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# SEVERAL IDENTITIES IN THE CATALAN TRIANGLE

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In this paper, we first establish several identities for the alternating sums in the Catalan triangle whose $(n, p)$ entry is defined by $B_{n,\ p} = \frac{p}{n}\binom{2n}{n-p}$. Second, we show that the Catalan triangle matrix $\mathcal{C}$ can be factorized by $\mathcal{C} = \mathcal{F}\mathcal{Y} = \mathcal{Z}\mathcal{F}$, where $\mathcal{F}$ is the Fibonacci matrix. From these formulas, some interesting identities involving $B_{n,\ p}$ and the Fibonacci numbers $F_n$ are given. As special cases, some new relationships between the well-known Catalan numbers $C_n$ and the Fibonacci numbers are obtained, for example:

$$C_n = F_{n+1} + \sum_{k=3}^{n}\left\{1 - \frac{(k+1)(5k-6)}{4(2k-1)(2k-3)}\right\} C_k F_{n-k+1},$$

and

$$\frac{n-1}{n+2}C_n = \frac{1}{2}F_n + F_{n-2} + \sum_{k=4}^{n}\left\{1 - \frac{(k+2)(5k^2-16k+9)}{4(k-1)(2k-1)(2k-3)}\right\} \frac{k-1}{k+2} C_k F_{n-k+1}.$$

**Key words :** Catalan triangle; Catalan number; sum; Fibonacci matrix; Fibonacci number.

## 1. Introduction

In [10], by considering lattice paths in the first quadrant, Shapiro derived the following triangle similar to Pascal's triangle, with entries given by

$$B_{n,p} = \frac{p}{n}\binom{2n}{n-p}, \ \ n, \ p \in N, \ p \le n. \tag{1}$$

The above triangle is called Catalan triangle because the well-known Catalan numbers are the entries in the first column. Catalan numbers are defined recursively by $C_0 = 1$ and $C_n = \sum_{i=0}^{n-1} C_i C_{n-1-i}, \ n \ge 1$. The general term of Catalan sequence is given by the formula

$$C_n = \frac{1}{n+1}\binom{2n}{n}, \ \ n \ge 1.$$

Catalan numbers have been widely encountered and widely investigated. In [11] Stanley lists some 70 examples of enumeration problems which are counted by the Catalan numbers. The research bibliography of Gould [4] also contains a wealth of references. Aigner [1] introduced a class of numbers, called Catalan-like numbers and provided a common framework for a series of coefficients with Catalan numbers [11] as special cases. Other generalized Catalan numbers are considered in [7]. Although the numbers $B_{n,\ p}$ are not as famous as Catalan numbers, they have also some different interpretations and applications. See [5].

In [10], Shapiro gave the following identities

$$\sum_{p=1}^{n} (B_{n,p})^2 = C_{2n-1}, \ n \in \mathbb{N}, \tag{2}$$

$$\sum_{p=1}^{n} B_{n,p} B_{n+1,p} = C_{2n}, \ n \ge 1. \tag{3}$$

In [5], Gutierrez, Hernandez, Miana and Romero gave an alternative proof of identities (2) and (3) and established the following identities:

$$\sum_{p=1}^{n} (pB_{n,p})^2 = (3n-2)C_{2(n-1)}, \ n \in \mathbb{N}, \tag{4}$$

$$\sum_{p=1}^{i} B_{n,p} B_{n,n+p-i}(n+2p-i) = (n+1)\binom{2(n-1)}{i-1} C_n, \ n \in \mathbb{N}. \tag{5}$$

In particular, if $i = n$ in (5), then

$$\sum_{p=1}^{n} p\,(B_{n,p})^2 = \frac{n(n+1)}{2} C_n C_{n-1},\; n \in \mathbb{N}. \tag{6}$$

The structure of the paper is as follows. In Section 2 we first establish several identities for the alternating sums in the Catalan triangle whose $(n, p)$ entry is defined by $B_{n,\,p} = \frac{p}{n}\binom{2n}{n-p}$. In Section 3 we show that Catalan triangle matrix can be factorized as the product of the Fibonacci matrix and a lower triangular matrix. From these factorizations, some interesting identities involving $B_{n,\,p}$ and the Fibonacci numbers $F_n$ are given. As special cases, some new relationships between the well-known Catalan numbers $C_n$ and the Fibonacci numbers are obtained.

## 2. SOME ALTERNATING SUMS

First of all, we give two identities which are to be used later.

