begin{aligned} 1 - \int_0^\infty \frac{e^{-u}}{1+u} du &= \int_0^\infty e^{-u} du - \int_0^\infty \frac{e^{-u}}{1+u} du \\ &= \int_0^\infty \frac{ue^{-u}}{1+u} du = 0.4036526, \end{aligned}$$

so that for the Euler series we get the value

$$0.5963474\ldots$$

It is interesting to note that the series is mentioned in Ramanujan's first letter to Hardy with the value

$$0.596\ldots$$

(see [6, 7, 8]). One can therefore say with considerable justification that Euler, and later, Ramanujan, had both discovered Borel summation. The success of the summation methods of Borel, Heine, and their modern successors in diverse areas such as quantum field theory and dynamical systems makes it clear that a deeper look at Euler's work may still yield a rich harvest of new ideas and results.

However it appears that Euler was not satisfied with the numerical evaluation of the integral, presumably because one cannot give a bound for the error. Desirous of a more accurate evaluation with error bounds he first obtained a remarkable continued fraction for the formal series f (see (2a)):

$$f = [1, x, x, 2x, 2x, 3x, 3x, \ldots, nx, nx, \ldots]. \tag{3}$$

He does not give a proof of this in his paper, contenting himself to the computation of a huge number of initial terms. Now, the continued fraction given above makes sense and is convergent when x is treated as a real variable > 0; moreover its successive convergents over and underestimate the true value, and so it can be numerically evaluated with error bounds. Euler does the numerical evaluation of the continued fraction. He uses a remarkable method of numerical computation and obtains for $x = 1$ the value accurate to 8 decimal places (at least), and gives this as the value of the factorial series (1). Implicit in his discussion are the convergence of the continued fraction, the oscillation of its successive convergents above and below the actual value, and the identity of the continued fraction (3) with the analytic function found earlier as a solution of the differential equation, namely

$$[1, x, x, 2x, 2x, \ldots] = \frac{e^{1/x}}{x} \int_0^x \frac{e^{-1/t}}{t} dt \quad (x > 0). \tag{4}$$

Numerical work with MAPLE suggests that (4) is true for all $x > 0$.

In [1] Euler mentions that a whole class of divergent series can be summed and numerically evaluated by this method, namely, the series

$$g_{pq} = 1 - px^q + p(p+q)x^{2q} - p(p+q)(p+2q)x^{3q} + \dots$$

with integers $p, q > 0$ for which the analytic approximation is

$$g_{pq} \sim \frac{e^{1/qx^q}}{x^p} \int_0^x e^{-1/qt^q} t^{p-q-1} dt \tag{5}$$

and the associated continued fraction

$$g_{pq} = [1, px^q, qx^q, (p+q)x^q, 2qx^q, (p+2q)x^q, 3qx^q, \dots]. \tag{6}$$

Clearly one should have a corresponding identity

$$[1, px^q, qx^q, (p+q)x^q, 2qx^q, \dots] = \frac{e^{1/qx^q}}{x^p} \int_0^x e^{-1/qt^q} t^{p-q-1} dt. \tag{7}$$

Euler does not give in [1] a proof of any of the relations (3) through (7).

In these notes which are mostly expository, we shall give a proof of (3) through (7) and establish even more general identities. Actually the proof of (3) at the level of formal series is already non trivial and may be found in Perron's classic treatise [9] while the analytical formulae can be resurrected from the treatment in [10].

Beyond the analytical aspects discussed above lies the numerical evaluation of the continued fractions. Euler uses a remarkable method of approximation which yields him a value accurate to several decimal places. It is not quite clear still to us why he is so successful.

The method given in [9] is to exhibit, following Gauss, the ratio of two hypergeometric series as a continued fraction at the level of formal power series. The continued fractions are a consequence of the contiguity relations satisfied by the hypergeometric series. After a change of variable $z \to -cz$ one finds that the hypergeometric series tends, coefficientwise, to the Euler divergent series, when $b = 0$ and $c \to \infty$ (here a, b, c are the usual parameters of the hypergeometric series). The idea behind our proofs is to follow this method for the hypergeometric *functions* defined by their Eulerian integral representations. The limiting process is the same but is a little more delicate because of the conditional convergence of the continued fractions. This will give (4) and (7). But if we keep $b > 0$ we get, by our procedure, additional identities evaluating continued fractions as ratios of exponential integrals (see Theorem at the end of §5).

There are many formulae in the classical literature for continued fractions

$$[1, u_1, u_2, \dots] := \frac{1}{1+} \frac{u_1}{1+} \frac{u_2}{1+} \dots$$

We shall see later that this is convergent if the $u_n > 0$ and $\sum_n u_n^{-1}$ is divergent. However the classical formulae involve cases in which the $\sum u_n^{-1} < \infty$, and it is not clear from the context if the value attributed is the limit of the even or odd convergents. For instance

$$\int_0^\infty \frac{\cosh at}{\cosh t} e^{-t} dt = [1, (1-a^2), 2^2, (2^2-a^2), \ldots],$$

due to Rogers, and

$$4\int_0^\infty \frac{xe^{-x\sqrt{5}}}{\cosh x} dx = [1, 1^2, 1^2, 2^2, 2^2, \ldots],$$

found in Ramanujan's first letter to Hardy.

For the classical theory of continued fractions [9] and [10] are very good sources where one can find references to the literature of the 18^{th}, 19^{th}, and 20^{th} centuries. Representing numbers by continued fractions is of course quite well known, but representing functions of a real or complex variable by continued fractions containing the variable appears to have begun with the great work of Stieltjes [11]. For continued fractions associated to the hypergeometric functions see the Ramanujan Notebooks [12] and the survey article by [13]. One of us (V) would like to acknowledge with great pleasure and gratitude discussions with professor John Coates of Cambridge, U.K., about the classical literature on continued fractions. He would also like to express his profound gratitude to professor Pierre Deligne, Institute for Advanced Study, Princeton, N.J. for the many letters that he wrote to him sharing his ideas on these continued fractions and Euler's work in general. Finally we like to thank one of the referees for a number of comments that led to improvements in the exposition.

2. Continued fractions as orbits of the group of fractional linear transformations

Let K be any field. Then we have a natural action of $\mathrm{GL}(2, K)$ on the projective line $P^1(K)$. In the usual coordinates that identify $P^1(K)$ with $\overline{K} := K \cup \{\infty\}$, the action is via fractional transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : t \longmapsto \frac{at+b}{ct+d} \qquad (t \in \overline{K}).$$

For any $a \in K$ let

$$F(a) = \begin{pmatrix} 0 & a \\ 1 & 1 \end{pmatrix}, \qquad F(a) : t \longmapsto \frac{a}{1+t} \qquad (t \in \overline{K}).$$

If

$$a_1, a_2, \ldots, a_n, \ldots$$

is a sequence of elements of K, we write

$$H_n = F(1)F(a_1)F(a_2)\ldots F(a_n).$$

If $t \in \overline{K}$ we define the finite continued fractions by

$$\frac{1}{1+}\frac{a_1}{1+}\frac{a_2}{1+}\cdots\frac{a_n}{1+t} = H_n[t].$$

In particular,

$$[1, a_1, a_2, \ldots, a_n] := \frac{1}{1+}\frac{a_1}{1+}\frac{a_2}{1+}\cdots\frac{a_{n-1}}{1+}\frac{a_n}{1} = H_n[0].$$

If K is a topological field it makes sense to ask if the sequence $H_n[0]$ converges as $n \to \infty$. If this happens we say that the infinite continued fraction

$$[1, a_1, a_2, \ldots] := \frac{1}{1+}\frac{a_1}{1+}\frac{a_2}{1+}\cdots$$

is convergent, and has the value

$$[1, a_1.a_2, \ldots] = \lim_{n\to\infty}[1, a_1, a_2, \ldots a_n].$$

This is the classical definition; but we may also ask what happens to orbits of other elements besides 0, namely, the behavior of $H_n[t]$ as $n \to \infty$ for $t \in \overline{K}$. We are primarily interested in two cases, when K is the field of formal Laurent series in an indeterminate x with coefficients in a field k, and when $K = \mathbf{R}$ or $\mathbf{C}$.

It would be more natural to work with

$$[a_1, a_2, \ldots] := 1 + \frac{a_1}{1+}\frac{a_2}{1+}\cdots ,$$

which is what is done in [9]. We prefer the slightly unorthodox $[1, a_1.a_2, \ldots]$ which begins with an inversion simply because many of the formulae that we want to discuss begin with the inversion.

