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AN EXTENSION OF A RESULT OF ZAHARESCU ON IRREDUCIBLE POLYNOMIALS[1]

Sudesh K. Khanduja and Ramneek Khassa

Department of Mathematics, Panjab University, Chandigarh 160 014, India
e-mails: skhand@pu.ac.in, ramneekkhassa@yahoo.co.in

Abstract It is well known that if $f(x)$ is a monic irreducible polynomial of degree d with coefficients in a complete valued field $(K, |\,|)$, then any monic polynomial of degree d over K which is sufficiently close to $f(x)$ with respect to $|\,|$ is also irreducible over K. In 2004, Zaharescu proved a similar result applicable to separable, irreducible polynomials over valued fields which are not necessarily complete. In this paper, the authors extend Zaharescu's result to all irreducible polynomials without assuming separability.

Key words Valued fields, non-Archimedean valued fields, irreducible polynomials.

1. Introduction

A classical result concerning irreducible polynomials over a valued field K which is complete with respect to a non-archimedean absolute value $|\,|$ says that if $f(x) = x^d + a_1x^{d-1} + \ldots + a_d$ belonging to $K[x]$ is irreducible over K, then there exists a positive real number ϵ such that any polynomial $g(x) = x^d + b_1x^{d-1} + \ldots + b_d$ belonging to $K[x]$ with $|b_j - a_j| < \epsilon$ for $1 \leqslant j \leqslant d$, is also irreducible over K (see [1, Chapter 2, Theorem 11]). An analogous result holds for separable, irreducible polynomials over henselian valued fields of arbitrary rank (cf. [2, Theorems 2.4.7, 4.1.7], [4, Chapter 5 (G)]). Examples are known which show that the above result fails to hold when K is not complete. In 2004, Zaharescu [5] proved a similar result for polynomials with coefficients in valued fields that are not necessarily complete but where the lack of completeness is compensated by the presence of a secondary valuation which satisfies a certain property in connection with the given primary valuation. Precisely stated, he proved the following theorem.

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Theorem A. *Let K be a field of characteristic zero equipped with two non-archimedean valuations v_1 and v_2 having value groups Γ_1 and Γ_2 respectively. Let A be a subring of K with field of fractions K which is integrally closed in K and $\widetilde{A}$ be the integral closure of A in the algebraic closure $\widetilde{K}$ of K. Let $\tilde{v}_1$ and $\tilde{v}_2$ be valuations on $\widetilde{K}$ whose restrictions to K coincide with v_1 and v_2 respectively. Assume that for any $\beta \in \widetilde{A} \setminus A$ and $\lambda_2 \in \Gamma_2$, there exists an element $\lambda_1 \in \Gamma_1$ such that*

$$\tilde{v}_1(u - \beta) \leqslant \lambda_1 \text{ for all } u \in A \text{ with } v_2(u) \geqslant \lambda_2. \tag{1}$$

Let $f(x) = x^d + a_1x^{d-1} + \ldots + a_d \in A[x]$ be an irreducible polynomial over K. Then given $\lambda_2 \in \Gamma_2$, there exists $\lambda_1 \in \Gamma_1$ such that for any $b_1, b_2, \ldots, b_d \in A$ for which

$$v_1(b_i - a_i) \geqslant \lambda_1, \;\; 1 \leqslant i \leqslant d, \tag{2}$$

and

$$v_2(b_i) \geqslant \lambda_2, \;\; 1 \leqslant i \leqslant d, \tag{3}$$

the polynomial $g(x) = x^d + b_1x^{d-1} + \ldots + b_d$ is irreducible over K.

In this paper, the above result has been extended to irreducible polynomials with coefficients in arbitrary valued fields without any condition on the characteristic of K. We prove the following theorem.

