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FINITE ELEMENT METHODS FOR SEMILINEAR ELLIPTIC PROBLEMS WITH SMOOTH INTERFACES

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The purpose of this paper is to study the finite element method for second order semilinear elliptic interface problems in two dimensional convex polygonal domains. Due to low global regularity of the solution, it seems difficult to achieve optimal order of convergence with straight interface triangles [*Numer. Math*., 79 (1998), pp. 175–202]. For a finite element discretization based on a mesh which involve the approximation of the interface, optimal order error estimates in L^2 and H^1-norms are proved for linear elliptic interface problem under practical regularity assumptions of the true solution. Then an extension to the semilinear problem is also considered and optimal error estimate in H^1 norm is achieved.

Key words : Elliptic, interface, semilinear, finite element method, optimal error estimate.

1. INTRODUCTION

Let Ω be a convex polygonal domain in $\mathbb{R}^2$ with boundary $\partial\Omega$. Let Γ be the C^2 smooth boundary of the open domain $\Omega_1 \subset \Omega$ and $\Omega_2 = \Omega \setminus \Omega_1$. We consider the following semilinear elliptic interface problem

$$-\nabla \cdot (\beta(x)\nabla u(x)) + u(x) = f(u) \quad \text{in } \Omega \tag{1.1}$$

with Dirichlet boundary condition

$$u(x) = 0 \quad \text{on } \partial\Omega \tag{1.2}$$

and interface conditions

$$[u] = 0, \quad \left[\beta \frac{\partial u}{\partial \mathbf{n}}\right] = 0 \quad \text{along } \Gamma, \tag{1.3}$$

where $[v]$ is the jump of a quantity v across the interface Γ and we define the jump by $[v] = v_1(x) - v_2(x)$, where v_i is the restriction of v on Ω_i. Here, $\mathbf{n}$ is the outward unit normal to $\partial\Omega_1$. For the simplicity of exposition, we assume that the coefficient function β is positive and piecewise constant, i.e.,

$$\beta(x) = \beta_1 \quad \text{in } \Omega_1; \qquad \beta(\mathrm{x}) = \beta_2 \quad \text{in } \Omega_2.$$

Semilinear interface problems with discontinuous coefficients are often encountered in the theory of magnetic field, heat conduction theory, the theory of elasticity and in reaction diffusion problems with localized autocatalytic chemical reactions (see, [9, 12, 17]). Because of the discontinuity of the coefficients along the interface, the solution of such problem has low regularity on the whole physical domain. Due to the low global regularity and the irregular shape of the interface, achieving higher order accuracy seems difficult with the classical finite element method, see [2] and [5]. For a more detailed description on the finite element analysis for interface problems, one may refer to [7]. Earlier, the finite element approximation to nonlinear elliptic problems with discontinuous coefficients are described in [9] and [17]. The results in [9] and [17] contained sub-optimal order of convergence of the finite element solution to the exact solution. The estimates in [9] are based on the use of Green's theorem and the pseudomonotone problem studied in [10]. The convergence results in [17] are proved under certain stringent assumptions which guarantee that the corresponding operator is strongly monotone and Lipschitz continuous. Recently, optimal error estimate in H^1 norm has been established in [15]. The algorithm in [15] require that the grid line follow the actual interface by assuming interface triangles to be curved triangles instead of straight triangles as in classical finite element methods. As it may be computationally inconvenient to fit the mesh exactly to the interface, a modification on finite element discretization based on a mesh which involve the approximation of the interface is proposed and analyzed in this work for both linear and semilinear elliptic problems.

Under certain hypotheses, the error of approximation of solutions of certain nonlinear problems is basically the same as the error of approximation of solutions of related linear problems [3, 11]. Therefore an essential improvement of the results of [5] for the linear elliptic interface problems have been obtained in this work and then semilinear problems are studied into the Brezzi-Rappaz-Raviart ([3]) framework. Optimal order error estimates in the H^1-norm (see, Theorem 4.2) is proved for the semilinear elliptic interface problem for a finite element discretization in which the grid line need not follow the actual interface exactly. The achieved estimates are analogous to the case with a regular solution, however, due to low regularity, their proof requires a lot of careful technical work coupled with an approximating operator framework. Other technical tools used in this paper are Sobolev embedding inequality, approximations properties for linear interpolation operator, duality arguments and some known results on elliptic interface problems and some auxiliary projections.

We have used standard notations for Sobolev spaces and norms in this paper. For $m \geq 0$ and real p with $1 \leq p \leq \infty$, we use $W^{m,p}(\Omega)$ to denote Sobolev space of order m with norms

$$\|u\|_{W^{m,p}(\Omega)} = \left(\sum_{0 \leq |\alpha| \leq m} \|D^\alpha u\|^p_{L^p(\Omega)} \right)^{1/p}, \quad 1 \leq p < \infty,$$

$$\|u\|_{W^{m,\infty}(\Omega)} = \max_{0 \leq |\alpha| \leq m} \|D^\alpha u\|_{L^\infty(\Omega)}, \quad p = \infty.$$

In particular for $p = 2$, we write $W^{m,2}(\Omega) = H^m(\Omega)$. $H_0^m(\Omega)$ is a closed subspace of $H^m(\Omega)$, which is also closure of $C_0^\infty(\Omega)$ (the set of all C^∞ functions with compact support) with respect to the norm of $H^m(\Omega)$. For a fractional number s, the Sobolev space H^s is defined in [1].