Formal Laurent Series. Here $K = k((x)) := k[[x]][x^{-1}]$ where k is a field of characteristic 0 and x is an indeterminate. $k[[x]]$ has the adic topology in which a basis for the topology at 0 is the family of ideals $(x^n k[[x]])$; a sequence f_n converges to f if and only if for any integer $m \geq 1$, there is an integer $N_m \geq 1$, such that $f_n \equiv f \pmod{x^{m+1}}$ for all $n \geq N_m$. In particular, a series $\sum_{n\geq 0} f_n$ converges if and only if $f_n \to 0$.

We assume that $a_n = u_n x$ where $u_n \in k$. The matrices H_n satisfy the recursion formulae

$$H_n = H_{n-1}\begin{pmatrix} 0 & u_n x \\ 1 & 1 \end{pmatrix}, \qquad H_0 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

If we write

$$H_n = \begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix}, \qquad H_n[0] = \frac{B_n}{D_n} = \frac{A_{n+1}}{C_{n+1}}$$

then we have the recursion formulae

$$A_n = B_{n-1}, \qquad B_n = B_{n-1} + A_{n-1} u_n x,$$

$$C_n = D_{n-1}, \qquad D_n = D_{n-1} + C_{n-1}u_n x,$$

with the initial conditions

$$A_0 = 0, B_0 = 1, C_0 = 1, D_0 = 1.$$

These are equivalent to

$$\begin{aligned} A_{n+1} &= A_n + A_{n-1}u_n x, \qquad A_0 = 0, A_1 = 1 \\ C_{n+1} &= C_n + C_{n-1}u_n x, \qquad C_0 = 1, C_1 = 1 \end{aligned}$$

with

$$B_n = A_{n+1}, \qquad D_n = C_{n+1}.$$

Clearly A_n, B_n, C_n, D_n are polynomials in x. Also $C_{n+1}(0) = C_n(0)$, and $A_{n+1}(0) = A_n(0)$ for $n \geq 1$ with $A_1(0) = C_1(0) = 1$, so that $A_n(0) = C_n(0) = 1$ for $n \geq 1$. If we write

$$A_n = 1 + a_{n1}x + a_{n2}x^2 + \ldots, \quad C_n = 1 + c_{n1}x + c_{n2}x^2 + \ldots,$$

then for the coefficients a_{nr}, c_{nr} we have the recursion formulae

$$\begin{aligned} a_{n+1,r} &= a_{nr} + u_n a_{n-1,r-1}, \quad a_{0r} = 0, a_{10} = 1, a_{1r} = 0\ (r \geq 1) \\ c_{n+1,r} &= c_{nr} + u_n c_{n-1,r-1}, \quad c_{00} = c_{10} = 1, , c_{0r} = c_{1r} = 0\ (r \geq 1). \end{aligned}$$

It is easy to see by an induction on n that

$$a_{nr}, c_{nr} \in \mathbf{Z}^+[u_1, u_2, \ldots, u_{n-1}],$$

where $\mathbf{Z}^+[T_1, T_2, \ldots, T_{n-1}]$ is the set of polynomials in the indeterminates $T_1, T_2, \ldots, T_{n-1}$ with coefficients which are integers ≥ 0, and further that

$$a_{nr} = 0 \left(r > \left[\frac{n-1}{2} \right] \right), \quad c_{nr} = 0 \left(r > \left[\frac{n}{2} \right] \right),$$

where $[t]$ is the largest integer $\leq t$. Thus

$$\deg(C_n) \leq \left[\frac{n}{2} \right], \qquad \deg(A_n) \leq \left[\frac{n-1}{2} \right].$$

More generally,

$$C_n = 1 + c_{n1}x + \ldots + c_{np}x^p, \qquad A_{n+1} = 1 + a_{n1}x + \ldots + a_{np}x^p \quad (n \geq 0);$$

where $p = [\frac{n}{2}]$ and the c_{nr} (resp. a_{nr}) are in $\mathbf{Z}^+[u_1, u_2, \ldots, u_{n-1}]$. In particular

$$\deg(C_n C_{n+1}) \leq n \quad (n \geq 1).$$

We also have

$$A_n(0) = 1 (n \geq 1), \qquad C_n(0) = 1 (n \geq 0).$$

Furthermore,

$$\det(H_n) = \det(H_{n-1})(-u_n x),$$

from which it follows that

$$\det(H_n) = (-1)^{n-1} u_1 u_2 \dots u_n x^n, \qquad \det(H_0) = -1.$$

Since

$$\det(H_n) = A_n C_{n+1} - C_n A_{n+1},$$

we have

$$\frac{A_{n+1}}{C_{n+1}} - \frac{A_n}{C_n} = (-1)^n \frac{u_1 u_2 \dots u_n}{C_n C_{n+1}} x^n \quad (n \geq 0).$$

Now $(C_n C_{n+1})^{-1} \in k[[x]]$ since $(C_n C_{n+1})(0) = 1$, and so the right side $\to 0$ in $k[[x]]$. It therefore follows that

$$E_0 := \sum_{n=0}^{\infty} \left(\frac{A_{n+1}}{C_{n+1}} - \frac{A_n}{C_n} \right)$$

is convergent and so is a well defined as an element of $k[[x]]$. If we write

$$E_0 := [1, u_1 x, u_2 x, \dots],$$

then

$$E_0 = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{u_1 u_2 \dots u_n}{C_n C_{n+1}} x^n = 1 + \gamma_1 x + \gamma_2 x^2 + \dots,$$

while

$$\frac{A_{N+1}}{C_{N+1}} = 1 + \sum_{n=1}^{N} (-1)^n \frac{u_1 u_2 \dots u_n}{C_n C_{n+1}} x^n \equiv E_0 \pmod{x^{N+1}}.$$

This can be rewritten as follows. If

$$E_0 = 1 + \gamma_1 x + \gamma_2 x^2 + \dots$$

then

$$\frac{A_{N+1}}{C_{N+1}} = 1 + \gamma_1 x + \dots + \gamma_N x^N + \delta_{N+1} x^{N+1} + \dots.$$

We thus have a map $k^\infty \longrightarrow k[[x]]$ given by

$$(u_1, u_2, \dots) \longmapsto E_0 = 1 + \gamma_1 x + \gamma_2 x^2 + \dots$$

which we write as

$$[1, u_1 x, u_2 x, \dots] = 1 + \gamma_1 x + \gamma_2 x^2 \dots.$$

The coefficients γ_r can be determined from the formula for E_0. We can write

$$C_nC_{n+1} = 1 + d_{n1}x + d_{n2}x^2 + \ldots + d_{nn}x^n, \qquad d_{nj} \in \mathbf{Z}^+[u_1, u_2, \ldots, u_n]$$

so that

$$(C_nC_{n+1})^{-1} = 1 + e_{n1}x + e_{n2}x^2 + \ldots, \qquad e_{nj} \in \mathbf{Z}[u_1, u_2, \ldots, u_n].$$

It then follows easily that

$$\gamma_1 = -u_1, \quad \gamma_r = (-1)^r u_1u_2 \ldots u_r + \delta_r, \quad \delta_r \in \mathbf{Z}[u_1, u_2, \ldots, u_{r-1}].$$

These formulae show that on the subset $k^{\infty\times}$ of k^∞ where all the $u_n \neq 0$, this map is one-one.

It is more complicated to determine the image of $k^{\infty\times}$ under this map and compute the inverse map on this image. In [9] this is done and the answer is quite involved. We do not need it for evaluating the Eulerian continued fractions. But we mention the following easy consequence of applying the transformation $z \to \rho z$ ($\rho \in k^\times = k \setminus \{0\}$): if

$$[1, u_1x, u_2x, \ldots] = 1 + \gamma_1 x + \gamma_2 x^2 \ldots$$

then, for any $\rho \neq 0$,

$$[1, \rho u_1x, \rho u_2x, \rho u_3x, \ldots] = 1 + \gamma_1\rho x + +\gamma_2\rho^2x^2 + \gamma_3\rho^3x^3 + \ldots.$$

To see this, note that

$$\frac{A_{N+1}}{C_{N+1}}(x) = 1 + \gamma_1 x + \ldots + \gamma_N x^N + \delta_{N+1}x^{N+1} + \ldots.$$

Since the left side is rational and thus has analytic meaning, we can make the substitution $x \longmapsto \rho x$ to get

$$\frac{A_{N+1}}{C_{N+1}}(\rho x) = 1 + \gamma_1 \rho x + \ldots + \gamma_N \rho^n x^N + \delta_{N+1}\rho^{N+1}x^{N+1} + \ldots,$$

which leads to

$$[1, \rho u_1x, \rho u_2x, \rho u_3x, \ldots] = 1 + \gamma_1\rho x + +\gamma_2\rho^2x^2 + \gamma_3\rho^3x^3 + \ldots.$$

We now have the following two lemmas from [9] which are useful because they are the ones, when generalized to the case of *positive continued fractions*, give us the mechanism to prove the identities we mentioned in the introduction. The proofs in [9] in the formal case are quite easy and we sketch them.