Theorem 1.1. *Let K be a field equipped with two non-archimedean valuations v_1 and v_2 of arbitrary rank. Let $A, \widetilde{K}, \widetilde{A}, \tilde{v}_1$ and $\tilde{v}_2$ be as in Theorem A. Assume that for any $\beta \in \widetilde{A} \setminus A$ and $\lambda_2 \in \Gamma_2$, there exists an element $\lambda_1 \in \Gamma_1$ such that (1) holds. Then for any polynomial $f(x) = x^d + a_1x^{d-1} + \ldots + a_d \in A[x]$ which is irreducible over K and any $\lambda_2 \in \Gamma_2$, there corresponds $\lambda_1 \in \Gamma_1$ depending upon f and λ_2 such that for any $b_1, b_2, \ldots, b_d \in A$ satisfying (2) and (3), the polynomial $g(x) = x^d + b_1x^{d-1} + \ldots + b_d$ is irreducible over K.*

As an application of Theorem 1.1, we shall deduce the following result.

Theorem 1.2. *Let K_0 be a field complete with respect to a real valuation v_0. Let $f(x, y) = x^d + P_1(y)x^{d-1} + \ldots + P_d(y)$ be an irreducible polynomial in two variables over K_0. Let v_0^y denote the Gaussian extension of v_0 to $K_0(y)$ defined by $v_0^y(\sum_i a_iy^i) = \min_i\{v_0(a_i) | \; a_i \in K_0\}$. Then given any integer M, there exists $N > 0$ (depending upon f and M) such that whenever $Q_i(y),\; 1 \leqslant i \leqslant d$, are polynomials over K_0 satisfying (i) degree $Q_i(y) \leqslant M$, (ii) $v_0^y(Q_i(y) - P_i(y)) > N$, then $g(x, y) = x^d + Q_1(y)x^{d-1} + \ldots + Q_d(y)$ is irreducible over K_0.*

An example has been given in Section 3 to show that the above theorem is not true in general if the polynomials $Q_i(y)$ fail to satisfy condition (i) stated above even if each $Q_i(y)$ is sufficiently close to $P_i(y)$ with respect to v_0^y.

2. Preliminary Results

Let v be a valuation of arbitrary rank of a field K. v^x will denote the Gaussian valuation of the field $K(x)$ of rational functions in an indeterminate x which extends the valuation v of K. We need the following theorem of Ershov proved in [3]. For reader's convenience, it is proved here.

Theorem B. *Let v be a valuation of arbitrary rank of an algebraically closed field K with value group Γ. Let $\epsilon > 0$ be an element of Γ. Let $f(x), g(x)$ belonging to $K[x]$ be monic polynomials of degree d such that $v^x(f-g) > d!\epsilon-(d+1)!v^x(f)$. If $f(x) = \prod_{i=1}^{d}(x-\alpha_i)$ is a factorization of $f(x)$ over K, then we have a factorization $\prod_{i=1}^{d}(x-\beta_i)$ of $g(x)$ such that $v(\alpha_i - \beta_i) > \epsilon$ for $1 \leqslant i \leqslant d$.*

We first prove the following lemma needed for the proof of Theorem B.

Lemma C. *Let K, v and Γ be as in Theorem B and $f(x), g(x) \in K[x]$ be monic polynomials of degree d such that $v^x(f-g) > d\epsilon - 2dv^x(f)$ for some positive ϵ in Γ. Then for each root α of $f(x)$, there corresponds a root β of $g(x)$ such that $v(\alpha-\beta) > \epsilon - v^x(f),\ v^x(f_0 - g_0) > \epsilon$, where $f_0 = f/(x-\alpha),\ g_0 = g/(x-\beta)$.*