*Lemma* 2.1 — The following hold :

$$\sum_{p=0}^{n-1}(-1)^p\binom{2n}{p}^2 = \frac{1}{2}(-1)^n\binom{2n}{n} - \frac{1}{2}(-1)^n\binom{2n}{n}^2, \quad (n \geq 1), \tag{7}$$

$$\sum_{p=0}^{n-2}(-1)^p\binom{2n-1}{p}\binom{2n-1}{p+1} = \frac{1}{2}(-1)^n\binom{2n-1}{n}\binom{2n-1}{n-1} - \frac{1}{2}(-1)^n\binom{2n-1}{n}, \quad (n \geq 2). \tag{8}$$

PROOF : Comparing the coefficients of $x^{2n}$ in both sides of the equalities $(1-x)^{2n}(1+x)^{2n} = (1-x^2)^{2n}$ and $(1-x)^{2n-1}(1+x)^{2n-1} = (1-x^2)^{2n-1}$ yields

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}^2 = (-1)^n\binom{2n}{n}, \tag{9}$$

and

$$\sum_{k=1}^{2n-1}(-1)^k\binom{2n-1}{k}\binom{2n-1}{k-1} = (-1)^n\binom{2n-1}{n}, \tag{10}$$

respectively. Since

$$\begin{aligned}\sum_{p=0}^{n-1}(-1)^p\binom{2n}{p}^2 &= \sum_{p=0}^{2n}(-1)^p\binom{2n}{p}^2-\sum_{p=n}^{2n}(-1)^p\binom{2n}{p}^2\\ &= (-1)^n\binom{2n}{n}-\sum_{p=0}^{n}(-1)^p\binom{2n}{p}^2,\end{aligned}$$

we have

$$2\sum_{p=0}^{n-1}(-1)^p\binom{2n}{p}^2=(-1)^n\binom{2n}{n}-(-1)^n\binom{2n}{n}^2.$$

The first formula is proved. The second formula can be proved similarly. □

***Theorem* 2.2** — *Let $n \geq 1$. Then*

$$\sum_{p=1}^{n}(-1)^p\left(B_{n,p}\right)^2=-\frac{1}{2}(n+1)C_n. \tag{11}$$

PROOF : Applying Lemma 2.1, we have

$$\begin{aligned}\sum_{p=1}^{n}(-1)^p\left(B_{n,p}\right)^2 &= \sum_{p=1}^{n}(-1)^p\frac{p^2}{n^2}\binom{2n}{n-p}^2\\ &= \frac{1}{n^2}\sum_{p=0}^{n-1}(-1)^{n-p}(n-p)^2\binom{2n}{p}^2\\ &= \frac{(-1)^n}{n^2}\sum_{p=0}^{n-1}(-1)^p\left(n^2-2np+p^2\right)\binom{2n}{p}^2\\ &= (-1)^n\sum_{p=0}^{n-1}(-1)^p\binom{2n}{p}^2-4(-1)^n\sum_{p=0}^{n-1}(-1)^p\\ &\quad \binom{2n-1}{p-1}\binom{2n}{p}+4(-1)^n\sum_{p=0}^{n-1}(-1)^p\binom{2n-1}{p-1}^2\\ &= \frac{1}{2}\binom{2n}{n}-\frac{1}{2}\binom{2n}{n}^2-4(-1)^n\sum_{p=1}^{n-1}(-1)^p\binom{2n-1}{p-1}\\ &\quad \left[\binom{2n}{p}-\binom{2n-1}{p-1}\right]\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2}\binom{2n}{n} - \frac{1}{2}\binom{2n}{n}^2 - 4(-1)^n \sum_{p=1}^{n-1} (-1)^p \binom{2n-1}{p-1}\binom{2n-1}{p} \\
&= \frac{1}{2}\binom{2n}{n} - \frac{1}{2}\binom{2n}{n}^2 - 2\binom{2n-1}{n} + 2\binom{2n-1}{n}\binom{2n-1}{n-1} \\
&= \frac{1}{2}\binom{2n}{n} - \frac{1}{2}\binom{2n}{n}^2 - 2 \times \frac{1}{2}\binom{2n}{n} + 2 \times \frac{1}{4}\binom{2n}{n}^2 \\
&= -\frac{1}{2}\binom{2n}{n} \\
&= -\frac{1}{2}(n+1)C_n.
\end{aligned}$$