Lemma 2.1. (1) *If* $\mathfrak{P}_n(x) = 1 + p_{n1}x + p_{n2}x^2 + \ldots$ *is an arbitrary sequence of elements in* $k[[x]]$, *we have*

$$H_n[\mathfrak{P}_n] \longrightarrow E_0 = [1, u_1x, u_2x, \ldots].$$

(2) *Let* $\mathfrak{P}_0(x) = 1/\mathfrak{P}(x)$. *For the power series* $\mathfrak{P}(x) = 1 + p_1x + p_2x^2 + \ldots$, *the relation*

$$[1, u_1x, u_2x, \ldots] = \mathfrak{P}(x)$$

is equivalent to the existence of a sequence of elements elements $\mathfrak{P}_0 = \mathfrak{P}^{-1}, \mathfrak{P}_1, \mathfrak{P}_2, \ldots$ *in* $k[[x]]$ *such that*

$$\mathfrak{P}_0(x) = 1 + \frac{u_1x}{\mathfrak{P}_1(x)}, \quad \mathfrak{P}_1(x) = 1 + \frac{u_2x}{\mathfrak{P}_2(x)}, \ldots.$$

In this case

$$\frac{1}{\mathfrak{P}_r(x)} = [1, u_{r+1}x, u_{r+2}x, \ldots].$$

Proof : (1) We have

$$H_n[\mathfrak{P}_n] - H_n[0] = \frac{A_n\mathfrak{P}_n + B_n}{C_n\mathfrak{P}_n + D_n} - \frac{B_n}{D_n} = \det(H_n)\mathfrak{P}_n\frac{1}{(C_n\mathfrak{P}_n + D_n)D_n}.$$

Now $(C_n\mathfrak{P}_n + D_n)D_n \equiv 2 \pmod{x}$ which implies that it is invertible in $k[[x]]$ and so

$$H_n[\mathfrak{P}_n] - H_n[0] \equiv 0 \pmod{x^n}.$$

As the right side $\rightarrow 0$ in $k[[x]]$ and as $H_n[0] \rightarrow E_0$ in $k[[x]]$, the assertion is proved.

(2) The non trivial part is to prove that $\mathfrak{P} = E_0$ if the $\mathfrak{P}_r$ exist. The data imply that $\mathfrak{P} = H_n[\mathfrak{P}_n] \rightarrow E_0$ by (1). Hence $\mathfrak{P} = E_0$. To show that $1/\mathfrak{P}_r(x) = [1, u_{r+1}x, u_{r+2}x, \ldots]$ we use induction and reduce it to the case $r = 0$. Now

$$[1, u_1x, u_2x, \ldots u_nx]^{-1} = 1 + u_1x[1, u_2x, u_3x, \ldots, u_nx],$$

so that, letting $n \rightarrow \infty$ we have

$$\mathfrak{P}_0(x) = [1, u_1x, u_2x, \ldots]^{-1} = 1 + u_1x[1, u_2x, u_3x, \ldots] = 1 + u_1x\mathfrak{P}_1(x)^{-1},$$

showing that $\mathfrak{P}_1(x)^{-1} = [1, u_2x, u_3x, \ldots]$. □

The next lemma, which contains the main idea behind our method, needs the assumption that k is a topological field [9].

Lemma 2.2. *Let* k *be a topological field. Suppose the* u_n *depend on a parameter* t *and that* $u_n(t) \rightarrow v_n$ *as* $t \rightarrow t_0$ *in the topology of* k. *Let*

$$[1, u_1(t)x, u_2(t)x, \ldots] = 1 + \varepsilon_1(t)x + \varepsilon_2(t)x^2 + \ldots.$$

Then the limits

$$\eta_r := \lim_{t\rightarrow t_0} \varepsilon_r(t)$$

exist in k. *Furthermore*

$$[1, v_1x, v_2x, \ldots] = 1 + \eta_1x + \eta_2x^2 + \ldots.$$

Proof : We know that $\varepsilon_r(t) = \gamma_r(u_1(t), \ldots, u_r(t))$, and so it follows that $\varepsilon_r(t) \rightarrow \gamma_r(v_1, v_2, \ldots, v_r)$ as $t \rightarrow t_0$. Hence $\eta_r = \gamma_r(v_1, v_2, \ldots, v_r)$, showing that $[1, v_1x, v_2x, \ldots] = 1 + \eta_1x + \eta_2x^2 + \ldots$. □

3. Positive Continued Fractions

We shall now study these questions when x is treated as a real variable > 0.

The group $\mathrm{GL}(2, \mathbf{C})$ acts on the extended complex plane by fractional linear transformations:

$$F = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : t \longmapsto \frac{at+b}{ct+d}.$$

We put $\mathbf{R}^+ = \{x|x > 0\}$, and $[0, \infty] = \mathbf{R}^+ \cup \{\infty\}$. Let us define, for $a > 0$,

$$F(a) = \begin{pmatrix} 0 & a \\ 1 & 1 \end{pmatrix}, \qquad F(a) : t \longmapsto \frac{a}{1+t}.$$

Then $F(a)$ leaves $[0, \infty]$ stable, mapping it onto $[0, a]$. Let $(x_r)_{r\geq 1}$ be a sequence of positive numbers. Let

$$H_n = F(1)F(x_1)F(x_2)\ldots F(x_n) = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}.$$

We are interested in the behavior, as $n \to \infty$, of the sequence $H_n[t]$ for various values of $t \in [0, \infty]$. Note that

$$H_n[t] = \frac{1}{1+}\frac{x_1}{1+}\frac{x_2}{1+}\cdots\frac{x_{n-1}}{1+}\frac{x_n}{1+t},$$

and that

$$H_n[0] = H_{n+1}[\infty] = \frac{1}{1+}\frac{x_1}{1+}\frac{x_2}{1+}\cdots\frac{x_{n-1}}{1+}\frac{x_n}{1}.$$

Then a_n, b_n, c_n, d_n satisfy the same recursion formulae as A_n, B_n, C_n, D_n previously, with x_n replacing $u_n x$. Thus

$$\begin{aligned} a_{r+1} &= a_r + x_r a_{r-1} \qquad a_0 = 0, \quad a_1 = 1, \qquad b_r = a_{r+1} \\ c_{r+1} &= c_r + x_r c_{r-1}, \qquad c_0 = 1, \quad c_1 = 1, \qquad d_r = c_{r+1}. \end{aligned}$$

Since the x_r are all > 0 it is immediate that $a_n = b_{n-1} > 0$ $(n \geq 1)$ and $c_n = d_{n-1} > 0$ $(n \geq 0)$. Clearly

$$H_n[t] = \frac{a_n t + b_n}{c_n t + d_n} \qquad (t \in [0, \infty]), \qquad H_n[\infty] = \frac{a_n}{c_n}.$$

Then

$$H_n[0] = \frac{b_n}{d_n} = \frac{a_{n+1}}{c_{n+1}} = H_{n+1}[\infty].$$

As before we have

$$\det H_n = a_n c_{n+1} - c_n a_{n+1} = (-1)^{n-1} x_1 x_2 \ldots x_n.$$

Now

$$\frac{a_{n+1}}{c_{n+1}} - \frac{a_n}{c_n} = H_{n+1}(\infty) - H_n(\infty) = (-1)^n \frac{x_1 x_2 \ldots x_n}{c_n c_{n+1}}.$$

Hence, as $H_1[\infty] = a_1/c_0 = 1$, we have

$$H_n[0] = H_{n+1}[\infty] = \frac{a_{n+1}}{c_{n+1}} = 1 + \sum_{m=1}^{n} (-1)^m \frac{x_1 x_2 \dots x_m}{c_m c_{m+1}} \qquad (n \geq 1).$$

Lemma 3.1. *Let*

$$\alpha_n = \frac{x_1 x_2 \dots x_n}{c_n c_{n+1}}.$$

Then $0 < \alpha_n < 1$ and $(\alpha_n)_{n \geq 1}$ is strictly decreasing.