In addition, we shall also work on the following spaces:

$$X = H^1(\Omega) \cap H^2(\Omega_1) \cap H^2(\Omega_2)$$

equipped with the norm

$$\|v\|_X = \|v\|_{H^1(\Omega)} + \|v\|_{H^2(\Omega_1)} + \|v\|_{H^2(\Omega_2)}.$$

The rest of the paper is organized as follows. Section 2 introduces the finite element discretization and some known results for elliptic interface problems. Section 3 is devoted to some new optimal *a priori* error estimates for linear elliptic interface problem. Convergence of finite element solution of semilinear elliptic interface problems are presented in section 4. Finally, a computational example for elliptic interface problem is presented in Section 5.

2. Finite Element Discretization and Preliminaries

For the purpose of finite element approximation of the problem (1.1)-(1.3), we use a finite element discretization based on [5]. We first approximate the domain Ω_1 by a domain Ω_1^h with the polygonal boundary Γ_h whose vertices all lie on the interface Γ. Let Ω_2^h be the approximation for the domain Ω_2 with polygonal exterior and interior boundaries as $\partial\Omega$ and Γ_h, respectively.

Triangulation $\mathcal{T}_h$ of the domain Ω satisfy the following conditions:

$(\mathcal{A}1)$ $\overline{\Omega} = \cup_{K\in\mathcal{T}_h} K$.

$(\mathcal{A}2)$ If K_1, $K_2 \in \mathcal{T}_h$ and $K_1 \neq K_2$, then either $K_1 \cap K_2 = \emptyset$ or $K_1 \cap K_2$ is a common vertex or edge of both triangles.

$(\mathcal{A}3)$ Each triangle $K \in \mathcal{T}_h$ is either in Ω_1^h or Ω_2^h and intersects Γ (interface) in at most two points.

$(\mathcal{A}4)$ For each triangle $K \in \mathcal{T}_h$, let h_K be the length of the largest side. Let $h = \max\{h_K : K \in \mathcal{T}_h\}$.

The triangles with one or two vertices on Γ are called the interface triangles, the set of all interface triangles is denoted by $\mathcal{T}_\Gamma^*$ and we write $\Omega_\Gamma^* = \cup_{K\in\mathcal{T}_\Gamma^*} K$. Let V_h be a family of finite dimensional subspaces of $H_0^1(\Omega)$ defined on $\mathcal{T}_h$ consisting of piecewise linear functions vanishing on the boundary $\partial\Omega$. Examples of such finite element spaces can be found in [4] and [6].

As a step towards finite element approximation, we now introduce the weak formulation of the problem (1.1)-(1.3). Define a bilinear form $A(\cdot,\cdot) : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ by

$$A(v, w) = \int_\Omega (\beta(x)\nabla v \cdot \nabla w + vw)\, dx.$$

Then it is immediate to derive the weak formulation of the interface problem as follows: Seek a function $u \in H_0^1(\Omega)$ such that

$$A(u,v) = (f(u),v) \quad \forall\, v \in H_0^1(\Omega), \tag{2.1}$$

where $(\cdot,\cdot)$ denotes the inner product of the $L^2(\Omega)$ space. We assume $f \in C^2(\mathbb{R},\mathbb{R})$ satisfying

$$|f'(s)| \le C|s| \text{ and } |f''(s)| \le C \quad \forall s \in \mathbb{R} \tag{2.2}$$

so that (2.1) admits a unique solution $u \in X$ (cf. [12, 17]).

For the purpose finite element approximation, we define the approximation $\beta_h(x)$ of the coefficient $\beta(x)$ as follows: For each triangle $K \in \mathcal{T}_h$, let $\beta_K(x) = \beta_i$ if $K \subset \Omega_i^h$, i=1 or 2 and we define β_h as

$$\beta_h(x) = \beta_K(x) \quad \forall K \in \mathcal{T}_h.$$

Then the finite element approximation to (2.1) is stated as follows: Find $u_h \in V_h$ such that

$$A_h(u_h,v_h) = (f(u_h),v_h) \quad \forall\, v_h \in V_h, \tag{2.3}$$

where $A_h(\cdot,\cdot) : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ is defined as

$$A_h(w,v) = \sum_{K\in\mathcal{T}_h} \int_K \{\beta_K(x)\nabla w \cdot \nabla v + wv\}dx \;\; \forall w,v \in H^1(\Omega).$$

Under the assumption (2.2) of f, the existence of a unique solution to (2.3) can be found in [17].

We now recall some existing results which will be frequently used in our analysis. Regarding the approximation of the bilinear form A_h, we have the following result. For a proof, we refer to [14].

Lemma 2.1 — For $w_h, v_h \in V_h$, we have

$$|A_h(w_h,v_h) - A(w_h,v_h)| \;\; \le \;\; Ch \sum_{K\in\mathcal{T}_\Gamma^*} \|\nabla v_h\|_{L^2(K)} \|\nabla w_h\|_{L^2(K)}.$$

The following interface approximation property plays a crucial role in our subsequent analysis. For a proof, we refer to [14] (cf. Lemma 3.3)

Lemma 2.2 — If Ω_Γ^* is the union of all interface triangles, then we have

$$\|u\|_{H^1(\Omega_\Gamma^*)} \leq Ch^{\frac{1}{2}}\|u\|_X.$$

Let $\Pi_h : C(\overline{\Omega}) \rightarrow V_h$ be the Lagrange interpolation operator corresponding to the space V_h. As the solutions concerned are only in $H^1(\Omega)$ globally, one cannot apply the standard interpolation theory directly. However, following the argument of [5] it is possible to obtain optimal error bounds for the interpolant Π_h (see Chapter 3, [7]). In [7], the authors have assumed that the solution $u \in X \cap W^{1,\infty}(\Omega_1 \cap \Omega_0) \cap W^{1,\infty}(\Omega_2 \cap \Omega_0)$, where Ω_0 is some neighborhood of the interface Γ. The following lemma shows that optimal approximation of Π_h can be derived for $u \in X$ with $[u] = 0$ along interface Γ.