□

***Theorem* 2.3** — *Let $n \geq 1$. Then*

$$\sum_{p=1}^{n} (-1)^p p^2 (B_{n,p})^2 = n(n-2)(2n-1)C_{n-1}. \tag{12}$$

PROOF : We use the identity

$$(n-p)^4 = p^2(p-1)^2 + (2-4n)p^2(p-1) + p^2(6n^2-4n+1) - 4n^3 p + n^4$$

to get that

$$\begin{aligned}
&\sum_{p=1}^{n} (-1)^p p^2 (B_{n,p})^2 = \sum_{p=1}^{n} (-1)^p p^2 \frac{p^2}{n^2} \binom{2n}{n-p}^2 \\
&= \frac{1}{n^2} \sum_{p=0}^{n-1} (-1)^{n-p} (n-p)^4 \binom{2n}{p}^2 \\
&= \frac{(-1)^n}{n^2} \sum_{p=0}^{n-1} (-1)^p \\
&\quad \left\{p^2(p-1)^2 + (2-4n)p^2(p-1) + p^2(6n^2-4n+1) - 4n^3 p + n^4\right\} \binom{2n}{p}^2 \\
&= 4(2n-1)^2(-1)^n \sum_{p=2}^{n-1} (-1)^p \binom{2n-2}{p-2}^2 + (-1)^n 4(2-4n) \sum_{p=1}^{n-1} (-1)^p (p-1) \\
&\quad \binom{2n-1}{p-1}^2 + (-1)^n 4(6n^2-4n+1) \sum_{p=1}^{n-1} (-1)^p \binom{2n-1}{p-1}^2
\end{aligned}$$

$$-(-1)^n 8n^2 \sum_{p=1}^{n-1}(-1)^p \binom{2n-1}{p-1}\binom{2n}{p} + (-1)^n n^2 \sum_{p=0}^{n-1}(-1)^p \binom{2n}{p}^2.$$

Applying Lemma 2.1 for $\sum_{p=2}^{n-1}(-1)^p\binom{2n-2}{p-2}^2$ and $\sum_{p=0}^{n-1}(-1)^p\binom{2n}{p}^2$, we have

$$\begin{aligned}
&\sum_{p=1}^{n}(-1)^p p^2 \left(B_{n,p}\right)^2 \\
= \; & 4(2n-1)^2\left[-\frac{1}{2}\binom{2n-2}{n-1}+\frac{1}{2}\binom{2n-2}{n-1}^2-\binom{2n-2}{n-2}^2\right] \\
&+n^2\left[\frac{1}{2}\binom{2n}{n}-\frac{1}{2}\binom{2n}{n}^2\right]+(-1)^n 4(2-4n)\sum_{p=1}^{n-1}(-1)^p(p-1)\binom{2n-1}{p-1}^2 \\
&+(-1)^n 4(6n^2-4n+1) \\
&\sum_{p=1}^{n-1}(-1)^p\binom{2n-1}{p-1}^2-(-1)^n 8n^2\sum_{p=1}^{n-1}(-1)^p\binom{2n-1}{p-1}\binom{2n}{p} \\
= \; & 4(2n-1)^2\left[-\frac{1}{2}\binom{2n-2}{n-1}+\frac{1}{2}\binom{2n-2}{n-1}^2-\binom{2n-2}{n-2}^2\right] \\
&+n^2\left[\frac{1}{2}\binom{2n}{n}-\frac{1}{2}\binom{2n}{n}^2\right]-(-1)^n 8(2n-1)^2\sum_{p=1}^{n-1}(-1)^p\binom{2n-1}{p-1}\binom{2n-2}{p-2} \\
&+(-1)^n 4(6n^2-4n+1) \\
&\sum_{p=1}^{n-1}(-1)^p\binom{2n-1}{p-1}^2-(-1)^n 8n^2\sum_{p=1}^{n-1}(-1)^p\binom{2n-1}{p-1}\left[\binom{2n-1}{p}+\binom{2n-1}{p-1}\right] \\
= \; & 4(2n-1)^2\left[-\frac{1}{2}\binom{2n-2}{n-1}+\frac{1}{2}\binom{2n-2}{n-1}^2-\binom{2n-2}{n-2}^2\right] \\
&+n^2\left[\frac{1}{2}\binom{2n}{n}-\frac{1}{2}\binom{2n}{n}^2\right]-(-1)^n 8(2n-1)^2\sum_{p=1}^{n-1}(-1)^p\binom{2n-1}{p-1}\binom{2n-2}{p-2} \\
&+(-1)^n 4(4n^2-4n+1) \\
&\sum_{p=1}^{n-1}(-1)^p\binom{2n-1}{p-1}^2-(-1)^n 8n^2\sum_{p=1}^{n-1}(-1)^p\binom{2n-1}{p-1}\binom{2n-1}{p}.
\end{aligned}$$