Proof : Write $f(x) = x^d + a_1x^{d-1} + \ldots + a_d,\ g(x) = x^d + b_1x^{d-1} + \ldots + b_d$ and denote $v^x(f)$ by $-v(c),\ c \in K$. As $f(x)$ is monic, $v(c) \geqslant 0$. Observe that for any root α of $f(x)$, $c\alpha$ is a root of the polynomial $x^d + a_1cx^{d-1} + \ldots + a_dc^d$ having coefficients in the valuation ring of v and hence $v(c\alpha) \geqslant 0$, which shows that

$$v(\alpha) \geqslant -v(c) = v^x(f). \tag{4}$$

Let α be a root of $f(x)$. Then it follows from the triangle law that

$$v(g(\alpha)) = v(g(\alpha) - f(\alpha)) \geqslant \min_{1\leqslant i\leqslant d}\{v(b_i - a_i) + (d-i)v(\alpha)\}. \tag{5}$$

In view of the hypothesis $v^x(f-g) > d\epsilon - 2dv^x(f)$ and (4), we see that

$$\begin{aligned} v(b_i - a_i) + (d-i)v(\alpha) &> d\epsilon - 2dv^x(f) + (d-i)v(\alpha) \\ &\geqslant d\epsilon - 2dv^x(f) + (d-i)v^x(f) \geqslant d\epsilon - dv^x(f). \end{aligned}$$

It is clear from the above inequality and (5) that $v(g(\alpha)) > d\epsilon - dv^x(f)$, which immediately shows that for at least one root β of $g(x)$

$$v(\alpha-\beta) > \epsilon - v^x(f). \tag{6}$$

To estimate $v^x(f_0 - g_0)$, write $f_0 - g_0 = \frac{f}{x-\alpha} - \frac{f}{x-\beta} + \frac{f}{x-\beta} - \frac{g}{x-\beta}$. Clearly

$$v^x\left(\frac{f}{x-\alpha} - \frac{f}{x-\beta}\right) = v^x(f) + v(\alpha-\beta) - v^x((x-\alpha)(x-\beta)) \geqslant v^x(f) + v(\alpha-\beta),$$

which in view of (6) gives

$$v^x\left(\frac{f}{x-\alpha}-\frac{f}{x-\beta}\right)>\epsilon. \tag{7}$$

Further by virtue of the hypothesis, we have

$$v^x\left(\frac{f}{x-\beta}-\frac{g}{x-\beta}\right)=v^x(f-g)-v^x(x-\beta)\geqslant v^x(f-g)>d\epsilon-2dv^x(f)\geqslant\epsilon. \tag{8}$$

It is immediate from (7) and (8) that $v^x(f_0-g_0)>\epsilon$ as desired. □

Proof of Theorem B : The theorem will be proved by induction on the degree d of $f(x)$. When $d=1$, $f=x-\alpha_1,\ g=x-\beta_1$ and $v(\beta_1-\alpha_1)=v^x(f-g)>\epsilon-2v^x(f)\geqslant\epsilon$. Consider now the case when $d=2$. Then by the hypothesis,

$$v^x(f-g)>2!\epsilon-3!v^x(f)=2(\epsilon-v^x(f))-4v^x(f).$$

Applying Lemma C (with ϵ replaced by $\epsilon-v^x(f)$), we see that there exists a root β_1 of $g(x)$ satisfying

$$v(\alpha_1-\beta_1)>(\epsilon-v^x(f))-v^x(f)=\epsilon-2v^x(f)\geqslant\epsilon$$

and $v^x(f_0-g_0)>\epsilon-v^x(f)\geqslant\epsilon$ where $f_0=f/(x-\alpha_1),\ g_0=g/(x-\beta_1)$. On writing $f_0=x-\alpha_2,\ g_0=x-\beta_2$, the last inequality shows that $v(\alpha_2-\beta_2)=v^x(f_0-g_0)>\epsilon$ proving the theorem in the case $d=2$.