Lemma 2.3 — Let $\Pi_h : X \rightarrow V_h$ be the linear interpolation operator and u be the solution for the interface problem (1.1)-(1.3), then the following approximation properties

$$\|u - \Pi_h u\|_{H^m(\Omega)} \leq Ch^{2-m}\|u\|_X,\ m = 0, 1,$$

hold true.

PROOF : For any $v \in X$, let v_i be the restriction of v on Ω_i for $i = 1, 2$. As the interface is of class C^2, we can extend the function $v_i \in H^2(\Omega_i)$ on to the whole Ω and obtain the function $\tilde{v}_i \in H^2(\Omega)$ such that $\tilde{v}_i = v_i$ on Ω_i and

$$\|\tilde{v}_i\|_{H^2(\Omega)} \leq C\|v_i\|_{H^2(\Omega_i)}, i = 1, 2. \tag{2.4}$$

For the existence of such extensions, we refer to Stein [16].

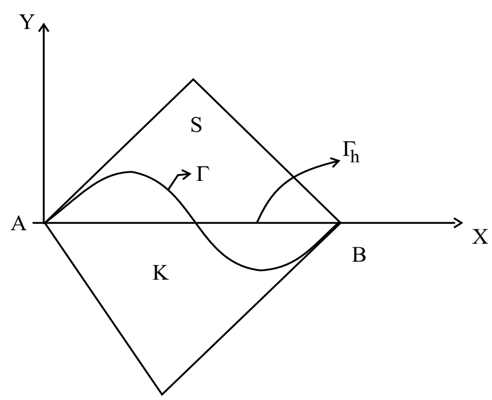

Figure 1: Interface Triangles K, S, along with interface Γ and its approximation Γ_h

Then, for $K \in \mathcal{T}_h$, we now define

$$\Pi_h u = \begin{cases} \Pi_h \tilde{u}_1 & \text{if } \mathrm{K} \subseteq \Omega_1^{\mathrm{h}} \\ \Pi_h \tilde{u}_2 & \text{if } \mathrm{K} \subseteq \Omega_2^{\mathrm{h}}. \end{cases}$$

We claim that $\Pi_h u \in V_h$ and this can be verified if we can prove that $\Pi_h u$ maintain the continuity along Γ_h. Let $\overline{AB}$ be a part of Γ_h as shown in the Figure 1 with $A(x_1, 0)$ and $B(x_2, 0)$. Along $\overline{AB}$,

$$\begin{aligned} \Pi_h u = \Pi_h \tilde{u}_1 = c_1 x + c_2 \ \text{ in } \mathrm{K} \subseteq \Omega_1^{\mathrm{h}} \\ \text{and} \quad \Pi_h u = \Pi_h \tilde{u}_2 = d_1 x + d_2 \ \text{ in } \mathrm{S} \subseteq \Omega_2^{\mathrm{h}}. \end{aligned}$$

At $A(x_1, 0)$ and $B(x_2, 0)$, we have

$$\begin{aligned} \tilde{u}_1(A) = c_1 x_1 + c_2 \ \text{ and } \ \tilde{\mathrm{u}}_1(\mathrm{B}) = \mathrm{c}_1 \mathrm{x}_2 + \mathrm{c}_2 \\ \text{and} \quad \tilde{u}_2(A) = d_1 x_1 + d_2 \ \text{ and } \ \tilde{\mathrm{u}}_2(\mathrm{B}) = \mathrm{d}_1 \mathrm{x}_2 + \mathrm{d}_2. \end{aligned}$$

Again $[u] = 0$ along Γ gives $\tilde{u}_1 = \tilde{u}_2 \ \text{ along } \Gamma$ and hence

$$\tilde{u}_1(A) = \tilde{u}_2(A), \ \ \tilde{u}_1(B) = \tilde{u}_2(B).$$

Thus, we have

$$(c_1 - d_1)x_1 + (c_2 - d_2) = 0.$$

Then comparing the coefficients, we have $c_1 = d_1$ and $c_2 = d_2$. Therefore $\Pi_h u$ maintain the continuity along Γ_h and hence $\Pi_h u \in V_h$.