Again, applying Lemma 2.1 for the final term on the right hand side of the above equation we have

$$\begin{aligned}
&\sum_{p=1}^{n}(-1)^p p^2 \left(B_{n,p}\right)^2 \\
= \;& 4(2n-1)^2\left[-\frac{1}{2}\binom{2n-2}{n-1}+\frac{1}{2}\binom{2n-2}{n-1}^2-\binom{2n-2}{n-2}^2\right] \\
&+n^2\left[\frac{1}{2}\binom{2n}{n}-\frac{1}{2}\binom{2n}{n}^2\right]-(-1)^n 8(2n-1)^2\sum_{p=1}^{n-1}(-1)^p\binom{2n-1}{p-1}\binom{2n-2}{p-2} \\
&+(-1)^n 4(4n^2-4n+1) \\
&\sum_{p=1}^{n-1}(-1)^p\binom{2n-1}{p-1}^2-(-1)^n 8n^2\sum_{p=1}^{n-1}(-1)^p\binom{2n-1}{p-1}\binom{2n-1}{p} \\
= \;& -2(2n-1)^2\binom{2n-2}{n-1}+2(2n-1)^2\binom{2n-2}{n-1}^2-4(2n-1)^2\binom{2n-2}{n-2}^2 \\
&+\frac{1}{2}n^2\binom{2n}{n}-\frac{1}{2}n^2\binom{2n}{n}^2 \\
&-(-1)^n 8n^2\left[\frac{1}{2}(-1)^n\binom{2n-1}{n}-\frac{1}{2}(-1)^n\binom{2n-1}{n}\binom{2n-1}{n-1}\right] \\
&+(-1)^n 4(2n-1)^2\sum_{p=1}^{n-1}(-1)^p\binom{2n-1}{p-1}\left[\binom{2n-1}{p-1}-2\binom{2n-2}{p-2}\right].
\end{aligned}$$

Noting that

$$\begin{aligned}
&\sum_{p=1}^{n-1}(-1)^p\binom{2n-1}{p-1}\left[\binom{2n-1}{p-1}-2\binom{2n-2}{p-2}\right] \\
= \;& \sum_{p=1}^{n-1}(-1)^p\left[\binom{2n-2}{p-1}+\binom{2n-2}{p-2}\right]\left[\binom{2n-2}{p-1}-\binom{2n-2}{p-2}\right] \\
= \;& \sum_{p=1}^{n-1}(-1)^p\left[\binom{2n-2}{p-1}^2-\binom{2n-2}{p-2}^2\right] \\
= \;& -\sum_{p=0}^{n-2}(-1)^p\binom{2n-2}{p}^2-\sum_{p=0}^{n-3}(-1)^p\binom{2n-2}{p}^2 \\
= \;& -2\sum_{p=0}^{n-2}(-1)^p\binom{2n-2}{p}^2+(-1)^n\binom{2n-2}{n-2}^2
\end{aligned}$$

$$\begin{aligned}
&= -2\left[\frac{1}{2}(-1)^{n-1}\binom{2n-2}{n-1} - \frac{1}{2}(-1)^{n-1}\binom{2n-2}{n-1}^2\right] + (-1)^n\binom{2n-2}{n-2}^2 \\
&= (-1)^n\binom{2n-2}{n-1} - (-1)^n\binom{2n-2}{n-1}^2 + (-1)^n\binom{2n-2}{n-2}^2,
\end{aligned}$$