Proof : From the recursion formula for c_n, namely, $c_{n+1} = c_n + x_n c_{n-1}$ with $c_0 = 1, c_1 = 1$, we first deduce that $c_n \geq 1$ for all $n \geq 0$, and then $c_{n+1} > x_n c_{n-1}$ for all $n \geq 1$. Hence $\alpha_n > 0$ for $n \geq 1$ while $\alpha_1 = x_1/(1 + x_1) < 1$. Moreover,

$$\frac{\alpha_{n+1}}{\alpha_n} = \frac{c_n x_{n+1}}{c_{n+2}} < 1.$$

□

Lemma 3.2. *If* $\lim_{n\to\infty} H_n[0] = \lambda$ *exists, then for any sequence* $(t_n), t_n \in [0, \infty]$*, the limit* $\lim_{n\to\infty} H_n[t_n]$ *exists and is equal to* λ*. Moreover, the convergence of* $H_n[t_n]$ *to* λ *is uniform over all sequences* (t_n)*.*

Proof : We have

$$H_n[0] = H_{n+1}[\infty] = \frac{a_{n+1}}{c_{n+1}} \to \lambda.$$

Let $0 < t_n < \infty$. We have

$$H_n[t_n] - \lambda = \frac{a_n t_n + a_{n+1}}{c_n t_n + c_{n+1}} - \lambda = \frac{(a_n - \lambda c_n) t_n + (a_{n+1} - \lambda c_{n+1})}{c_n t_n + c_{n+1}}$$

so that

$$H_n[t_n] - \lambda = \frac{\left(\frac{a_n}{c_n} - \lambda\right)}{1 + \frac{c_{n+1}}{c_n t_n}} + \frac{\left(\frac{a_{n+1}}{c_{n+1}} - \lambda\right)}{1 + \frac{c_n t_n}{c_{n+1}}}.$$

Using the fact that $t_n > 0$ and all the c_n are > 0 we get, for $t_n \in [0, \infty]$,

$$|H_n[t_n] - \lambda| \leq \left| \frac{a_n}{c_n} - \lambda \right| + \left| \frac{a_{n+1}}{c_{n+1}} - \lambda \right| \to 0.$$

The convergence is clearly uniform over all sequences (t_n). □

Theorem 3.3. *The sequence* $\{H_n[0]\} = \{H_{n+1}[\infty]\}$ *is increasing for odd* n *and decreasing for even* n*, and any odd term is less than any even term. Hence* $\lim_{n\to\infty} H_n[0]$ *exists when* $n \to \infty$ *through even or odd* n*. For* $\lim_{n\to\infty} H_n[0]$ *to exist, these two limits must be the same, in which case the limit is this common value, say* λ*. The necessary and sufficient condition for this is that*

$$\alpha_n = \frac{x_1 x_2 \dots x_n}{c_n c_{n+1}} \to 0.$$

The limit is then given by

$$\lambda = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{x_1 x_2 \ldots x_n}{c_n c_{n+1}}, \quad 0 < \frac{1}{1+x_1} < \lambda < 1.$$

In particular this is the case when

$$\sum_{n \geq 1} \frac{1}{x_n} = \infty.$$

Proof : We have seen that

$$H_{n+1}[\infty] = 1 + \sum_{m=1}^{n} (-1)^m \alpha_m,$$

where the α_m are strictly decreasing positive numbers < 1. All statements of the theorem (except the last one) are now immediate consequences of the standard facts about alternating series. The limit λ satisfies $0 < 1 - \alpha_1 < \lambda < 1$. But $\alpha_1 = x_1/c_1c_2 = x_1/1 + x_1$ so that $1/(1 + x_1) < \lambda < 1$.

For the last one we must estimate $\gamma_n := c_n c_{n+1}$. Now $c_{n+1} = c_n + x_n c_{n-1}$ so that

$$c_n c_{n+1} = c_n^2 + x_n c_{n-1} c_n > c_{n-1} c_n + x_n c_{n-1} c_n = (1 + x_n) c_{n-1} c_n$$

showing that

$$\gamma_n > (1 + x_n)\gamma_{n-1}.$$

Hence

$$c_n c_{n+1} = \gamma_n > \prod_{2 \leq i \leq n} (1 + x_i)$$

and so

$$\frac{x_1 x_2 \ldots x_n}{c_n c_{n+1}} < (1 + x_1) \prod_{1 \leq i \leq n} \left(1 + \frac{1}{x_i}\right)^{-1} = (1 + x_1) e^{-\sum_1^n \log(1+1/x_i)}.$$

Since $\log(1 + u) > (1/2)u$ for positive small u, we have, assuming $x_i \to \infty$, that

$$\frac{x_1 x_2 \ldots x_n}{c_n c_{n+1}} < \text{const}\ e^{-(1/2) \sum_1^n 1/x_i} \to 0$$

under the assumption that

$$\sum_{n \geq 1} \frac{1}{x_n} = \infty.$$

This finishes the proof of the theorem. □

Lemma 3.4. *Suppose that* $[1, x_1, x_2, \ldots]$ *is convergent and has the value* λ*. Then, for any* $r \geq 1, [1, x_{r+1}, x_{r+2}, \ldots]$ *is also convergent. If* λ_r *is its value, we have* $\lambda = [1, x_1, \ldots x_{r-1}, x_r \lambda_r]$.

Proof. Let $\lambda_{rn} = [1, x_{r+1}, \ldots, x_{r+n}]$. Then $H_{r-1}[x_r \lambda_{rn}] = H_{n+r}[0] \to \lambda$ as $n \to \infty$. Hence $\lambda_{rn} = x_r^{-1} H_{r-1}^{-1} H_{n+r}[0] \to x_r^{-1} H_{r-1}^{-1}[\lambda]$. So $\lambda_r = [1, x_{r+1}, x_{r+2}, \ldots]$ $= x_r^{-1} H_{r-1}^{-1}[\lambda]$, giving $H_{r-1}[x_r \lambda_r] = \lambda$. □

Let

$$G(a) = F_a \circ F(a), \qquad F(a) = \begin{pmatrix} 0 & a \\ 1 & 1 \end{pmatrix},$$

so that

$$G(a) = \begin{pmatrix} a & a \\ 1 & a+1 \end{pmatrix}, \qquad G(a)[t] = \frac{a^2 + a^2 t}{t + a^2 + 1}.$$

Corollary 3.5. *For any sequence* (u_n) *of positive numbers*

$$\lim_{n \to \infty} F_1 G_{u_1} G_{u_2} \ldots G_{u_n}[t]$$

exists for all $t \in [0, \infty]$*. In particular,*

$$\lim_{n \to \infty} [1, u_1, u_1, u_2, u_2, \ldots, u_n, u_n] = \lim_{n \to \infty} \frac{1}{1+} \frac{u_1}{1+} \frac{u_1}{1+} \frac{u_2}{1+} \frac{u_2}{1+} \cdots \frac{u_n}{1+} \frac{u_n}{1}$$

exists.

Proof : The even limits of the second continued fraction exist by Theorem 3.3.

Remark. Note the absence of any conditions on the sequence. By abuse of notation we write

$$[1, u_1, u_1, u_2, u_2, \ldots] = \lim_{n \to \infty} [1, u_1, u_1 . u_2, u_2, \ \ldots, u_n, u_n].$$

Lemma 3.6. *Suppose* $x_n > 0$ *are such that* $[1, x_1, x_2, \ldots]$ *is convergent with value* λ*. If* $g, g_0 = g^{-1}, g_1, g_2, \ldots$ *are numbers* > 0 *such that we have*

$$g_0 = 1 + \frac{x_1}{g_1}, \quad g_1 = 1 + \frac{x_2}{g_2}, \ \ldots$$

then

$$g = \lambda = [1, x_1, x_2, \ldots], \qquad \frac{1}{g_r} = \lambda_r = [1, x_{r+1}, x_{r+2}, \ldots] \quad (r \geq 0).$$

Proof : We have $g = H_{n-1}[x_n g_n^{-1}]$. Since the limit $\lim_{n \to \infty} H_n[0] = \lambda$ exits, the limit $\lim_{n \to \infty} H_{n-1}[x_n g_n^{-1}]$ also exists and is equal to λ (Lemma 3.2). But $H_{n-1}[x_n g_n^{-1}] = g$ for all n and so $g = \lambda$. Since $g = \lambda = H_{r-1}[x_r g_r^{-1}]$ and $g = H_{r-1}[x_r \lambda_r]$, we must have $g_r^{-1} = \lambda_r$. □

We now introduce the variable $x > 0$ and consider the continued fraction of the form

$$[1, u_1 x, u_2 x, \ldots] \qquad (u_n > 0)$$

which are convergent. By Theorem 3.3, we know that this will be the case as soon as

$$\sum_n \frac{1}{u_n} = \infty.$$

Let us write

$$E_0(x) = [1, u_1 x, u_2 x, \ldots].$$

Lemma 3.7. *Suppose that the u_n depend on a parameter t and that $u_n(t) \to v_n > 0$ as $t \to t_0$. Assume that the continued fractions*

$$E_0(x,t) = [1, u_1(t)x, u_2(t)x, \ldots] \quad \text{and } E_0(x) = [1, v_1 x, v_2 x, \ldots]$$

are convergent. Then, if the limit

$$\lim_{t \to t_0} E_0(x,t) = F(x)$$

exists for all $x > 0$, this limit must be equal to $E_0(x)$.

