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ASYMPTOTIC TOEPLITZ AND HANKEL OPERATORS ON THE BERGMAN SPACE

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In this paper the concept of asymptotic Toeplitz and asymptotic Hankel operators on the Bergman space are introduced and properties of these classes of operators are studied. The importance of this notion is that it associates with a class of operators a Toeplitz operator and with a class of operators a Hankel operator where the original operators are not even Toeplitz or Hankel. Thus it is possible to assign a symbol to an operator that is not Toeplitz or Hankel and hence a symbol calculus is obtained. Further a relation between Toeplitz operators and little Hankel operators on the Bergman space is established in some asymptotic sense.

Key words: Toeplitz operators, Hankel operators, Bergman space, Hardy space, Bergman shift operator.

1. INTRODUCTION

In this paper we introduce the concept of asymptotic Toeplitz operators and asymptotic Hankel operators on the Bergman space. Such concepts on the Hardy space were introduced by Barria and Halmos [4] and by Feintuch [6], [7]. The importance of this notion is that it associates with a class of operators a Toeplitz operator and with a class of operators a Hankel operator where the original operators are not even Toeplitz or Hankel and hence a symbol calculus is obtained.

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disc in the complex plane $\mathbb{C}$. Let $dA(z)$ be the area measure on $\mathbb{D}$ normalised so that the area of the disc $\mathbb{D}$ is 1. In rectangular and polar coordinates,

$$dA(z) = \frac{1}{\pi} dxdy = \frac{1}{\pi} rdrd\theta.$$

Let $L^2(\mathbb{D}, d\mathbb{A})$ denote the Hilbert space of complex-valued, absolutely square-integrable, Lebesgue measurable functions f on $\mathbb{D}$ with the inner product

$$\langle f, g \rangle = \int f(z)\overline{g(z)} dA(z).$$

Let $L^\infty(\mathbb{D}, d\mathbb{A})$ denote the Banach space of Lebesgue measurable functions f on $\mathbb{D}$ with

$$\|f\|_\infty = \text{esssup}\{|f(z)| : z \in \mathbb{D}\} < \infty.$$

Let $H^\infty(\mathbb{D})$ be the space of bounded analytic functions on $\mathbb{D}$. Let $L^2_a(\mathbb{D})$(the subscript "a" stands for analytic) be the subspace of $L^2(\mathbb{D}, d\mathbb{A})$ consisting of analytic functions. The space $L^2_a(\mathbb{D})$ is called the Bergman space. Since point evaluation at $z \in \mathbb{D}$ is a bounded linear functional on the Hilbert space $L^2_a(\mathbb{D})$, the Riesz representation theorem implies that there exists a unique function K_z in $L^2_a(\mathbb{D})$ such that

$$f(z) = \int_{\mathbb{D}} f(w)\overline{K_z(w)} dA(w)$$

for all f in $L^2_a(\mathbb{D})$. Let $K(z, w)$ be the function on $\mathbb{D} \times \mathbb{D}$ defined by

$$K(z, w) = \overline{K_z(w)}.$$

The function $K(z, w)$ is thus the reproducing kernel for the Bergman space $L^2_a(\mathbb{D})$ and is called the Bergman kernel. It can be shown that the sequence of functions $\{e_n(z)\} = \{\sqrt{n+1} z^n\}_{n \geq 0}$ form the standard orthonormal basis for $L^2_a(\mathbb{D})$ and $K(z, w) = \sum_{n=1}^\infty e_n(z)\overline{e_n(w)}$. The Bergman kernel is independent of the choice of orthonormal basis and $K(z, w) = \frac{1}{(1 - z\bar{w})^2}$. Since $L^2_a(\mathbb{D})$ is a closed subspace of $L^2(\mathbb{D}, d\mathbb{A})$(see [3]); there exists an orthogonal projection P from $L^2(\mathbb{D}, d\mathbb{A})$ onto $L^2_a(\mathbb{D})$. For $\phi \in L^\infty(\mathbb{D})$, we define the Toeplitz operator T_ϕ on $L^2_a(\mathbb{D})$ by $T_\phi f = P(\phi f), f \in L^2_a(\mathbb{D})$ and the little Hankel operator S_ϕ on L^2_a

is defined by $S_\phi f = PJ(\phi f)$ where $J : L^2(\mathbb{D}, d\mathbb{A}) \to L^2(\mathbb{D}, d\mathbb{A})$ is such that $Jf(z) = f(\bar{z})$.