Assume now that $f(x),g(x)$ have degree $d\geqslant 3$. Then $d!\geqslant d(d-1)\geqslant 2d$. In view of the hypothesis, $v^x(f-g)>d!\epsilon-(d+1)!v^x(f)=d[(d-1)!\epsilon-d!v^x(f)]-d!v^x(f)\geqslant d[(d-1)!\epsilon-d!v^x(f)]-2dv^x(f)$. By Lemma C (applied with ϵ replaced by $(d-1)!\epsilon-d!v^x(f)$) given a root α_1 of $f(x)$, there exists a root β_1 of $g(x)$ such that

$$v(\alpha_1-\beta_1)>(d-1)!\epsilon-d!v^x(f)-v^x(f)>\epsilon$$

and

$$v^x(f_0-g_0)>(d-1)!\epsilon-d!v^x(f)\geqslant(d-1)!\epsilon-d!v^x(f_0),$$

where $f_0=f/(x-\alpha_1),\ g_0=g/(x-\beta_1)$. The theorem now follows by the induction hypothesis applied to $f_0,\ g_0$.

Notations. Let (K,v) be as in Theorem B. The constant $d!\epsilon-(d+1)!v^x(f)$ occurring in this theorem will be denoted by ϵ_f and will be referred to as Ershov's constant with respect to v associated to a polynomial $f(x)\in K[x]$ corresponding to ϵ belonging to the value group of v. For any polynomial $f(x)\in K[x]$, we shall denote by ω_f the constant defined by

$$\omega_f=\max\{v(\alpha),v(\alpha-\alpha')\mid \alpha\neq\alpha',\ \alpha,\ \alpha' \text{ run over the roots of } f(x)\},$$

which will be referred to as the Generalized Krasner's constant associated to f with respect to v. Note that in case $f(x)$ has a single root α, then $\omega_f=v(\alpha)$.

3. Proof of Theorems 1.1 and 1.2.

Let $p^t \geqslant 1$ denote the multiplicity of each root of $f(x)$, p being the characteristic of K or 1. Let $\alpha_1, \alpha_2, \dots, \alpha_d$ be the roots of $f(x)$ in $\widetilde{K}$ not necessarily distinct. For any non-empty proper subset S of {1,2,…,d} having r elements, we shall denote the elementary symmetric sums by

$$\sigma_{S,1} = \sum_{j\in S} \alpha_j \ , \ \ \sigma_{S,2} = \sum_{\substack{i,j\in S\\ i<j}} \alpha_i\alpha_j \ , \dots , \ \sigma_{S,r} = \prod_{j\in S} \alpha_j.$$

Also $f_S(x)$ will stand for the polynomial $\prod_{j\in S}(x-\alpha_j)$, i.e.

$$f_S(x) = x^r - \sigma_{S,1}\, x^{r-1} + ... + (-1)^r \sigma_{S,r}.$$

Since $f(x)$ is irreducible over K and A is integrally closed, at least one of the coefficients of $f_S(x)$ belongs to $\widetilde{A} \setminus A$. For any such S, we shall denote by j_S the smallest positive integer for which σ_{S,j_S} belongs to $\widetilde{A} \setminus A$. Set

$$\delta = \min\{\tilde{v}_1(\alpha_i) \mid 1 \leqslant i \leqslant d\} \ \text{ and } \ \Delta = \max\{\tilde{v}_1(\alpha_i) \mid 1 \leqslant i \leqslant d\}.$$

Define δ'= min $\{0, (d-1)\delta\}$ and $\mu_f = \frac{\delta'}{p^t} + (\frac{p^t-d}{p^t})\omega_f$, where ω_f is the Generalized Krasner's constant associated to f with respect to $\tilde{v}_1$. We shall denote by ϵ_f the Ershov's constant with respect to $\tilde{v}_1$ associated to $f(x)$ corresponding to a fixed positive element $\epsilon \geqslant \omega_f$ belonging to the value group $\widetilde{\Gamma}_1$ of $\tilde{v}_1$. We divide the proof into two steps.