Now, for any triangle $K \in \mathcal{T}_h \backslash \mathcal{T}_\Gamma^*$, the standard finite element interpolation theory (cf. [4, 6]) implies that

$$\|u - \Pi_h u\|_{H^m(K)} \leq C h^{2-m} \|u\|_{H^2(K)}, \ m = 0, 1. \tag{2.5}$$

For any element $K \in \mathcal{T}_\Gamma^*$, we write $K_i = K \cap \Omega_i, i = 1, 2$, for our convenience. Again it follows from the standard analysis that dist$(\Gamma, \Gamma_h) \leq \mathrm{O}(h^2)$. Thus, without loss of generality, we can assume that meas$(K_2) \leq Ch^3$. Further, using the Hölder's inequality and the fact meas$(K_2) \leq Ch^3$ we derive that for any $p > 2$, and $m = 0, 1$,

$$\begin{aligned}
\|u - \Pi_h u\|_{H^m(K_2)} &\leq Ch^{\frac{3(p-2)}{2p}} \|u - \Pi_h u\|_{W^{m,p}(K_2)} \\
&\leq Ch^{\frac{3(p-2)}{2p}} \|u - \Pi_h u\|_{W^{m,p}(K)} \\
&\leq Ch^{\frac{3(p-2)}{2p}+1-m} \|u\|_{W^{1,p}(K)},
\end{aligned} \tag{2.6}$$

in the last inequality, we used the standard interpolation theory (cf. [6]). On the other hand

$$\begin{aligned}
\|u - \Pi_h u\|_{H^m(K_1)} &= \|\tilde{u}_1 - \Pi_h \tilde{u}_1\|_{H^m(K_1)} \\
&\leq C\|\tilde{u}_1 - \Pi_h \tilde{u}_1\|_{H^m(K)} \\
&\leq Ch^{2-m}\|\tilde{u}_1\|_{H^2(K)} \\
&\leq Ch^{2-m}\|u\|_X,
\end{aligned} \tag{2.7}$$

in the last inequality, we used (2.4).

In view of (2.6)-(2.7), it now follows that

$$\begin{aligned}
&\|u - \Pi_h u\|^2_{H^m(\Omega^*_\Gamma)} \\
&\leq Ch^{4-2m}\|u\|^2_X + C \sum_{K \in \mathcal{T}^*_\Gamma} h^{\frac{3(p-2)}{p}+2-2m} \|u\|^2_{W^{1,p}(K)} \\
&\leq Ch^{4-2m}\|u\|^2_X + C \sum_{K \in \mathcal{T}^*_\Gamma} h^{5-2m-\frac{6}{p}} \|u\|^2_{W^{1,p}(K)} \\
&\leq Ch^{4-2m}\|u\|^2_X + C \sum_{K \in \mathcal{T}^*_\Gamma} h^{5-2m-\frac{6}{p}} \{\|u\|^2_{W^{1,p}(K_1)} + \|u\|^2_{W^{1,p}(K_2)}\} \\
&\leq Ch^{4-2m}\|u\|^2_X + C \sum_{K \in \mathcal{T}^*_\Gamma} h^{5-2m-\frac{6}{p}} \\
&\quad \{\|\tilde{u}_1\|^2_{W^{1,p}(K_1)} + \|\tilde{u}_2\|^2_{W^{1,p}(K_2)}\}.
\end{aligned} \tag{2.8}$$

We now recall Sobolev embedding inequality for two dimensions (cf. Ren and Wei [13])

$$\|v\|_{L^p(\Omega)} \leq Cp^{\frac{1}{2}} \|v\|_{H^1(\Omega)} \ \ \forall v \in H^1(\Omega), \ \ p > 2. \tag{2.9}$$

Now, setting $p = 6$ in the Sobolev embedding inequality (2.9), we obtain

$$\begin{aligned} \|\tilde{u}_i\|_{L^6(K_i)} &\le \|\tilde{u}_i\|_{L^6(\Omega_i)} \le C\|\tilde{u}_i\|_{H^1(\Omega_i)}, \\ \|\nabla\tilde{u}_i\|_{L^6(K_i)} &\le \|\nabla\tilde{u}_i\|_{L^6(\Omega_i)} \le C\|\nabla\tilde{u}_i\|_{H^1(\Omega_i)}. \end{aligned}$$

In view of the above estimates, it now follows that

$$\|\tilde{u}_i\|_{W^{1,6}(K_i)} \le C\|\tilde{u}_i\|_{H^2(\Omega_i)}.$$

This together with (2.8), we have

$$\|u - \Pi_h u\|^2_{H^m(\Omega^*_\Gamma)} \le Ch^{4-2m}\|u\|^2_X, \quad m = 0, 1. \tag{2.10}$$

Then Lemma 2.3 follows immediately from the estimates (2.5) and (2.10). □

3. Linear Elliptic Interface Problem

In this section, we will establish some new optimal error estimates for linear elliptic interface problem which will be useful in the error analysis of semilinear problems.

Consider a linear elliptic interface problem of the form

$$-\nabla.(\beta(x)\nabla u) + u = f(x) \quad \text{in } \Omega \tag{3.1}$$

subject to the boundary condition

$$u(x) = 0 \quad \text{on } \partial\Omega \tag{3.2}$$

and interface conditions

$$[u] = 0, \qquad \left[\beta\frac{\partial u}{\partial \mathbf{n}}\right] = 0 \quad \text{along } \Gamma. \tag{3.3}$$

The solution $u \in X \cap H^1_0(\Omega)$ satisfies the following *a priori* estimate (cf. [5])

$$\|u\|_X \le C\|f\|_{L^2(\Omega)}. \tag{3.4}$$

As a first step towards the finite element approximation, we now introduce the weak formulation of the problem (3.1)-(3.3) as follows: Seek a function $u \in H^1_0(\Omega)$ such that

$$A(u, v) = (f, v) \quad \forall v \in H^1_0(\Omega) \tag{3.5}$$

For $f \in L^2(\Omega)$, the finite element approximation to (3.5) is stated as follows: Find $u_h \in V_h$ such that

$$A_h(u_h, v_h) = (f, v_h) \quad \forall v_h \in V_h. \tag{3.6}$$

From (3.5) and (3.6), we note that

$$\begin{aligned} A(u_h - \Pi_h u, v_h) &= A(u - \Pi_h u, v_h) \\ &\quad + \{A(u_h, v_h) - A_h(u_h, v_h)\} \\ &\equiv: (I)_1 + (I)_2 \end{aligned} \tag{3.7}$$