Proof : We have

$$[1, u_1(t)x, , \ldots, u_{2n+1}(t)x] < E_0(x,t) < [1, u_1(t)x, , \ldots, u_{2n}(t)x]$$

for all $x > 0, t, n \geq 1$. Letting $t \to t_0$ we get

$$[1, v_1 x, \ldots, v_{2n+1} x] \leq F(x) \leq [1, v_1 x, \ldots, v_{2n} x]$$

for all $n \geq 1, x > 0$. Letting $n \to \infty$ we get

$$E_0(x) \leq F(x) \leq E_0(x)$$

which proves what we want. □

Corollary 3.8. *Suppose $E_0(x) = [1, u_1 x, u_2 x, \ldots]$ is convergent for all $x > 0$. If $G = G_0^{-1}, G_1, G_2, \ldots$ are functions defined for $x > 0$ which are > 0 and satisfy*

$$G_0(x) = 1 + \frac{u_1 x}{G_1(x)}, \quad G_1(x) = 1 + \frac{u_2 x}{G_2(x)}, \ldots$$

then

$$G(x) = [1, u_1 x, u_2 x, \ldots], \qquad \frac{1}{G_r(x)} = [1, u_{r+1} x, \ldots].$$

Proof : Immediate from Lemma 3.6. □

4. Hypergeometric Continued Fractions

We shall now look at the continued fractions generated by the hypergeometric functions and their deformations. At the level of formal or convergent power series the results are in [9]. For our purposes we need them for the case of positive continued fractions also.

We start with

$$F(a,b,c;z) = 1 + \frac{ab}{c.1}z + \frac{a(a+1)b(b+1)}{c(c+1)2!}z^2 + \dots$$

defined for all a, b, c with $c \notin \{0, -1, -2, \dots\}$. The series is convergent for $|z| < 1$ and satisfies the differential equation

$$z(z-1)\frac{d^2F}{dF^2} + [(a+b+1)z - c]\frac{dF}{dz} + abF = 0.$$

The coefficients of the differential equation (after normalizing so that the leading coefficient is 1) have no singularities in the simply connected cut plane $\mathbf{C}_{[0,\infty]} := \mathbf{C} \setminus [0,\infty]$. Hence the above series continues analytically to $\mathbf{C}_{[1,\infty]}$. If $c > b > 0$ this analytic continuation is given by Euler's integral

$$F(a,b,c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt,$$

which is convergent on the cut plane $\mathbf{C}_{[1,\infty]}$; here $(1-zt)^{-a} = e^{-a\log(1-zt)}$ and we take the branch of $\log$ that is 0 at 1 and is analytic on the cut plane $\mathbf{C}_{[-\infty,0]}$. The branch of $(1-zt)^{-a}$ thus defined on $\mathbf{C}_{[1,\infty]}$ coincides with the binomial series expansion of $(1-zt)^{-a}$ or $|z| < 1$. Integrating this series identifies the integral, which is analytic on $\mathbf{C}_{[1,\infty]}$, with the series $F(a,b,c;z)$ on the unit disc. The integral (with the gamma factors) $\to 1$ when $b \to 0+$, uniformly for z in compact subsets of $\mathbf{C}_{[1,\infty]}$, and so we may allow b to be 0 provided we define the integral (with the gamma factors) to be 1 when $b = 0$. It follows from this integral representation that

$$F(a,b,c;x) > 0 \quad (a \in \mathbf{R}, c > b \geq 0, x \in \mathbf{R}, x < 1).$$

So from now on we assume

$$a \in \mathbf{R}, c > b \geq 0, z \in \mathbf{C}_{[1,\infty]}.$$

Any polynomial relation between hypergeometric functions established by using power series is valid on $\mathbf{C}_{[1,\infty]}$ provided $a \in \mathbf{R}, c > b \geq 0$.

We follow [9] is associating continued fractions to the *ratio* of hypergeometric series that go back to Gauss. One starts with the *contiguity relations* of Gauss:

$$F(a,b,c;z) \quad = F(a,b+1,c+1;z) - \tfrac{a(c-b)}{c(c+1)}zF(a+1,b+1,c+2;z),$$

$$F(a,b,c;z) \quad = F(a+1,b,c+1;z) - \tfrac{b(c-a)}{c(c+1)} z F(a+1,b+1,c+2;z).$$

These relations are valid as formal or convergent power series with the only restriction that $c \notin \{0,-1,-2,\ldots\}$. Hence

$$\frac{F(a,b,c;z)}{F(a,b+1,c+1;z)} = 1 - \frac{a(c-b)}{c(c+1)} z \frac{F(a+1,b+1,c+2;z)}{F(a,b+1,c+1;z)}$$

from the first relation, and then

$$\frac{F(a,b+1,c+1;z)}{F(a+1,b+1,c+2;z)} = 1 - \frac{(b+1)(c+1-a)}{(c+1)(c+2)} z \frac{F(a+1,b+2,c+3;z)}{F(a+1,b+1,c+2;z)}$$

from the second relation (after changing b, c to $b+1, c+1$). Write

$$\begin{aligned} R_0(a,b,c;z) &= \frac{F(a,b,c;z)}{F(a,b+1,c+1;z)} \\ R_1(a,b,c;z) &= \frac{F(a,b+1,c+1;z)}{F(a+1,b+1,c+2;z)}. \end{aligned}$$

Then the above relations can be written as

$$\begin{aligned} R_0(a,b,c;z) &= 1 - \frac{a(c-b)}{c(c+1)} z \frac{1}{R_1(a,b,c;z)} \\ R_1(a,b,c;z) &= 1 - \frac{(b+1)(c+1-a)}{(c+1)(c+2)} z \frac{1}{R_0(a+1,b+1,c+2;z)} \end{aligned}$$

Let

$$\begin{aligned} R_{2n}(a,b,c;z) &= R_0(a+n,b+n,c+2n;z) &= \frac{F(a+n,b+n,c+2n;z)}{F(a+n,b+n+1,c+2n+1;z)} \\ R_{2n+1}(a,b,c;z) &= R_1(a+n,b+n,c+2n;z) &= \frac{F(a+n,b+n,c+2n+1;z)}{F(a+n,b+n,c+2n+1;z)}. \end{aligned}$$

Then

$$R_0 = 1 + \frac{a_1' z}{R_1}, \quad R_1 = 1 + \frac{a_2' z}{R_2}, \ \ldots$$

with

$$a_{2n}' = -\frac{(b+n)(c-a+n)}{(c+2n-1)(c+2n)}, \qquad a_{2n+1}' = -\frac{(a+n)(c-b+n)}{(c+2n)(c+2n+1)}.$$

By Lemma 2.1 we thus have, as formal power series,

$$\frac{F(a,b+1,c+1;z)}{F(a,b,c;z)} = [1, a_1' z, a_2' z, \ldots].$$

If we now change z to $-cz$ we get

$$\frac{F(a,b+1,c+1;-cz)}{F(a,b,c;-cz)} = [1, a_1 z, a_2 z, \ldots],$$

where

$$a_{2n} = \frac{(b+n)(c-a+n)c}{(c+2n-1)(c+2n)}, \qquad a_{2n+1} = \frac{(a+n)(c-b+n)c}{(c+2n)(c+2n+1)}.$$