Step I. In this step, we show that with $\lambda_1 > \epsilon_f$, if $g(x) = x^d + b_1x^{d-1} + \dots + b_d$ belonging to $A[x]$ is any polynomial satisfying (2), then there exists a factorization $\prod_{i=1}^{d}(x-\theta_i)$ of $g(x)$ over $\widetilde{K}$ such that

$$\tilde{v}_1(\theta_i - \alpha_i) \geqslant \frac{\lambda_1}{p^t} + \mu_f \text{ for } 1 \leqslant i \leqslant d. \tag{9}$$

Since $\lambda_1 > \epsilon_f$, by Theorem B the roots $\theta_1, \theta_2, \dots, \theta_d$ of $g(x)$ can be arranged so that

$$\tilde{v}_1(\theta_i - \alpha_i) > \omega_f \text{ for } 1 \leqslant i \leqslant d. \tag{10}$$

We are going to prove that (10) implies (9). Fix one i and denote $\theta_i,\ \alpha_i$ by θ, α respectively. Since $\omega_f \geqslant \tilde{v}_1(\alpha)$, it is immediate from (10) and the strong triangle law that $\tilde{v}_1(\theta) = \tilde{v}_1(\alpha) \geqslant \delta$. To prove (9), consider first the case when $f(x)$ has at least two distinct roots. For any root $\alpha' \neq \alpha$ of $f(x)$, in view of (10) and the strong triangle law, it follows that

$$\tilde{v}_1(\theta - \alpha') = \min\{\tilde{v}_1(\theta-\alpha), \tilde{v}_1(\alpha-\alpha')\} = \tilde{v}_1(\alpha - \alpha'). \tag{11}$$

Keeping in mind that $v_1(a_j - b_j) \geqslant \lambda_1$ and $\tilde{v}_1(\theta) = \tilde{v}_1(\alpha) \geqslant \delta$, we obtain

$$\begin{aligned}\tilde{v}_1(f(\theta)) = \tilde{v}_1(f(\theta) - g(\theta)) &\geqslant \min_{1\leqslant j\leqslant d}\{\tilde{v}_1(a_j - b_j) + (d-j)\tilde{v}_1(\theta)\}\\ &\geqslant \lambda_1 + \min_j\{(d-j)\delta\} = \lambda_1 + \delta'.\end{aligned}$$

Therefore on substituting $f(\theta) = \prod_{\alpha'}(\theta - \alpha')^{p^t}$, where α' runs over distinct roots of $f(x)$, the above inequality shows that

$$\tilde{v}_1(f(\theta)) = p^t\tilde{v}_1(\theta - \alpha) + p^t \sum_{\alpha' \neq \alpha} \tilde{v}_1(\theta - \alpha') \geqslant \lambda_1 + \delta'.$$

Using (11), the above inequality can be rewritten as

$$\tilde{v}_1(\theta - \alpha) + \sum_{\alpha' \neq \alpha} \tilde{v}_1(\alpha - \alpha') \geqslant (\lambda_1 + \delta')/p^t,$$

which in view of $\tilde{v}_1(\alpha - \alpha') \leqslant \omega_f$ implies that

$$\tilde{v}_1(\theta - \alpha) \geqslant \frac{\lambda_1 + \delta'}{p^t} + \left(\frac{p^t - d}{p^t}\right)\omega_f = \frac{\lambda_1}{p^t} + \mu_f.$$

This proves (9) when $f(x)$ has at least two distinct roots.

Consider now the case when $f(x)$ has a single root α repeated p^t times. In this situation $\omega_f = \tilde{v}_1(\alpha)$. For any root θ of $g(x)$, arguing as in the first case, we see that

$$\tilde{v}_1(f(\theta) - g(\theta)) \geqslant \min_j\{\tilde{v}_1(a_j - b_j) + (d - j)\tilde{v}_1(\theta)\} \geqslant \lambda_1 + \delta'.$$

As $f(\theta) = (\theta - \alpha)^{p^t}$, the above inequality gives $\tilde{v}_1(\theta - \alpha) \geqslant \frac{\lambda_1+\delta'}{p^t} = \frac{\lambda_1}{p^t} + \mu_f$ as desired. This completes the proof of Step I.