By Lemma 2.3, we can bound the term $(I)_1$ by

$$\begin{aligned} |(I)_1| &\leq C\|u - \Pi_h u\|_{H^1(\Omega)}\|\nabla v_h\|_{L^2(\Omega)} \\ &\leq Ch\|u\|_X\|v_h\|_{H^1(\Omega)} \end{aligned} \tag{3.8}$$

For the term $(I)_2$, use Lemma 2.1 to have

$$\begin{aligned} |(I)_2| &\leq Ch\|\nabla u_h\|_{L^2(\Omega)}\|\nabla v_h\|_{L^2(\Omega)} \\ &\leq Ch\|\nabla u_h\|_{L^2(\Omega)}\|v_h\|_{H^1(\Omega)} \\ &\leq Ch\|f\|_{L^2(\Omega)}\|v_h\|_{H^1(\Omega)} \end{aligned} \tag{3.9}$$

where we have used the inequality

$$\|\nabla u_h\|_{L^2(\Omega)} \leq C\|f\|_{L^2(\Omega)}$$

which follows directly from (3.6) by taking $v_h = u_h$ and using coercivity.

From the estimates (3.8)-(3.9), we conclude by taking $v_h = u_h - \Pi_h u$ in (3.7) that

$$\|u_h - \Pi_h u\|_{H^1(\Omega)} \leq Ch(\|u\|_X + \|f\|_{L^2(\Omega)}). \tag{3.10}$$

The above estimate (3.10) together with Lemma 2.3 and (3.4) leads to the following optimal order error estimate in H^1 norm.

***Theorem* 3.1** — *Let u and u_h be the solutions of the problem (3.1)-(3.3) and (3.6), respectively. Then, for $f \in L^2(\Omega)$, the following H^1-norm error estimate holds:*

$$\|u - u_h\|_{H^1(\Omega)} \leq Ch\|f\|_{L^2(\Omega)}. \qquad \square$$

For the L^2 norm error estimate we shall use the Nitsche's trick. We consider the following elliptic interface problem

$$-\nabla \cdot (\beta \nabla w) + w = u - u_h \quad \text{in } \Omega$$

with Dirichlet boundary condition

$$w(x) = 0 \quad \text{on } \partial\Omega$$

and interface conditions

$$[w] = 0, \quad \left[\beta \frac{\partial w}{\partial \mathbf{n}}\right] = 0 \quad \text{along } \Gamma.$$

Then clearly $w \in X \cap H_0^1(\Omega)$ and satisfies the weak form

$$A(w, v) = (u - u_h, v) \;\; \forall v \in H_0^1(\Omega). \tag{3.11}$$

Further, w satisfies the *a priori* estimate (cf. [5])

$$\|w\|_X \leq C\|u - u_h\|_{L^2(\Omega)}. \tag{3.12}$$

We then define its finite element approximation to be the function $w_h \in V_h$ such that

$$A_h(w_h, v_h) = (u - u_h, v_h) \;\; \forall v_h \in V_h. \tag{3.13}$$

Arguing as in the derivation of Theorem 3.1 and further using the *a priori* estimate (3.12), we have

$$\|w - w_h\|_{H^1(\Omega)} \leq Ch\|u - u_h\|_{L^2(\Omega)}. \tag{3.14}$$

Setting $v = u - u_h \in H_0^1(\Omega)$ in (3.11) and using (3.5) and (3.6), we obtain

$$\begin{aligned} \|u - u_h\|_{L^2(\Omega)}^2 &= A(w, u - u_h) \\ &= A(w - w_h, u - u_h) + A(w_h, u - u_h) \\ &= A(w - w_h, u - u_h) + [A_h(u_h, w_h) - A(u_h, w_h)] \\ &\equiv: (II)_1 + (II)_2 \end{aligned} \tag{3.15}$$

By Theorem 3.1 and (3.14) we immediately have

$$|(II)_1| \leq Ch^2\|f\|_{L^2(\Omega)}\|u - u_h\|_{L^2(\Omega)} \tag{3.16}$$

Arguing as deriving (3.9) we can deduce

$$
\begin{aligned}
|(II)_2| &\leq Ch \sum_{K\in\mathcal{T}_\Gamma^*} \|\nabla u_h\|_{L^2(K)} \|\nabla w_h\|_{L^2(K)} \\
&\leq Ch\|\nabla u_h\|_{L^2(\Omega_\Gamma^*)} \|\nabla w_h\|_{L^2(\Omega_\Gamma^*)} \\
&\leq Ch\|\nabla(u-u_h)\|_{L^2(\Omega_\Gamma^*)} \|\nabla w_h\|_{L^2(\Omega_\Gamma^*)} \\
&\quad +Ch\|\nabla u\|_{L^2(\Omega_\Gamma^*)} \|\nabla(w-w_h)\|_{L^2(\Omega_\Gamma^*)} \\
&\quad +Ch\|\nabla u\|_{L^2(\Omega_\Gamma^*)} \|\nabla w\|_{L^2(\Omega_\Gamma^*)} \\
&\leq Ch^2\|f\|_{L^2(\Omega)} \|u-u_h\|_{L^2(\Omega)} \\
&\quad +Ch^{\frac{5}{2}}\|u\|_X \|u-u_h\|_{L^2(\Omega)} + Ch^2\|u\|_X\|w\|_X
\end{aligned}
$$