Note that all the a_n are > 0 if $c > a \geq 0, c > b \geq 0$. Write

$$G(a,b,c;z) = G_0(a,b,c;z)^{-1} = \frac{F(a,b+1,c+1;-cz)}{F(a,b,c;-cz)}.$$

Further let

$$\begin{aligned} G_{2n}(a,b,c;z) &= G_0(a+n,b+n,c+2n,z) \\ G_{2n+1}(a,b,c;z) &= G_1(a+n,b+n,c+2n;z). \end{aligned}$$

Then the continued fraction for G is equivalent to the identities, as formal power series,

$$G_0 = 1 + \frac{a_1 z}{G_1}, \quad G_1 = 1 + \frac{a_2 z}{G_2}\ , \ \ldots .$$

We now want to make the transition to the case when z is no longer formal but varies in the cut plane $\mathbf{C}_{[-\infty,-1]}$. Because we changed z to $-cz$, the functions G_n are now meromorphic on $\mathbf{C}_{[-\infty,-1]}$ and finite and > 0 for $x > -1$, in particular for $x > 0$. The above relations then yield, by Corollary 3.8, the continued fractions:

$$\frac{F(a,b+1,c+1;-cx)}{F(a,b,c;-cx)} = [1, a_1 x, a_2 x, \ldots] \qquad (x > 0).$$

Substituting the integrals for the hypergeometric functions we get

$$[1, a_1 x, a_2 x, \ldots] = \frac{c\int_0^1 t^b (1-t)^{c-b-1}(1+cxt)^{-a} dt}{b\int_0^1 t^{b-1}(1-t)^{c-b-1}(1+cxt)^{-a} dt}.$$

Here $c > a > 0, c > b \geq 0$ with the proviso that when $b = 0$ we replace the denominator by 1. Changing ct to u we get finally

$$[1, a_1 x, a_2 x, \ldots] = \frac{\int_0^c u^b (1-(u/c))^{c-b-1}(1+xu)^{-a} du}{b\int_0^c u^{b-1}(1-(u/c))^{c-b-1}(1+xu)^{-a} du},$$

with

$$a_{2n} = \frac{(b+n)(c-a+n)c}{(c+2n-1)(c+2n)}, \qquad a_{2n+1} = \frac{(a+n)(c-b+n)c}{(c+2n)(c+2n+1)}.$$

When $b = 0$ this becomes

$$[1, a_1 x, a_2 x, \ldots] = \int_0^c (1-(u/c))^{c-1}(1+xu)^{-a} du,$$

with

$$a_{2n} = \frac{n(c-a+n)c}{(c+2n-1)(c+2n)}, \qquad a_{2n+1} = \frac{(a+n)(c+n)c}{(c+2n)(c+2n+1)}.$$

5. Deformations

We now treat c as a real parameter and let it go to $+\infty$.
The case $b = 0$. We have

$$F(a,1,c+1;-cz) = 1 - \frac{a}{c+1}cz + \ldots + (-1)^n \frac{a\ldots(a+n-1)}{(c+1)\ldots(c+n)}c^n z^n + \ldots .$$

When $c \to \infty$, the condition $c > a > 0$ is not disturbed, and the series tends, coefficientwise to the Euler divergent series

$$1 - az + a(a+1)z^2 - \ldots + (-1)^n a(a+1)\ldots(a+n-1)z^n + \ldots .$$

Since

$$a_{2n} \longrightarrow n, \qquad a_{2n+1} \longrightarrow a+n,$$

Lemma 2.2 gives us

$$[1, az, z, (a+1)z, 2z, \ldots] = 1 - az + \ldots + (-1)^n a(a+1)\ldots(a+n-1)z^n + \ldots .$$

At the analytical level we can now assert that (after making the change of variables $u = ct$)

$$[1, ax, x, (a+1)x, 2x, \ldots] = \lim_{c\to\infty} \int_0^c \left(1 - \frac{u}{c}\right)^{c-1} (1+ux)^{-a} du.$$

Hence we obtain

$$[1, ax, x, (a+1)x, 2x, \ldots] = \int_0^\infty e^{-u}(1+xu)^{-a} du \qquad (x > 0).$$

Taking $a = 1$ we get

$$[1, x, z, 2x, 2x, \ldots] = \int_0^\infty \frac{e^{-u}}{(1+xu)} du = \frac{e^{1/x}}{x} \int_0^x \frac{e^{1/t}}{t} \qquad (x > 0).$$

If we take $a = p/q$ and change x to qx we get

$$[1, px, qx, (p+q)x, 2qx, \ldots] = \int_0^\infty \frac{e^{-u}}{(1+qxu)^{p/q}} du,$$

or, using the change of variables $(1/qx) - (1/qt) = u$,

$$[1, px, qx, (p+q)x, 2qx, \ldots] = \frac{e^{1/qx}}{qx^{p/q}} \int_0^x e^{-1/qt} t^{(p/q)-2} dt.$$

The case $b > 0$. The method is the same. We start with

$$\begin{aligned}[1.a_1x, a_2x, \ldots] \quad &= \frac{c\int_0^1 t^b(1-t)^{c-b-1}(1+cxt)^{-a}dt}{b\int_0^1 t^{b-1}(1-t)^{c-b-1}(1+cxt)^{-a}dt} \\ &= \frac{\int_0^c u^b(1-\frac{u}{c})^{c-b-1}(1+xu)^{-a}du}{b\int_0^c u^{b-1}(1-\frac{u}{c})^{c-b-1}(1+xu)^{-a}du},\end{aligned}$$

where

$$a_{2n} = \frac{(b+n)c(c-a+n)}{(c+2n-1)(c+2n)}, \qquad a_{2n+1} = \frac{(a+n)c(c-b+n)}{(c+2n)(c+2n+1)}.$$

When $c \to \infty$, we have

$$a_{2n} \to b+n, \qquad a_{2n+1} \to a+n.$$

Hence

$$[1, ax, (b+1)x, (a+1)x, (b+2)x, \ldots] = \frac{\int_0^\infty u^b e^{-u}(1+xu)^{-a}du}{b\int_0^\infty u^{b-1}e^{-u}(1+xu)^{-a}du}.$$

The formal series underlying this convergent continued fraction is

$$\lim_{c\to\infty} \frac{F(a, b+1, c+1; -cz)}{F(a,b,c;-cz)} = \frac{\Omega(a, b+1; z)}{\Omega(a,b;z)},$$

where

$$\Omega(a,b;z) = 1 - abz + a(a+1)b(b+1)\frac{z^2}{2!} - \ldots.$$

We thus have the following.

Theorem 5.1. *Let $p, q \geq 1$ be integers. Then the infinite continued fraction*

$$[1, px^q, qx^q, (p+q)x^q, 2qx^q, \ldots] = \frac{1}{1+}\,\frac{px^q}{1+}\,\frac{qx^q}{1+}\,\frac{(p+q)x^q}{1+}\,\frac{2qx^q}{1+}\ldots$$

is convergent for any $x > 0$ and has the value

$$[1, px^q, qx^q, (p+q)x^q, 2qx^q, \ldots] = \frac{e^{1/qx^q}}{x^p}\int_0^x e^{-1/qt^q}t^{p-q-1}dt.$$

In particular

$$[1, x, x, 2x, 2x, 3x, 3x, \ldots] = \frac{e^{1/x}}{x}\int_0^x \frac{e^{-1/t}}{t}dt.$$

6. Numerical Evaluation

The numerical evaluation of the continued fraction above for $x = 1$ is central to Euler's paper. He does not restrict himself to just stopping at some large finite stage but makes a serious effort to estimate the tail neglected. In interpreting Euler's ingenious method we are grateful to the letters that Professor Pierre Deligne wrote to one of us (V) [14]. The discussion that follows is entirely due to him.