Let $\mathbb{T}$ denote the unit circle in $\mathbb{C}$. Let $L^\infty(\mathbb{T})$ be the space of essentially bounded measurable functions on $\mathbb{T}$. In the sequel $L_{\tilde{\phi}}$ denotes an operator in $\mathcal{L}(L_a^2(\mathbb{D}))$ with a classical Toeplitz matrix with respect to the standard orthonormal basis of $L_a^2(\mathbb{D})$ with symbol $\widetilde{\phi}$ in $L^\infty(\mathbb{T})$, i.e., $\langle L_{\widetilde{\phi}} e_j, e_i\rangle = \widehat{\widetilde{\phi}}(i-j)$ where $\widehat{\widetilde{\phi}}(k)$ is the kth Fourier coefficient of $\tilde{\phi}$. Similarly $B_{\tilde{\phi}}$ denotes an opertor in $\mathcal{L}(L_a^2(D))$ with a classical Hankel matrix with respect to the standard orthonormal basis of $L_a^2(\mathbb{D})$ with symbol $\tilde{\phi}$ in $L^\infty(\mathbb{T})$, i.e., $\langle B_{\tilde{\phi}} e_j, e_i\rangle = \widehat{\tilde{\phi}}(i+j)$ where $\hat{\tilde{\phi}}(k)$ is the kth Fourier coefficient of $\tilde{\phi}$. It is important to note that if $\tilde{\phi}$ and $\tilde{\psi}$ belong to $L^\infty(\mathbb{T})$ then $L_{\widetilde{\phi\psi}} = B_{\tilde{\phi}^+} B_{\tilde{\psi}} + L_{\tilde{\phi}} L_{\tilde{\psi}}$ and $B_{\widetilde{\phi\psi}} = L_{\tilde{\phi}^+} B_{\tilde{\psi}} + B_{\tilde{\phi}} L_{\tilde{\psi}}$ where $\tilde{\phi}^+(\mathbf{z}) = \tilde{\phi}(\bar{\mathbf{z}})$. Moreover if $\phi = \sum_{k=0}^\infty \hat{\phi}(k) z^k$ belongs to $L^\infty(\mathbb{D})$ then $\tilde{\phi}$ will denote the function $\sum_{k=0}^\infty \hat{\phi}(k) e^{ik\theta}$ in $L^\infty(\mathbb{T})$.

Let $\mathcal{M}$ denote the maximal ideal space of H^∞ and $\mathcal{C}(\mathcal{M})$ denote the space of continuous functions on $\mathcal{M}$. The following algebras are identical [10].

1. $\mathcal{C}(\mathcal{M})$; 2. The supremum norm closure of the algebra generated by $H^\infty(\mathbb{D})$ and $\overline{H^\infty(\mathbb{D})}$; 3. The supremum norm closure of the algebra generated by the bounded harmonic functions on $\mathbb{D}$. Let $\mathcal{A}$ be the set $\{\sum_{k=1}^m \bar{\phi}_k \psi_k : \phi_k, \psi_k \in H^\infty(\mathbb{D})\}$. The set $\mathcal{A}$ is dense in $\mathcal{C}(\mathcal{M})$. If $\phi \in \mathcal{A}$ and $\phi = \sum_{k=1}^m \bar{\phi}_k \psi_k$ then $\tilde{\phi}$ will denote the function $\sum_{k=1}^m \bar{\tilde{\phi}}_{\mathbf{k}} \tilde{\psi}_{\mathbf{k}}$ in $L^\infty(\mathbb{T})$.