where we have used Theorem 3.1, Lemma 2.2 and (3.14), and the following inequality

$$
\|\nabla w_h\|_{L^2(\Omega)} \leq C\|u-u_h\|_{L^2(\Omega)}.
$$

Thus, for the term $(II)_2$, we have

$$
\begin{aligned}
|(II)_2| &\leq Ch^2\|f\|_{L^2(\Omega)} \|u-u_h\|_{L^2(\Omega)} \\
&\quad +Ch^2\|u\|_X \|u-u_h\|_{L^2(\Omega)} \\
&\leq Ch^2\|f\|_{L^2(\Omega)} \|u-u_h\|_{L^2(\Omega)}. \qquad (3.17)
\end{aligned}
$$

Finally using (3.16)-(3.17) in (3.15), we obtain

$$
\|u-u_h\|^2_{L^2(\Omega)} \leq Ch^2\|f\|_{L^2(\Omega)} \|u-u_h\|_{L^2(\Omega)}.
$$

Thus, we have proved the following optimal order estimates in L^2 norm.

***Theorem* 3.2** — *Let u and u_h be the solutions of the problem (3.1)-(3.3) and (3.6), respectively. Then, for $f \in L^2(\Omega)$, there exist a positive constant C independent of h such that*

$$
\|u-u_h\|_{L^2(\Omega)} \leq Ch^2\|f\|_{L^2(\Omega)}. \quad \square
$$

Remark 3.1 : The present work improves the results of [5] and [14] for elliptic interface problems under minimum regularity assumptions of the true solution and for a practical finite element discretization. To the best of authors knowledge Theorem 3.1 and Theorem 3.2 have not been established before for conforming finite element method with straight interface triangles. For quadrature finite element method with higher smoothness assumption of $f \in H^2(\Omega)$, similar result can be found in Deka ([8]).

4. Error Analysis for Semilinear Elliptic Interface Problems

This section deals with the error analysis for semilinear elliptic interface problems. Before proceeding further, below, we quote some results concerning the approximation of a class of nonlinear problems from [3, 11] for our future use.

Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces. Consider the following class of nonlinear problems: Find $\psi \in \mathcal{X}$ such that

$$\psi + \mathcal{T}\mathcal{G}(\psi) = 0, \tag{4.1}$$

where $\mathcal{T} \in \mathcal{L}(\mathcal{Y}; \mathcal{X})$ and $\mathcal{G}$ is a C^2-mapping from $\mathcal{X}$ into $\mathcal{Y}$. We assume that there exists another Banach space $\mathcal{Z}$, contained in $\mathcal{Y}$, with continuous embedding, such that

$$D\mathcal{G}(\psi) \in \mathcal{L}(\mathcal{X}; \mathcal{Z}) \quad \forall\, \psi \in \mathcal{X}, \tag{4.2}$$

where $D\mathcal{G}$ denotes the Fréchet derivative of $\mathcal{G}$.

Approximations are defined by introducing a subspace $\mathcal{X}_h \subset \mathcal{X}$ and an approximating operator $\mathcal{T}_h \in \mathcal{L}(\mathcal{Y}; \mathcal{X}_h)$. That is, seek a $\psi_h \in \mathcal{X}_h$ such that

$$\psi_h + \mathcal{T}_h\mathcal{G}(\psi_h) = 0. \tag{4.3}$$

Concerning the linear operator $\mathcal{T}_h$, we assume the following approximation properties:

$$\lim_{h\to 0} \|(\mathcal{T}_h - \mathcal{T})\phi\|_{\mathcal{X}} = 0 \quad \forall\, \phi \in \mathcal{Y} \tag{4.4}$$

and

$$\lim_{h\to 0} \|(\mathcal{T}_h - \mathcal{T})\|_{\mathcal{L}(\mathcal{Z};\mathcal{X})} = 0. \tag{4.5}$$

Note that (4.5) is a consequence of (4.4) when the imbedding of $\mathcal{Z}$ into $\mathcal{Y}$ is compact.

We now recall the result of [3] (see also [11]) in the following theorem.

***Theorem* 4.1** — *Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces. Assume that $\mathcal{G}$ is a C^2-mapping from $\mathcal{X}$ into $\mathcal{Y}$ and that $D^2\mathcal{G}$(the second Fréchet derivatives of $\mathcal{G}$) is bounded on all bounded sets of $\mathcal{X}$. Assume that (4.2), (4.4) and (4.5) hold. Further, let ψ and ψ_h satisfy (4.1) and (4.3), respectively. Then there exists a constant $C > 0$, independent of h, such that*

$$\|\psi_h - \psi\|_{\mathcal{X}} \leq C\|(\mathcal{T}_h - \mathcal{T})\mathcal{G}(\psi)\|_{\mathcal{X}}. \tag{4.6}$$

The main concern of this section is to estimate the error involves in the semilinear finite element approximation. We first recast the problem into Brezzi-Rappaz-Raviart framework. For this purpose, we set

$$\mathcal{X} = H_0^1(\Omega), \quad \text{and} \quad \mathcal{Y} = \mathcal{Z} = L^2(\Omega).$$

It is well-known (cf. [2] and [5]) that, for $\tilde{f} \in L^2(\Omega)$, there exists a unique solution $\tilde{u} \in \mathcal{X}$ satisfying the following linear interface problem

$$A(\tilde{u}, v) = (\tilde{f}, v) \quad \forall\, v \in H_0^1(\Omega). \tag{4.7}$$

We introduce the linear operator $\mathcal{T} : \mathcal{Y} \to \mathcal{X}$ to be the solution operator for the linear elliptic interface problems, i.e. $\mathcal{T}\tilde{f} = \tilde{u}$ for $\tilde{f} \in \mathcal{Y}$ and $\tilde{u} \in \mathcal{X}$ if and only if (4.7) holds.