we get

$$\begin{aligned}
&\sum_{p=1}^{n}(-1)^p p^2 \left(B_{n,p}\right)^2 \\
&= -2(2n-1)^2\binom{2n-2}{n-1} + 2(2n-1)^2\binom{2n-2}{n-1}^2 - 4(2n-1)^2\binom{2n-2}{n-2}^2 \\
&\quad + \frac{1}{2}n^2\binom{2n}{n} - \frac{1}{2}n^2\binom{2n}{n}^2 - 8n^2\left[\frac{1}{2}\binom{2n-1}{n} - \frac{1}{2}\binom{2n-1}{n}\binom{2n-1}{n-1}\right] \\
&\quad + 4(2n-1)^2\binom{2n-2}{n-1} - 4(2n-1)^2\binom{2n-2}{n-1}^2 + 4(2n-1)^2\binom{2n-2}{n-2}^2 \\
&= 2(2n-1)^2\binom{2n-2}{n-1} - 2(2n-1)^2\binom{2n-2}{n-1}^2 + \frac{1}{2}n^2\binom{2n}{n} - \frac{1}{2}n^2\binom{2n}{n}^2 \\
&\quad - 4n^2\binom{2n-1}{n} + 4n^2\binom{2n-1}{n}\binom{2n-1}{n-1} \\
&= 2(2n-1)^2\binom{2n-2}{n-1} - 2(2n-1)^2\binom{2n-2}{n-1}^2 + n(2n-1)\binom{2n-2}{n-1} \\
&\quad - n(2n-1)\binom{2n}{n}\binom{2n-2}{n-1} - 4n(2n-1)\binom{2n-2}{n-1} + 4(2n-1)^2\binom{2n-2}{n-1}^2 \\
&= (2n^2-5n+2)\binom{2n-2}{n-1} \\
&= n(n-2)(2n-1)C_{n-1}.
\end{aligned}$$

The proof is complete. □

***Theorem* 2.4** — *Let $n \geq 2$. Then*

$$\begin{aligned}
&\sum_{p=1}^{n}(-1)^p p(B_{n,p})^2 + 4(n-1)^3\sum_{p=0}^{n-2}\frac{(-1)^p}{(p+1)(p+2)}B_{n-1,p+1}B_{n-1,p+2} \\
&= n(2n^2-6n+3)C_{n-1}. \qquad (13)
\end{aligned}$$

PROOF: Using

$$(n-p)^3 = -p^2(p-1) + (3n-1)p^2 - 3n^2p + n^3,$$

we have

$$\sum_{p=1}^{n}(-1)^p p(B_{n,p})^2 = \frac{1}{n^2}\sum_{p=1}^{n}(-1)^p p^3\binom{2n}{n-p}^2 = \frac{(-1)^n}{n^2}\sum_{p=0}^{n-1}(-1)^p(n-p)^3\binom{2n}{p}^2$$

$$= \frac{(-1)^n}{n^2}\sum_{p=0}^{n-1}(-1)^p\left(-p^2(p-1)+(3n-1)p^2-3n^2p+n^3\right)\binom{2n}{p}^2$$

$$\begin{aligned} &= 4(-1)^n\sum_{p=0}^{n-2}(-1)^p p\binom{2n-1}{p}^2 - 4(-1)^n(3n-1)\sum_{p=0}^{n-2}(-1)^p\binom{2n-1}{p}^2 \\ &\quad -3(-1)^n\sum_{p=0}^{n-1}(-1)^p p\binom{2n}{p}^2 + (-1)^n n\sum_{p=0}^{n-1}(-1)^p\binom{2n}{p}^2 \\ &= (-1)^n 4(2n-1)\sum_{p=1}^{n-2}(-1)^p\binom{2n-1}{p}\binom{2n-2}{p-1} - (-1)^n 4(3n-1)\sum_{p=0}^{n-2}(-1)^p \\ &\quad \binom{2n-1}{p}^2 - (-1)^n 6n\sum_{p=1}^{n-1}(-1)^p\binom{2n}{p}\binom{2n-1}{p-1} + (-1)^n n\sum_{p=0}^{n-1}(-1)^p\binom{2n}{p}^2. \end{aligned} \tag{14}$$

By combinatorial computation, the first term on the right hand side of (14) equals

$$\begin{aligned} \sum_{p=1}^{n-2}(-1)^p\binom{2n-1}{p}\binom{2n-2}{p-1} &= \sum_{p=1}^{n-2}(-1)^p\binom{2n-2}{p}\binom{2n-2}{p-1} + \sum_{p=1}^{n-2}(-1)^p\binom{2n-2}{p-1}^2 \\ &= \sum_{p=1}^{n-2}(-1)^p\binom{2n-2}{p}\binom{2n-2}{p-1} \\ &\quad -\sum_{p=0}^{n-3}(-1)^p\binom{2n-2}{p}^2. \end{aligned}$$

The second term on the right hand side of (14) equals

$$\begin{aligned} \sum_{p=0}^{n-2}(-1)^p\binom{2n-1}{p}^2 &= \sum_{p=0}^{n-2}(-1)^p\left[\binom{2n-2}{p}+\binom{2n-2}{p-1}\right]^2 \\ &= (-1)^n\binom{2n-2}{n-2}^2 + 2\sum_{p=0}^{n-2}(-1)^p\binom{2n-2}{p}\binom{2n-2}{p-1}. \end{aligned}$$