Note also that if $\phi \in H^\infty(\mathbb{D})$ then $T_\phi = D_1 L_{\tilde{\phi}} D_2$ where D_1 is the operator on $L_a^2(\mathbb{D})$ given by $D_1 e_j = \frac{1}{\sqrt{j+1}} e_j$ and D_2 the operator on $L_a^2(\mathbb{D})$ given by $D_2 e_j = \sqrt{j+1} e_j$. The operator D_1 is bounded but D_2 is an unbounded operator. Similarly if $\phi \in \overline{H^\infty(\mathbb{D})}$ then $T_\phi = D_2 L_{\tilde{\phi}} D_1$. Let S_ϕ be the little Hankel operator on $L_a^2(\mathbb{D})$. If $\phi \in H^\infty$ then $S_\phi = 0$. For $\phi \in \overline{H^\infty}$ the operator $S_\phi = D_2 B_{\tilde{\psi}} D_2$ where $\tilde{\psi}(\mathbf{e}^{\mathbf{i}\theta}) = \sum_{\mathbf{k=0}}^\infty \frac{\mathbf{1}}{\mathbf{k+1}} \hat{\phi}(-\mathbf{k}) \mathbf{e}^{-\mathbf{ik}\theta}$ i.e., $\tilde{\psi}$ is the convolution on the circle of $\tilde{\phi} = \sum_{\mathbf{k=0}}^\infty \hat{\phi}(-\mathbf{k}) \mathbf{e}^{-\mathbf{ik}\theta}$ with the function $\tilde{\phi}_{\mathbf{1}} = \sum_{\mathbf{k=0}}^\infty \frac{\mathbf{1}}{\mathbf{k+1}} \mathbf{e}^{-\mathbf{ik}\theta}$, and $B_{\tilde{\psi}}$ is the operator in $\mathcal{L}(L_a^2(\mathbb{D}))$ having a classical Hankel matrix with symbol $\tilde{\psi} \in \mathbf{L}^\infty(\mathbb{T})$. Now consider the operator $W : L_a^2(\mathbb{D}) \longrightarrow L_a^2(\mathbb{D})$ such that $W e_j = e_{j+1}, j \geq 0$. It can easily be checked that $(D_1 W D_2)^n = D_1 W^n D_2$ and $\|D_1 W^n D_2\| = 1$. Similarly one can also check that $\|D_1 W^{*n} D_2\| = \|(D_1 W^* D_2)^n\| = \sqrt{n+1}$. Let $R : L_a^2(\mathbb{D}) \longrightarrow L_a^2(\mathbb{D})$ such that $Rf(z) = \frac{f(z)-f(0)}{z}$. Let $S = T_z$, the Bergman shift operator. It is to be noted that $RS = I, R = D_1 W^* D_2$ and $S = D_1 W D_2$. Further observe that $R = T_{\bar{\mathbf{z}}} + K$ where K is a compact operator on $L_a^2(\mathbb{D})$ and for all $\phi \in L^\infty(\mathbb{D}), T_\phi - T_z^* T_\phi T_z = T_\phi - T_{\bar{z}} T_\phi T_z = T_{\phi - \bar{z}\phi z} = T_{(1-|z|^2)\phi(z)}$ is a compact operator (see [5]) in $\mathcal{L}(L_a^2(\mathbb{D}))$.

The following result will be used often in the remaining part of our discussion which explains when two Toeplitz operators with bounded harmonic symbols commute.

It is shown in [2] that if ϕ and ψ are bounded harmonic on $\mathbb{D}$ then $T_\phi T_\psi = T_\psi T_\phi$ if and only if

1. the function ϕ and ψ are both analytic on $\mathbb{D}$; or

2. the functions $\bar{\phi}$ and $\bar{\psi}$ are both analytic on $\mathbb{D}$; or

3. there exist constants $a, b \in \mathbb{C}$, not both zero, such that $a\phi + b\psi$ is constant on $\mathbb{D}$.