Further, we now define an approximating operator $\mathcal{T}_h : \mathcal{Y} \to \mathcal{X}$ to be the finite element solution operator for the linear elliptic interface problem (4.7), i.e., $\mathcal{T}_h\tilde{f} = \tilde{u}_h$ for $\tilde{f} \in \mathcal{Y}$ and $\tilde{u}_h \in V_h$ if and only if

$$A_h(\tilde{u}_h, v_h) = (\tilde{f}, v_h) \quad \forall\, v_h \in V_h. \tag{4.8}$$

Thus, the problem (2.1) is now equivalent to

$$\mathcal{T}\mathcal{G}(u) = u, \tag{4.9}$$

and (2.3) is equivalent to

$$\mathcal{T}_h\mathcal{G}(u_h) = u_h, \tag{4.10}$$

where we define $\mathcal{G} : \mathcal{X} \to \mathcal{Y}$ by

$$\mathcal{G}(v) = f(v) \;\; \text{for all } v \in \mathcal{X}. \tag{4.11}$$

Concerning the linear operator $\mathcal{T}_h$, we have the following approximation properties from the previous section:

$$\|(\mathcal{T} - \mathcal{T}_h)w\|_{\mathcal{X}} \leq Ch\|w\|_{\mathcal{Y}}, \tag{4.12}$$

for all $w \in \mathcal{Y}$.

We now prove the main result of this section in the following theorem.

***Theorem* 4.2** — *Let u and u_h be the solutions of (4.9) and (4.10), respectively. Then, for $f \in C^2(\mathbb{R}, \mathbb{R})$ satisfying*

$$|f'(s)| \leq C|s| \; and \; |f''(s)| \leq C, \quad \forall s \in \mathbb{R},$$

there exists a positive constant C independent of h such that

$$\|u - u_h\|_{H^1(\Omega)} \leq Ch(\|u\|_{H^2(\Omega_1)} + \|u\|_{H^2(\Omega_2)}).$$

PROOF : For any $\tilde{f} \in \mathcal{Z} = L^2(\Omega)$, a regularity theorem (cf. [5], page 178) for the solution $\tilde{u}$ of linear elliptic interface problem (4.7) imply that $\tilde{u} = \mathcal{T}\tilde{f} \in \mathcal{X}$ and

$$\|\tilde{u}\|_{H^1(\Omega)} + \|\tilde{u}\|_{H^2(\Omega_1)} + \|\tilde{u}\|_{H^2(\Omega_2)} \leq C\|\tilde{f}\|_{L^2(\Omega)}.$$

It now follows from the results of previous section for linear elliptic interface problem that

$$\begin{aligned}
\|(\mathcal{T} - \mathcal{T}_h)\tilde{f}\|_{\mathcal{X}} &= \|\tilde{u} - \tilde{u}_h\|_{\mathcal{X}} \\
&= \|\tilde{u} - \tilde{u}_h\|_{H^1(\Omega)} \\
&\leq Ch(\|\tilde{u}\|_{H^2(\Omega_1)} + \|\tilde{u}\|_{H^2(\Omega_2)}) \\
&\leq Ch\|\tilde{f}\|_{L^2(\Omega)} = Ch\|\tilde{f}\|_{\mathcal{Z}}.
\end{aligned}$$

Here, $\tilde{u}_h$ is the finite element approximation to $\tilde{u}$ given by (4.8). Then taking the supremum over $\tilde{f} \in \mathcal{Z}$, we obtain

$$\|(\mathcal{T} - \mathcal{T}_h)\|_{\mathcal{L}(\mathcal{Z},\mathcal{X})} \leq Ch \to 0 \;\; as \; h \to 0.$$

Let $v, w \in \mathcal{X}$ be given. By the definition of the operator $\mathcal{G}$, we have $\mathcal{G}(v) = f(v)$ for all $v \in \mathcal{X}$. The Fréchet derivative of $\mathcal{G}$ at v becomes $D\mathcal{G}(v) = f'(v)$ and hence,

$$D\mathcal{G}(v)w = (f'(v))w.$$

Now,

$$\begin{aligned} \|D\mathcal{G}(v)\| &= \sup_{w\in H_0^1(\Omega)} \frac{\|D\mathcal{G}(v)w\|_{\mathcal{Z}}}{\|w\|_{H^1(\Omega)}} \\ &= \sup_{w\in H_0^1(\Omega)} \frac{\|f'(v)w\|_{L^2(\Omega)}}{\|w\|_{H^1(\Omega)}}. \end{aligned} \tag{4.13}$$

Further, we note that

$$\begin{aligned} \|f'(v)w\|^2_{L^2(\Omega)} &= \int_\Omega |f'(v)|^2|w|^2dx \\ &\le \left(\int_\Omega |f'(v)|^4dx\right)^{1/2}\left(\int_\Omega |w|^4dx\right)^{1/2} \\ &= \left\{\left(\int_\Omega |f'(v)|^4dx\right)^{1/4}\right\}^2\left\{\left(\int_\Omega |w|^4dx\right)^{1/4}\right\}^2. \end{aligned}$$