Let us denote the Euler continued fraction by E so that

$$E = \frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\frac{2}{1+}\frac{2}{1+}\cdots\frac{a-1}{1+P},$$

where

$$P = P(a) = \frac{a}{1+}\frac{a}{1+}\frac{a+1}{1+}\frac{a+1}{1+}\cdots.$$

The function P satisfies the recursion

$$P(a) = \frac{aP(a+1)+a}{P(a+1)+(a+1)}.$$

Euler's calculations become clearer under the assumption that $P(a)$ is a slowly varying function for large a. In the first approximation we take $P(a) = P(a+1)$ to find that $P(a)$ satisfies the quadratic equation

$$x^2 + x = a.$$

So we get

$$P(a) \sim \frac{-1+\sqrt{1+4a}}{2} = \sqrt{a} - \frac{1}{2} + O(a^{-1/2}).$$

This suggests that $P(a) \sim \sqrt{a}$ agreeing with the assumption that it is slowly varying, and also that the derivative of P is $\sim 1/2\sqrt{a}$. In the second approximation we use the recursion at $a-1$ and a to get

$$P(a-1) = \frac{(a-1)P(a)+(a-1)}{P(a)+a} \qquad P(a+1) = \frac{(a+1)P(a)-a}{a-P(a)}.$$

The approximate linearity means

$$P(a) = \frac{P(a+1)+P(a-1)}{2},$$

which gives for $P(a)$ the cubic equation

$$2x^3 + 2x^2 - (2a-1)x - a = 0.$$

Euler takes $a = 22$ and for $P(a)$ the root of this cubic equation near $\sqrt{a}$ (between 4 and 5), computing it by Newton's method to get his value

$$E = 0.5963473621372,$$

which should be compared with calculation by Mathematica taking 640 and 641 terms of F that gives

$$E = 0.5963473624.$$

To examine this method more closely we write $P(a+1) = P(a) + \varepsilon$ and get the equation

$$P(a)^2 + P(a)(1+\varepsilon) - a(1+\varepsilon) = 0.$$

Solving this we get

$$P(a) = \sqrt{a}\Big(1 + \varepsilon + (1+\varepsilon)^2/4a\Big)^{1/2} - \frac{1}{2} - \frac{\varepsilon}{2}.$$

If we start from $P(a) = \sqrt{a} + c + O(a^{-1/2})$ we take $\varepsilon = \frac{1}{2\sqrt{a}} + O(a^{-3/2})$, and so we obtain

$$\begin{aligned} P(a) &= \sqrt{a}\Big(1 + \frac{1}{2\sqrt{a}} + \frac{1}{4a} + \dots\Big)^{1/2} \\ &= \sqrt{a}\Big(1 + \frac{1}{4\sqrt{a}} + \frac{1}{8a} - \frac{1}{32a}\Big) - \frac{1}{2} - \frac{1}{4\sqrt{a}} + \dots \\ &= \sqrt{a} - \frac{1}{4} - \frac{5}{32}\frac{1}{\sqrt{a}} + \dots . \end{aligned}$$

This can be continued to get an asymptotic expansion for $P(a)$ as a series in $a^{k/2}$. But this argument only suggests that such an expansion exists, starting as

$$P(a) = \sqrt{a} - \frac{1}{4} - \frac{5}{32}\frac{1}{\sqrt{a}} + \dots .$$

Numerically we can then try to evaluate the fraction E by taking a small number of terms but substituting

$$P(a) = \sqrt{a} - \frac{1}{4}.$$

For instance, taking $a = 30, 40, 50$ we get

$$E = 0.5963473620, \quad E = 0.5963473620 \quad E = 0.5963473622.$$

This discussion does not treat the second approximation that Euler makes that leads to a cubic equation. There are many questions here that need to be understood: the existence of asymptotic expansions for $P(a)$, the estimation of the errors committed, and above all, to extend the theory to cover continued fractions of a similar type–for instance,

$$[1,1,1,1,1,\dots 2^a,2^a,2^a,2^a,\dots,3^a,3^a,3^a,3^a,\dots\dots]$$

where each integer is repeated k times.

It may not be without interest to give an explicit formula for the tails of the Euler continued fraction. We have, for $a, b \in \mathbf{R}, b > 0$,

$$\lim_{c\to\infty}\left[\frac{F(a,b+1,c+1;-cz)}{F(a,b,c;-cz)}\right]_{z=1} = [1,a,b+1,a+1,b+2,\ldots].$$

We now substitute for the hypergeometric functions their integral representations to get

$$\left[\frac{F(a,b+1,c+1;-cz)}{F(a,b,c;-cz)}\right]_{z=1} = \frac{c\int_0^1 t^b(1-t)^{c-b-1}(1+ct)^{-a}dt}{b\int_0^1 t^{b-1}(1-t)^{c-b-1}(1+ct)^{-a}dt}.$$

Going to the new variable $\tau = ct$ we get

$$\left[\frac{F(a,b+1,c+1;-cz)}{F(a,b,c:-cz)}\right]_{z=1} = \frac{\int_0^c \tau^b(1-(\tau/c))^{c-b-1}(1+\tau)^{-a}d\tau}{b\int_0^c \tau^{b-1}(1-(\tau/c))^{c-b-1}(1+\tau)^{-a}d\tau}.$$

Letting $c \to \infty$ we thus obtain

$$[1,a,b+1,a+1,b+2,\ldots] = \frac{\int_0^\infty e^{-\tau}\tau^b(1+\tau)^{-a}d\tau}{b\int_0^\infty e^{-\tau}\tau^{b-1}(1+\tau)^{-a}d\tau}.$$

Integrating by parts the denominator it can be seen to be equal to

$$\int_0^\infty \tau^b\bigg(a(1+\tau)^{-a-1}+(1+\tau)^{-a}\bigg)e^{-\tau}d\tau.$$

Hence, if we set

$$P(a,b) = \frac{a}{1+}\,\frac{b+1}{1+}\,\frac{a+1}{1+}\,\frac{b+2}{1+}\cdots$$

then

$$\frac{1}{1+P(a,b)} = \frac{\int_0^\infty \tau^b(1+\tau)^{-a}e^{-\tau}d\tau}{\left(a\int_0^\infty \tau^b(1+\tau)^{-a-1}e^{-\tau}d\tau + \int_0^\infty \tau^b(1+\tau)^{-a}e^{-\tau}d\tau\right)}.$$

Set

$$\Phi(a,b) = \int_0^\infty \tau^b(1+\tau)^{-a}e^{-\tau}d\tau.$$

Then

$$\frac{1}{1+P(a,b)} = \frac{\Phi(a,b)}{a\Phi(a+1,b)+\Phi(a,b)}.$$

Hence finally

$$P(a,b) = \frac{a\Phi(a+1,b)}{\Phi(a,b)} = \frac{a\int_0^\infty \tau^b(1+\tau)^{-a-1}e^{-\tau}d\tau}{\int_0^\infty \tau^b(1+\tau)^{-a}e^{-\tau}d\tau}.$$

To get the tails of the Euler continued fraction we notice that when $b = a - 1$, we have

$$P(a, a-1) = P(a) = \frac{a}{1+}\frac{a}{1+}\frac{a+1}{1+}\frac{a+1}{1+}\dots.$$

Thus

$$P(a) = \frac{a}{1+}\frac{a}{1+}\frac{a+1}{1+}\frac{a+1}{1+}\dots = \frac{a\int_0^\infty \tau^{a-1}(1+\tau)^{-a-1}e^{-\tau}d\tau}{\int_0^\infty \tau^{a-1}(1+\tau)^{-a}e^{-\tau}d\tau}.$$

For the asymptotic expansion

$$P(a) = \sqrt{a} - \frac{1}{4} - \frac{5}{32}\frac{1}{\sqrt{a}} + \dots$$

there is considerable numerical evidence, but the existence of the asymptotic expansion in powers of $a^{-1/2}$ remains to be proved.

7. The Sturmian Nature of the Sequence (C_n)

As in Sturm's classical theorem we are interested in the following situation. Let

$$P_0, P_1, \dots, P_k$$

be a sequence of *non-zero real* polynomials of a real variable x that are successive terms of an algorithm:

$$P_0 = Q_1P_1 - P_2, P_1 = Q_2P_2 - P_3, \dots P_{k-1} = Q_kP_k.$$

We are interested in counting the number and locations of roots of the polynomials P_j. For any real number a at which none of the $P_j(a)$ vanish we define $V(a)$ as the number of sign changes in the sequence

$$P_0(a), P_1(a), \dots P_k(a).$$

The function V is defined on $\mathbf{R}$ except for a finite set. Clearly V is locally constant on the open set where it is defined. If $b \in \mathbf{R}$ is a point where some of the P_j vanish, there is an open interval $I = (c, d)$ containing b such that V is defined and constant on each of the two intervals (c, b) and (b, d) so that the jump at b, namely,

$$J(b) = V(b+0) - V(b-0),$$

is well-defined. Following Smiley [15] we have the following lemma.