The idea of an asymptotic Toeplitz operator and asymptotic Hankel operator on the Hardy space was introduced and first studied in [4] and [7] respectively. The significance of these results is that it gives distance formulae which can be viewed as operator theoretic analogues of results [7] of Nehari, Hartman and Adamjan, Arov and Krein. In this paper we introduce the concept of asymptotic Toeplitz and Hankel operators on the Bergman space. We show that every operator in the algebra generated by Toeplitz operators with symbols in $\mathcal{A}$ are both asymptotic Toeplitz and asymptotic Hankel operators. If $\phi \in \mathcal{A}$ then the little Hankel operator S_ϕ is an asymptotic Hankel operator. Further if $\phi, \psi \in \mathcal{A}$ then $T_\phi S_\psi$ and $S_\phi S_\psi$ are asymptotic Toeplitz operators. We find necessary and sufficient conditions on $T \in \mathcal{L}(L_a^2(\mathbb{D}))$ such that T is weakly asymptotic Toeplitz and weakly asymptotic Hankel. We also relate the concept of asymptotic Toeplitz and Hankel operators on Hardy and Bergman spaces and established relation between Toeplitz operators and little Hankel operators in some asymptotic sense.

Our approach is similar to that of Barria and Halmos [4] and Feintuch [7], [6]. Most of the results by Feintuch and Barria and Halmos were obtained by examining the corresponding Toeplitz and Hankel matrices with respect to the standard orthonormal basis on $H^2(\mathbb{T})$. On the Bergman space $L_a^2(\mathbb{D})$, the Toeplitz and Hankel operators do not have nice matrices and many of the techniques used to study these operators on the Hardy space do not seem to work in this context. In this paper we have restricted our symbol space to the algebra generated by the bounded harmonic functions on $\mathbb{D}$ rather than $L^\infty(\mathbb{D})$. Functions in $L^\infty(\mathbb{T})$ correspond, via the Poisson integral, to bounded harmonic functions on $\mathbb{D}$, so perhaps the restriction to consideration only of Toeplitz and Hankel operators with bounded harmonic symbols is natural.

Toeplitz and Hankel operators on the Hardy space have been studied intensively for a long time and have widespread applications to both system theory and approximation theory. Both these classes of operators can be represented as "pieces" of a multiplication operator and in control theory these representations correspond to frequency domain and time domain representations of a time invariant system. While the Toeplitz operator is the system the associated Hankel operator gives a lot of information about the system [11]. Hankel operators play an essential role in the theory of Toeplitz operators [11] and many problems about Toeplitz operators can also be formulated in terms of Hankel operators and vice-versa. These operators on the Bergman space is less understood and the results are very often strikingly different to that on the Hardy space. Toeplitz operators with bounded harmonic symbols on $L^2_a(\mathbb{D})$ behave more like Toeplitz operator on the Hardy space [3]. In the Bergman space setting, there are two very different notions of Hankel operator, the big and little Hankel operators. Little Hankel operators on the Bergman space behave more like Hankel operators on the Hardy space. However no strong connection between Toeplitz operators and little Hankel operators on $L^2_a(\mathbb{D})$ is established yet. In this paper we relate Toeplitz operators on $L^2_a(\mathbb{D})$ and little Hankel operators on $L^2_a(\mathbb{D})$ in some asymptotic sense.

Feintuch in [6], [7] obtained the operator theoretic analogues of the distance formulae of Adamjan-Arov-Krein, Hartman and Nehari (see [11]). These distance formulae play an important role [7] in system theory and approximation theory. In this paper we relate the concept of asymptotic Toeplitz and Hankel operators on Hardy and Bergman spaces via unitary maps in a restricted sense. Thus for this subset of the set of asymptotic Hankel operators on the Bergman space we have similar distance formulae valid. We have also shown that the more general distance formulae as obtained by Feintuch [7] is not possible for all asymptotic Hankel operators on the Bergman space.

2. Asymptotic Toeplitz Operators

In this section we shall proceed to make a suitable definition of asymptotic Toeplitz operator on the Bergman space. On the Hardy space one considers the backward shift operator and the forward shift operator to define the concept of asymptotic

Toeplitz operators. In case of the Bergman space since $\|R^n\| = \sqrt{n+1}$ and $\|S^n\| = 1$ the sequence $\{R^nTS^n\}$ may not be uniformly bounded for all operators T in $\mathcal{L}(L^2_a(\mathbb{D}))$. We define an asymptotic Toeplitz operator in the following way.