Therefore,

$$\|f'(v)w\|_{L^2(\Omega)} \le \|f'(v)\|_{L^4(\Omega)}\|w\|_{L^4(\Omega)}. \tag{4.14}$$

Applying $|f'(s)| \le C|s|$, we have $|f'(v)| \le C|v|$, which immediately implies

$$\left(\int_\Omega |f'(v)|^4dx\right)^{1/4} \le C\|v\|_{L^4(\Omega)}. \tag{4.15}$$

Combining (4.14) and (4.15), we obtain

$$\|f'(v)w\|_{L^2(\Omega)} \le C\|v\|_{L^4(\Omega)}\|w\|_{L^4(\Omega)}.$$

Now, using the Sobolev embedding inequality for two dimensions (cf. [13], page 184):

$$\|v\|_{L^p(\Omega)} \le Cp^{\frac{1}{2}}\|v\|_{H^1(\Omega)} \quad \forall p > 2,\ v \in H^1(\Omega), \tag{4.16}$$

we obtain

$$\|f'(v)w\|_{L^2(\Omega)} \le C\|v\|_{H^1(\Omega)}\|w\|_{H^1(\Omega)},$$

which together with (4.13) immediately implies $D\mathcal{G}(v) \in \mathcal{L}(X;Z)$. Similarly, under the assumption $|f''(s)| \le C$ on $\mathbb{R}$ we can easily prove that $D^2\mathcal{G}(v)$ is bounded. Then it follows from (4.6) that

$$\|u-u_h\|_{H^1(\Omega)} \le C\|(\mathcal{T}-\mathcal{T}_h)\mathcal{G}(u)\|_{H^1(\Omega)}. \tag{4.17}$$

Using the estimate for linear elliptic interface problem from section 3, we obtain

$$\|(\mathcal{T} - \mathcal{T}_h)\mathcal{G}(u)\|_{H^1(\Omega)} \leq Ch\|\mathcal{G}(u)\|_{L^2(\Omega)} = Ch\|f(u)\|_{L^2(\Omega)}. \tag{4.18}$$

Again it follows immediately from equation (1.1) that

$$\|f(u)\|_{L^2(\Omega)} \leq C\|u\|_{H^2(\Omega_1)} + C\|u\|_{H^2(\Omega_2)}.$$

This together with the estimate (4.18) leads to the desired error estimate. □

Remark : In comparison to the results of [13], the main advantage with Theorem 4.2 is that it is obtained for a more practical finite element discretization in which the grid line need not follow the actual interface exactly.

5. Numerical Results

In this section we report the results of computations of a two-dimensional elliptic interface problem. We take for the domain the rectangle $\Omega = (0, 2) \times (0, 1)$. The interface occurs at $x = 1$ so that $\Omega_1 = (0, 1) \times (0, 1)$, $\Omega_2 = (1, 2) \times (0, 1)$ and the interface $\Gamma = \bar{\Omega}_1 \cap \bar{\Omega}_2$.

Consider the following elliptic boundary value problem on Ω:

$$-\nabla \cdot (\beta_i \nabla u_i) + u_i = f_i \quad \text{in} \ \ \Omega_i, \ i = 1, 2, \tag{5.1}$$
$$u_i = 0 \quad \text{on} \ \ \partial\Omega \cap \bar{\Omega}_i, \ \ i = 1, 2, \tag{5.2}$$
$$u_1|_\Gamma = u_2|_\Gamma, \quad (\beta_1 \nabla u_1 \cdot \mathbf{n}_1)|_\Gamma + (\beta_2 \nabla u_2 \cdot \mathbf{n}_2)|_\Gamma = 0, \tag{5.3}$$

where $\mathbf{n}_i$ denotes the unit outer normal vector on Ω_i, $i = 1, 2$.

For the exact solution, we choose

$$u_1(x, y) = \sin(\pi x)\sin(\pi y), \quad (x, y) \in \Omega_1$$

and

$$u_2(x, y) = -\sin(2\pi x)\sin(\pi y), \quad (x, y) \in \Omega_2.$$

The right-hand sides f_1 and f_2 in (5.1) are determined from the choice for u_1 and u_2, respectively with $\beta_1 = 1$ and $\beta_2 = \frac{1}{2}$.

For our numerical results, globally continuous piecewise linear finite element functions based on uniform triangulations of $\Omega_i,\ i = 1, 2$, were used. The nodes of the triangulations of Ω_1 and Ω_2 coincide on the interface Γ as stated in Section 2. Let h_x and h_y be the discretization parameters along x and y axes, respectively. Then we choose our mesh parameter h such that $h^2 = {h_x}^2 + {h_y}^2$. From Table 5.1, we see the convergence of the finite element solution to the exact solution in L^2 and H^1 norms.

Table 5.1: *Numerical results for the test problem (5.1)-(5.3).*

h^2	$(h_x,\, h_y)$	$\|u - u_h\|_{L^2(\Omega)}$	$\|u - u_h\|_{H^1(\Omega)}$
$1/8$	$(1/2,\, 1/2)$	.056165	.158861
$1/32$	$(1/4,\, 1/4)$	.014041	.079430
$1/128$	$(1/8,\, 1/8)$	.003510	.039709
$1/512$	$(1/16,\, 1/16)$	.000877	.019854

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