Lemma 7.1. *If b is such that $P_0(b) \neq 0$, then $J(b) = 0$.*

Proof : If none of the P_j vanish at b this is obvious. Suppose $P_j(b) = 0$ for some $j \geq 1$. We claim that $j < k$. Indeed, if $P_k(b) = 0$ the equation $P_{k-1} = Q_kP_k$ shows that $P_{k-1}(b) = 0$ and so, in succession, all the P_j vanish at b, contradicting $P_0(b) \neq 0$. The equation $P_{j-1} = Q_jP_j - P_{j+1}$ shows that $P_{j-1}(b)$ and $P_{j+1}(b)$ are either both 0 or both $\neq 0$. If $P_{j-1}(b) = 0$ we can use the recursive formulae successively to get $P_0(b) = 0$, a contradiction. Hence $P_{j-1}(b)P_{j+1}(b) \neq 0$. The equation $P_{j-1}(b) = -P_{j+1}(b)$ now shows that $P_{j-1}(b)P_{j+1}(b) < 0$. For $b' < b$ and close to b, the sequence $P_{j-1}(b'), P_j(b'), P_{j+1}(b')$ has exactly one sign change, and the same is true when we replace b' by $b'' > b$ and close to b. So in computing the jump at b the contribution to it by P_j is 0. Since this applies to all possible j such that $P_j(b) = 0$ the lemma follows. □

Lemma 7.2. *Suppose b is such that $P_k(b) \neq 0$ but $P_0(b) = 0$. Then $J(b) \leq 1$.*

Proof : In this case $P_1(b) \neq 0$. For otherwise we will have $P_2(b) = 0$ and so on so that we have, in succession, $P_3, P_4, \ldots$ all vanish at b, hence finally, $P_k(b) = 0$, a contradiction. To the sequence $P_1, P_2, \ldots, P_k$ we can apply Lemma 7.1 and conclude that the number of sign changes for this sequence stays constant near b. We thus have to consider only the sequence P_0, P_1. Since P_1 has the same sign across b, we can have at most one sign change, proving the Lemma. □

Theorem 7.3. *Suppose $[\alpha, \beta]$ is an interval such that P_0 does not vanish at α or β and P_k does not vanish on the entire interval $[\alpha, \beta]$. Then*

$$|V(\alpha + 0) - V(\beta - 0)| \leq \text{the number of roots of } P_0 \text{ in } [\alpha, \beta].$$

Proof : Let R be the set of roots of P_0 in $[\alpha, \beta]$. Clearly

$$|V(\alpha+) - V(\beta-)| \leq \sum_{b \in R} |J(b)| \leq \#R.$$

The theorem follows at once from Lemmas 7.1 and 7.2. □

As an application we have the following theorem.

Theorem 7.4. *Suppose $(P_j)_{0 \leq j \leq k}$ is a sequence as above, with $\deg(P_j) = k - j$. If all the P_j have coefficients ≥ 0 and $P_j(0) > 0$, then all roots of all the P_j are simple, real and negative. The roots of P_j are moreover interlaced by the roots of P_{j+1}.*

Proof : Clearly P_k is a constant > 0. For $b << 0$ we have $V(b) = k$ while for $b < 0$ and close to 0, we have $V(b) = 0$. Hence

$$|V(-\infty) - V(0 - 0)| = k \leq \#R.$$

So P_0 already has k distinct roots in $(-\infty, 0)$. So all roots of P_0 are simple, real, and negative. The same argument applied to the sequence $P_j, P_{j+1}, \ldots$ gives the result for P_j.

The interlacing needs additional arguments. It is enough to do it for $j = 0$. So we shall prove that between two successive roots of P_0 there is a root of P_1. First of all the inequality

$$|V(-\infty) - V(0 - 0)| = k \leq \#R$$

now becomes an equality:

$$|V(-\infty) - V(0-0)| = \#R.$$

Since $V(b) = k$ for $b << 0$ and $V(0-0) = 0$, it follows that at each root V drops by exactly 1, namely

$$V(b-0) - V(b+0) = 1 \qquad (b \in R).$$

Let us now consider two successive roots b_1, b_2 of P_0 with $b_1 < b_2 < 0$. Suppose that P_1 does not have a root in the open interval (b_1, b_2). Notice that $P_1(b_j) \neq 0$ for $j = 1, 2$ since otherwise, both P_0 and P_1 would vanish at one of the b_j and so, by the argument given earlier, all the P_j would vanish there, contradicting the assumption that P_k is a non-zero constant. So P_1 has no root in $[b_1, b_2]$ from which we conclude that the sequence $P_1(c), \ldots, P_k(c)$ has no sign change as c varies in (b'_1, b'_2) where b'_1 is slightly to the left of b_1 and b''_2 is slightly to the right of b_2. So the variation of $V(c)$ is exactly the number of sign changes in the sequence $P_0(c)P_1(c)$. If the sign of P_1 in the interval is $+$ (the argument is the same if the sign is $-$) then the condition that the drop of V at b_1 is exactly 1 means that the sign changes of P_0, P_1 have to be the following: $(-, +)$ at b'_1 slightly to the left of b_1 and $(+, +)$ at b''_1 slightly to the right of b_1. The same argument at the right end point b_2 shows that the signs are $(-, +)$ at b'_2 slightly to the left of b_2 and $(+, +)$ at b''_2 slightly to the right of b_2. But this shows that P_0 changes sign within the open interval (b_1, b_2), a contradiction, since we are assuming that P_0 has no root there. This finishes the proof. □

This result has applications to the polynomials that occur in the continued fractions of Euler. Let $(u_n)_{n\geq 1}$ be a sequence of numbers > 0 and let $C_n(n = 0, 1, 2, \ldots)$ be the sequence of polynomials defined by the recursion formula

$$C_n = C_{n-1} + u_{n-1}xC_{n-2}(n \geq 2), \qquad C_0 = a, C_1 = b$$

where a, b are constants and $a \geq 0, b \geq 0, a + b > 0$. The first few terms of the sequence are

$$\begin{aligned}
C_2 &= b + au_1x \\
C_3 &= b + (au_1 + bu_2)x \\
C_4 &= b + (au_1 + bu_2 + bu_3)x + au_1u_3x^2 \\
C_5 &= b + (au_1 + bu_2 + bu_3 + bu_4)x + (au_1u_3 + au_1u_4 + bu_2u_4)x^2 \\
&\ldots .
\end{aligned}$$

It is immediate that all the C_n $(n \geq 2)$ have strictly positive coefficients and that $C_n(0) = b$ for $n \geq 1$ and $C_1(0) = a$. One can show by induction on n that the degree of C_n for $n \geq 2$ is $[\frac{n}{2}]$ when $a > 0$ and $[\frac{n-1}{2}]$ when $a = 0$. We now find that we have the following recursion

$$C_n = [1 + (u_{n-1} + u_{n-2})x]C_{n-2} - u_{n-2}u_{n-3}x^2C_{n-4} \qquad (n \geq 4).$$

To get this into the Sturmian form we make a transformation. For and polynomial $p(x)$ of degree d we write $p^*(x) = x^d p(1/x)$. If p has strictly positive coefficients the same is true for p^* and further p^* also has degree d. The C_n^* have now the recursion formula

$$C_n^* = [(u_{n-1} + u_{n-2}) + x]C_{n-2}^* - u_{n-2}u_{n-3}C_{n-4}^* \qquad (n \geq 4).$$

These are now in the form we have been discussing above. We therefore have 2 families

$$C_{2k}^*, C_{2k-2}^*, \dots C_2^*, C_0^*$$

and

$$C_{2k+1}^*, C_{2k-1}^*, \dots C_3^*, C_1^*.$$

Their degrees are

$$k, k-1, \dots 0$$

in both cases, assuming that $a > 0, b > 0$. If one or the other of a, b is zero, we have to stop at an earlier stage. If $a = 0$, then the even sequence is terminated at C_2 and the degrees are

$$k-1, k-2, \dots, 0,$$

while the odd sequence needs no change. If $b = 0$, neither sequence needs a change. We thus have:

Theorem 7.5. *The roots of C_n are all simple, real, and negative. Moreover the roots of C_{n-1} interlace the roots of C_n.*

8. Summary

Euler's attempt to sum the factorial series led him to an integral and a continued fraction which have been shown to be equal, as Euler surmised. The numerical evaluation of the continued fraction was carried out by Euler to a very high order of accuracy taking only 20+ terms and using an ingenious approximation for the portion omitted. It is necessary to understand this approximation more rigorously and prove that the omitted tail parts have asymptotic expansions, as well as to extend this theory for more general continued fractions of the Euler type, and evaluating them, perhaps obtaining generalizations of the formulas of Ramanujan and Rogers. It is also worthwhile attempting to prove that the polynomials in the denominators of these more general continued fractions are Sturmian. Finally it would be interesting to relate this to the work of Stieltjes.

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