Step II. In this step, we prove the irreducibility of $g(x)$ giving the final choice of λ_1. Fix an element λ_2 in Γ_2. Define λ_2'= min$\{\lambda_2/d, \lambda_2\}$. Let $g(x) = x^d + b_1x^{d-1} + \ldots + b_d$ belonging to $A[x]$ be any monic polynomial satisfying (2) and (3) with $\lambda_1 > \epsilon_f$. We first show that for each root θ of $g(x)$,

$$\tilde{v}_2(\theta) \geqslant \lambda_2'. \tag{12}$$

On writing $\theta^d = -(b_1\theta^{d-1} + \ldots + b_d)$ and using the triangle law, we have

$$\begin{aligned} d\tilde{v}_2(\theta) \geqslant \min_{1\leqslant j\leqslant d}\{v_2(b_j) + (d-j)\tilde{v}_2(\theta)\} \quad &\geqslant \quad \lambda_2 + \min_{1\leqslant j\leqslant d}\{(d-j)\tilde{v}_2(\theta)\} \\ &= \quad \lambda_2 + (d-i)\tilde{v}_2(\theta)(\text{say}) \end{aligned}$$

which implies $\tilde{v}_2(\theta) \geqslant \lambda_2/i \geqslant \lambda_2'$ proving (12).

Recall that for any non-empty proper subset S of {1,2,...,d}, the coefficient σ_{S,j_S} of $f_S(x) = \prod_{j\in S}(x - \alpha_j)$ belongs to $\widetilde{A} \setminus A$. On applying (1) with β replaced by σ_{S,j_S} and λ_2 by $j_S\lambda_2'$, there exists an element $\lambda_{1,S}$ belonging to Γ_1 such that

$$\tilde{v}_1(u - \sigma_{S,j_S}) \leqslant \lambda_{1,S} \tag{13}$$

for all $u \in A$ with $v_2(u) \geqslant j_S\lambda_2'$.

Suppose that $g(x)$ is reducible over K. It will be shown that this will give an upper bound (depending upon λ_2 and $f(x)$) on λ_1. Write $g(x) = G(x)H(x)$, with $G(x)$, $H(x)$ monic, non-constant polynomials belonging to $K[x] \cap \widetilde{A}[x] = A[x]$. Denote $G(x)$ by $x^m + c_1x^{m-1} + \ldots + c_m$. It is immediate from (12) that

$$v_2(c_i) \geqslant i\lambda_2', \ \ 1 \leqslant i \leqslant m. \tag{14}$$

Recall that by Step I, $\theta_1, \theta_2, \ldots, \theta_d$ is an arrangement of roots of $g(x)$ satisfying (10). Write $G(x) = \prod\limits_{j\in S}(x - \theta_j)$ where S is a proper subset of $\{1, 2, \ldots, d\}$.

Consider $f_S(x) = \prod\limits_{j\in S}(x - \alpha_j)$. One of the coefficients of $f_S(x)$ is

$$(-1)^{j_S}\sigma_{S,j_S} = (-1)^{j_S} \sum_{\substack{i_1,\ldots,i_{j_S}\in S\\ i_1<\ldots<i_{j_S}}} \alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_{j_S}},$$

and the corresponding coefficient in $G(x)$, say u_0 is given by:

$$(-1)^{j_S}u_0 = (-1)^{j_S} \sum_{\substack{i_1,\ldots,i_{j_S}\in S\\ i_1<\ldots<i_{j_S}}} \theta_{i_1}\theta_{i_2}\ldots\theta_{i_{j_S}}.$$