The third term on the right hand side of (14) equals

$$\begin{aligned}\sum_{p=0}^{n-1}(-1)^p\binom{2n}{p}\binom{2n-1}{p-1} &= \sum_{p=0}^{n-1}(-1)^p\binom{2n-1}{p}\binom{2n-1}{p-1}+\sum_{p=1}^{n-1}(-1)^p\binom{2n-1}{p-1}^2\\ &= \sum_{p=0}^{n-1}(-1)^p\binom{2n-1}{p}\binom{2n-1}{p-1}-\sum_{p=0}^{n-2}(-1)^p\binom{2n-1}{p}^2\\ &= \sum_{p=0}^{n-1}(-1)^p\binom{2n-1}{p}\binom{2n-1}{p-1}-(-1)^n\\ &\quad \binom{2n-2}{n-2}^2-2\sum_{p=0}^{n-2}(-1)^p\binom{2n-2}{p}\binom{2n-2}{p-1}.\end{aligned}$$

Hence we have

$$\begin{aligned}&\sum_{p=1}^{n}(-1)^p p(B_{n,p})^2\\ = &-2(3n-2)\binom{2n-2}{n-2}^2+(-1)^n n\sum_{p=0}^{n-1}(-1)^p\binom{2n}{p}^2-(-1)^n 6n\sum_{p=0}^{n-1}(-1)^n\\ &\binom{2n-1}{p}\binom{2n-1}{p-1}-(-1)^n 4(2n-1)\sum_{p=0}^{n-3}(-1)^p\binom{2n-2}{p}^2-(-1)^n 4(n-1)\\ &\sum_{p=0}^{n-2}(-1)^p\binom{2n-2}{p}\binom{2n-2}{p-1}.\end{aligned}$$

Using Lemma 2.1 it follows that

$$\begin{aligned}\sum_{p=1}^{n}(-1)^p p(B_{n,p})^2 &= \frac{1}{2}n\binom{2n}{n}-\frac{1}{2}n\binom{2n}{n}^2-2(3n-2)\binom{2n-2}{n-2}^2-3n\binom{2n-1}{n}\\ &\quad +3n\binom{2n-1}{n}\binom{2n-1}{n-1}+2(2n-1)\binom{2n-2}{n-1}-2(2n-1)\\ &\quad \binom{2n-2}{n-1}^2+4(2n-1)\binom{2n-2}{n-2}^2\\ &\quad -(-1)^n 4(n-1)\sum_{p=0}^{n-2}(-1)^p\binom{2n-2}{p}\binom{2n-2}{p-1}\\ &= \frac{2n^2-6n+3}{n}\binom{2n-2}{n-1}^2-(-1)^n 4(n-1)\end{aligned}$$

$$\begin{aligned}
&\sum_{p=0}^{n-2}(-1)^p\binom{2n-2}{p}\binom{2n-2}{p-1} \\
= \quad & n(2n^2-6n+3)C_{n-1} - 4(n-1)^3\sum_{p=0}^{n-2}\frac{(-1)^p}{(p+1)(p+2)} \\
& B_{n-1,\ p+1}B_{n-1,\ p+2}.
\end{aligned}$$

The proof is complete. □

***Theorem* 2.5** — *Let* $n \geq 1$. *Then*

$$\sum_{p=1}^{n}(-1)^p B_{n,\ p}B_{n+1,\ p} \quad + \quad \sum_{p=1}^{n}(-1)^p\frac{n^2-p^2}{p(p+2)} B_{n,\ p}B_{n+1,\ p+2} = -n(2n+1)C_n^2 + nC_n. \quad (15)$$

PROOF: First, we have

$$\begin{aligned}
&\sum_{p=1}^{n}(-1)^p B_{n,\ p}B_{n+1,\ p} = \frac{1}{n(n+1)}\sum_{p=1}^{n}(-1)^p p^2\binom{2n}{n-p}\binom{2n+2}{n+1-p} \\
= \quad & \frac{(-1)^n}{n(n+1)}\sum_{p=0}^{n-1}(-1)^p(n-p)^2\binom{2n}{p}\binom{2n+2}{p+1} \\
= \quad & \frac{(-1)^n}{n(n+1)}\sum_{p=0}^{n-1}(-1)^p(n^2-2np+p^2)\binom{2n}{p}\binom{2n+2}{p+1} \\
= \quad & \frac{(-1)^n}{n(n+1)}\left\{n^2\sum_{p=0}^{n-1}(-1)^p\binom{2n}{p}\binom{2n+2}{p+1} - 2n\sum_{p=0}^{n-1}(-1)^p p\binom{2n}{p}\binom{2n+2}{p+1}\right. \\
& \left. + \sum_{p=0}^{n-1}(-1)^p p^2\binom{2n}{p}\binom{2n+2}{p+1}\right\}.
\end{aligned}$$