We are going to prove that

$$\tilde{v_1}(u_0 - \sigma_{S,j_S}) > \tfrac{\lambda_1}{p^t} + \mu_f - \Delta - \rho_S,$$

where $\Delta = \max\limits_{1\leqslant i\leqslant d}\{\tilde{v}_1(\alpha_i)\}$ and ρ_S is an element of $\widetilde{\Gamma}_1$ depending upon f and S, to be specified later. For any subset $\{i_1, i_2, \ldots, i_k\}$ of $\{1, 2, \ldots, d\}$, we can write

$$\begin{aligned}\theta_{i_1}\theta_{i_2}\ldots\theta_{i_k} - \alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_k} = \theta_{i_1}\ldots\theta_{i_{k-1}}(\theta_{i_k} - \alpha_{i_k}) + \theta_{i_1}\ldots\theta_{i_{k-2}}\\ (\theta_{i_{k-1}} - \alpha_{i_{k-1}})\alpha_{i_k} + \ldots + (\theta_{i_1} - \alpha_{i_1})\alpha_{i_2}\ldots\alpha_{i_k}\end{aligned} \tag{15}$$

where θ_{ik}'s and α_{ik}'s may not all be distinct. Recall that $\tilde{v}_1(\theta_i) = \tilde{v}_1(\alpha_i)$. Using (9), it follows that

$$\begin{aligned}\tilde{v}_1(\theta_{i_1}\ldots\theta_{i_{k-1}}(\theta_{i_k} - \alpha_{i_k})) &= \tilde{v}_1(\alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_k}) - \tilde{v}_1(\alpha_{i_k}) + \tilde{v}_1(\theta_{i_k} - \alpha_{i_k})\\ &\geqslant \tilde{v}_1(\alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_k}) - \Delta + \frac{\lambda_1}{p^t} + \mu_f.\end{aligned}$$

Arguing similarly for other summands on the right hand side of (15), we see that

$$\tilde{v}_1(\theta_{i_1}\theta_{i_2}\ldots\theta_{i_k} - \alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_k}) \geqslant \tilde{v}_1(\alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_k}) - \Delta + \frac{\lambda_1}{p^t} + \mu_f.$$

It immediately follows from the above inequality and the triangle law that

$$\tilde{v}_1(u_0 - \sigma_{S,j_S}) \geqslant \frac{\lambda_1}{p^t} + \mu_f - \Delta + \min_{\substack{i_1,\ldots,i_{j_S}\in S\\ i_1<\ldots<i_{j_S}}}\{\tilde{v}_1(\alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_{j_S}})\},$$

i.e.,

$$\tilde{v}_1(u_0 - \sigma_{S,j_S}) \geqslant \frac{\lambda_1}{p^t} + \mu_f - \Delta + \rho_S. \tag{16}$$

where $\rho_S = \min_{\substack{i_1,\ldots,i_{j_S}\in S \\ i_1<\ldots<i_{j_S}}} \{\tilde{v}_1(\alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_{j_S}})\}$ is in $\widetilde{\Gamma}_1$. Recall that by virtue of (14), the coefficient u_0 of x^{m-j_S} in $G(x)$ satisfies $v_2(u_0) \geqslant j_S\lambda_2'$. Therefore it follows from (13) that

$$\tilde{v}_1(u_0 - \sigma_{S,j_S}) \leqslant \lambda_{1,S}. \tag{17}$$

The inequalities (16) and (17) imply that $\lambda_1 \leqslant p^t(\lambda_{1,S} + \Delta - \mu_f - \rho_S)$. Thus it follows that if we start with an element $\lambda_1 > \epsilon_f$ and $\lambda_1 > p^t(\Delta - \mu_f) + p^t \max_S\{\lambda_{1,S} - \rho_S\}$ where S runs over all non-empty proper subsets of $\{1, 2, \ldots, d\}$, then each polynomial $g(x)$ satisfying (2) and (3) must be irreducible over K. □

Proof of Theorem 1.2 : Denote $K_0(y)$ by K and the Gaussian valuation v_0^y of K by v_1. Let v_2 stand for the valuation of K defined for any polynomial $h(y)$ by $v_2(h(y)) = -deg\ h(y)$. Let $\tilde{v}_1$ and $\tilde{v}_2$ be any prolongations of v_1, v_2 to the algebraic closure $\widetilde{K}$ of K. Set $A = K_0[y]$ and denote by $\tilde{A}$ the integral closure of A in $\widetilde{K}$.