Applying Lemma 2.1, it follows that

$$\begin{aligned}
\sum_{p=0}^{n-1}(-1)^p\binom{2n}{p}\binom{2n+2}{p+1} \quad &= \quad \sum_{p=0}^{n-1}(-1)^p\binom{2n}{p}\left[\binom{2n}{p+1} + 2\binom{2n}{p} + \binom{2n}{p-1}\right] \\
&= \quad \sum_{p=0}^{n-1}(-1)^p\binom{2n}{p}\binom{2n}{p+1} + 2\sum_{p=0}^{n-1}(-1)^p\binom{2n}{p}^2
\end{aligned}$$

$$\begin{aligned}
& -\sum_{p=0}^{n-2}(-1)^p\binom{2n}{p}\binom{2n}{p+1} \\
= \quad & (-1)^{n-1}\binom{2n}{n-1}\binom{2n}{n}+(-1)^n\binom{2n}{n}-(-1)^n\binom{2n}{n}^2 \\
= \quad & -(2n+1)(n+1)(-1)^nC_n^2+(n+1)(-1)^nC_n.
\end{aligned}$$

Noting that

$$\sum_{p=0}^{n-1}(-1)^p p\binom{2n}{p}\binom{2n+2}{p+1}=(-1)^n n(n+1)\sum_{p=1}^{n}(-1)^p\frac{n-p}{p(p+2)}B_{n,\,p}B_{n+1,\,p+2}$$

and

$$\sum_{p=0}^{n-1}(-1)^p p^2\binom{2n}{p}\binom{2n+2}{p+1}=(-1)^n n(n+1)\sum_{p=1}^{n}(-1)^p\frac{(n-p)^2}{p(p+2)}B_{n,\,p}B_{n+1,\,p+2},$$

the proof of the theorem can be completed. □

## 3. $B_{n,\,p}$ AND FIBONACCI NUMBERS

The $n\times n$ Catalan triangle matrix $\mathcal{C}$ is defined by

$$\mathcal{C}=[B_{i,j}]_{\,i,j=1,2,\ldots,n},$$

where $B_{i,j}$ is defined in (1).

The Fibonacci numbers have been discussed in so many papers and books. Let $F_n$ be the $n$th Fibonacci number. The $n\times n$ Fibonacci matrix $\mathcal{F}=[f_{i,j}]$ $(i,j=1,2,\ldots,n)$ is defined by

$$f_{i,j}=\begin{cases} F_{i-j+1}, & \text{if } i-j\geq 0,\\ 0, & \text{if } i-j<0.\end{cases} \tag{16}$$

The inverse of $\mathcal{F}$ is given as follows: if $\mathcal{F}^{-1}=[f'_{i,j}]$ $(i,j=1,2,\ldots,n)$, then

$$f'_{i,j}=\begin{cases} 1, & i=j,\\ -1, & i=j+1,j+2,\\ 0, & \text{otherwise.}\end{cases} \tag{17}$$

We define two $n \times n$ matrices $\mathcal{Y} = [y_{i,j}]$ and $\mathcal{Z} = [z_{i,j}]$ $(i, j = 1, 2, \ldots, n)$ as follows:

$$y_{i,j}(x) = B_{i,\,j} - B_{i-1,\,j} - B_{i-2,\,j}, \tag{18}$$

here we make the convention that $B_{-1,j} = B_{-2,j} = 0$, and

$$z_{i,j}(x) = B_{i,\,j} - B_{i,\,j+1} - B_{i,\,j+2}, \tag{19}$$

respectively.

From the definition of $\mathcal{C}$ and the inverse of $\mathcal{F}$, it is easy to see that $\mathcal{F}^{-\infty}\mathcal{C} = \mathcal{Y}$ and $\mathcal{C}\mathcal{F}^{-\infty} = \mathcal{Z}$. Hence we have

$$\mathcal{C} = \mathcal{F}\mathcal{Y} = \mathcal{Z}\mathcal{F}. \tag{20}$$

From (20), we have the following identities.