In view of Theorem 1.1, the desired result is proved once we verify that for any $\beta \in \tilde{A} \setminus A$ and any integer λ_2, there exists an integer λ_1 satisfying (1). Suppose to the contrary (1) is not satisfied for some $\beta \in \tilde{A} \setminus A$ and some integer λ_2. Then there exists a sequence $\{u_n\}$ in A such that

$$\tilde{v}_1(u_n - \beta) \geq n,\ v_2(u_n) \geq \lambda_2 \ \text{ for } n \geq 1. \tag{18}$$

The first inequality above shows that $\{u_n\}$ will be a Cauchy sequence in K with respect to v_1 and the second says that the sequence $\{\deg\ u_n\}$ is bounded, say by D. Write $u_n = \sum_{i=0}^{D} c_{ni}y^i$. Note that $\{c_{ni}\}_n$ is a Cauchy sequence with respect to v_0 and hence converges to an element c_i of the complete field K_0. Therefore $\{u_n\}$ converges to an element $\sum_{i=0}^{D} c_i y^i$ of A with respect to v_1. But the first inequality of (18) implies that $\{u_n\}$ converges to β which does not belong to A. This contradiction shows that the hypothesis of Theorem 1.1 is satisfied. □

The following example shows that condition (i) of Theorem 1.2 cannot be dispensed with.

Example 3.1. Let K_0 be a field of characteristic zero which is complete with respect to a non-trivial real valuation v_0. Fix an element α in K_0 satisfying $v_0(\alpha) > 0$ if the characteristic of the residue field of v_0 is zero and $v_0(\alpha) > 2v_0(p)$ if $p > 0$ is the characteristic of the residue field of v_0. Set $f(x,y) = x^2 - (1 + \alpha y)$ and for any $m \geqslant 1$, define $g_m(x,y) = x^2 - (A_m(y))^2$ where $A_m(y) = 1 +$

$\sum_{k=1}^{m} \binom{1/2}{k} \alpha^k y^k \in K_0[y]$ and $\binom{1/2}{k} = \frac{(\frac{1}{2})(\frac{1}{2}-1)...(\frac{1}{2}-k+1)}{k!}$. Note that $A_m(y)$ are the partial sums of Taylor series expansion of $\sqrt{1+\alpha y}$. Hence $(A_m(y))^2 - (1+\alpha y)$ as a polynomial in y has only terms of degree $> m$. For $k > m$, the coefficient c_k of y^k in $(A_m(y))^2 - (1+\alpha y)$ is $\sum_{\substack{i+j=k \\ i,j \geqslant 0}} \binom{1/2}{i} \binom{1/2}{j} \alpha^k$. If the characteristic of the residue field of v_0 is zero, then clearly $v_0(c_k) \geqslant kv_0(\alpha) \to \infty$; when this characteristic is $p > 0$, then keeping in mind the fact that $v_0(i!) \leqslant \frac{i}{p-1} v_0(p)$, it can be easily seen that $v_0(c_k) \geqslant kv_0(\alpha) - \frac{k}{p-1} v_0(p) - v_0(2^k) \geqslant k[v_0(\alpha) - 2v_0(p)]$ which tends to infinity as k approaches infinity in view of the choice of α. Hence $(A_m(y))^2$ converges to $(1+\alpha y)$ with respect to v_0^y. Therefore the coefficients of $f(x,y)$ and $g_m(x,y)$ are sufficiently close for large m, but each $g_m(x,y)$ is reducible over K_0 while $f(x,y)$ is irreducible.

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