***Theorem* 3.1** — *For* $1 \le p \le n-2$,

$$B_{n,\,p} = F_{n-p+1} + (2p-1)F_{n-p} + \sum_{k=p+2}^{n} \left\{ 1 - \frac{(k^2-p^2)(5k^2-16k+13-p^2)}{4(k-1)(k-2)(2k-1)(2k-3)} \right\} B_{k,\,p} F_{n-k+1} \tag{21}$$

*and*

$$B_{n,\,p} = F_{n-p+1} + (2n-3)F_{n-p} + \sum_{k=p}^{n-2} \left\{ 1 - \frac{n(n-k)(2k+3)}{k(n+k+1)(n+k+2)} \right\} B_{n,\,k} F_{k-p+1}. \tag{22}$$

PROOF: From (18), it is easy to see that

$$y_{p,\,p} = 1, \quad y_{p+1,\,p} = 2p-1$$

and for $k \ge p+2$,

$$y_{k,\,p} \;=\; B_{k,\,p} - B_{k-1,\,p} - B_{k-2,\,p}$$

$$
\begin{aligned}
&= \frac{p}{k}\binom{2k}{k-p} - \frac{p}{k-1}\binom{2k-2}{k-p-1} - \frac{p}{k-2}\binom{2k-4}{k-p-2} \\
&= \frac{p}{k}\binom{2k}{k-p}\left\{1 - \frac{(k^2-p^2)(5k^2-16k+13-p^2)}{4(k-1)(k-2)(2k-1)(2k-3)}\right\} \\
&= B_{k,\,p}\left\{1 - \frac{(k^2-p^2)(5k^2-16k+13-p^2)}{4(k-1)(k-2)(2k-1)(2k-3)}\right\}.
\end{aligned}
$$

Then, from (20) it follows that

$$
\begin{aligned}
B_{n,\,p} &= \sum_{k=p}^{n} F_{n-k+1} y_{k,p} \\
&= F_{n-p+1} y_{p,p} + F_{n-p} y_{p+1,p}(x) + \sum_{k=p+2}^{n} F_{n-k+1} y_{k,p} \\
&= F_{n-p+1} + (2p-1)F_{n-p} \\
&+ \sum_{k=p+2}^{n}\left\{1 - \frac{(k^2-p^2)(5k^2-16k+13-p^2)}{4(k-1)(k-2)(2k-1)(2k-3)}\right\} B_{k,\,p} F_{n-k+1},
\end{aligned}
$$

and (21) holds. Similarly, we can obtain identity (22). □

As consequences of Theorem 3.1, we derive the following interesting new relationships between the Catalan numbers and the Fibonacci numbers.

*Corollary* 3.2 — For $n \geq 3$,

$$
C_n = F_{n+1} + \sum_{k=3}^{n}\left\{1 - \frac{(k+1)(5k-6)}{4(2k-1)(2k-3)}\right\} C_k F_{n-k+1}, \tag{23}
$$

and

$$
C_n = F_n + (2n-3)F_{n-1} + \sum_{k=1}^{n-2}\left\{1 - \frac{n(n-k)(2k+3)}{k(n+k+1)(n+k+2)}\right\} B_{n,\,k} F_k. \tag{24}
$$

PROOF: Take $p = 1$ in Theorem 3.1. □

*Corollary* 3.3 — For $n \geq 4$,

$$
\begin{aligned}
\frac{n-1}{n+2} C_n = \frac{1}{2} F_n &+ F_{n-2} \\
&+ \sum_{k=4}^{n}\left\{1 - \frac{(k+2)(5k^2-16k+9)}{4(k-1)(2k-1)(2k-3)}\right\} \\
&\quad \frac{k-1}{k+2} C_k F_{n-k+1},
\end{aligned} \tag{25}
$$

and

$$\frac{2(n-1)}{n+2}C_n = F_{n-1} + (2n-3)F_{n-2} + \sum_{k=2}^{n-2}\left\{1-\frac{n(n-k)(2k+3)}{k(n+k+1)(n+k+2)}\right\} B_{n,\,k}F_{k-1}. \tag{26}$$

PROOF : Take $p = 2$ in Theorem 3.1. □

*Note*: Applying the idea and method of the Lucas matrix introduced in [13], some identities involving Catalan numbers $C_n$ and Lucas numbers can be obtained.

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