Definition 2.1 — An operator $T \in \mathcal{L}(L^2_a(\mathbb{D}))$ is strongly asymptotic Toeplitz if the sequence $\{W^{*n}D_2TD_1W^n\} = \{D_2R^nTS^nD_1\}$ is strongly convergent. Similarly we define the operator T is weakly(uniformly) asymptotic Toeplitz if the sequence $\{W^{*n}D_2TD_1W^n\}$ is weakly(uniformly) convergent.

The simplest examples of asymptotic Toeplitz operators are Toeplitz operators in $\mathcal{L}(L^2_a(\mathbb{D}))$ with symbols in $H^\infty(\mathbb{D})+\overline{\mathbb{H}^\infty(\mathbb{D})}$. In case $\phi \in H^\infty$, $W^{*n}D_2T_\phi D_1W^n = L_{\tilde{\phi}}$, $\tilde{\phi} \in \mathbf{H}^\infty(\mathbb{T})$ for all n. In case $\phi \in \overline{H^\infty}$, the operator $D_2T_\phi D_1$ is bounded since the matrix of T_ϕ is upper triangular with respect to the standard orthonormal basis on $L^2_a(\mathbb{D})$ and $|\langle D_2T_\phi D_1e_j, e_i\rangle| = |\sqrt{\frac{i+1}{j+1}}\langle T_\phi e_j, e_i\rangle| \leq |\langle T_\phi e_j, e_i\rangle|$ if $j \geq i$. Hence $\{W^{*n}D_2T_\phi D_1W^n\}$ is uniformly bounded. It is also easy to check that $\{W^{*n}D_2T_\phi D_1W^ne_j\}$ converges to $L_{\tilde{\phi}}e_j$. Therefore the operator T_ϕ is strongly asymptotic Toeplitz. We shall show in Theorem 2.3 that if an operator T in $\mathcal{L}(L^2_a(\mathbb{D}))$ is asymptotic Toeplitz (strong, weak or uniform sense) the limit is an operator of the form $L_{\tilde{\psi}}$ where $L_{\tilde{\psi}}$ is the operator on $L^2_a(\mathbb{D})$ with a classical Toeplitz matrix with symbol $\tilde{\psi} \in \mathbf{L}^\infty(\mathbb{T})$. If T is an asymptotic Toeplitz operator (strong, weak or uniform sense) we define the symbol of T as $\tilde{\psi}$ if $\{W^{*n}D_2TD_1W^n\}$ converges to $L_{\tilde{\psi}}$ and we denote the symbol of T as $\sigma(T)$. Henceforth whenever we say an operator is asymptotic Toeplitz we mean it in the strong sense unless otherwise stated. Let R and S be as defined before. First we shall verify what condition on the entries of the matrix of the operator T in $\mathcal{L}(L^2_a(\mathbb{D}))$ imply the uniform boundedness of the sequence of operators $\{R^nTS^n\}$. The following theorem gives a sufficient condition.

***Theorem* 2.2** — *Let (t_{ij}) be the (i,j)th entry of the matrix of the operator T. If sup$_n \sum_i \sum_j (n+1)|t_{i+n,j+n}|^2 < \infty$ then $\{R^nTS^n\}$ is uniformly bounded.*

PROOF: It is easy to see that for all $i, j \geq 0$,

$$\langle R^nTS^ne_j, e_i\rangle = \frac{\sqrt{j+1}}{\sqrt{i+1}}\frac{\sqrt{i+n+1}}{\sqrt{j+n+1}}t_{i+n,j+n}$$

where (t_{ij}) is the (i,j)th entry of the matrix of ther operator T. The square of the

Hilbert Schmidt norm of the operator R^nTS^n is equal to

$$\sum_i \sum_j \frac{j+1}{i+1}\frac{i+n+1}{j+n+1}|t_{i+n,j+n}|^2.$$

The last sum is finite if $\sum_i \sum_j (n+1)|t_{i+n,j+n}|^2 < \infty$. The theorem follows. □

If $\phi \in H^\infty, R^nT_\phi S^n = T_\phi$ for all n. Hence $\|R^nT_\phi S^n\|$ is uniformly bounded by $\|T_\phi\|$. If $\phi \in \overline{H^\infty}$ and $\phi(z) = \sum_{k=0}^\infty \hat{\phi}(-k)\bar{z}^k$, then $\{R^nT_\phi S^n\}$ is uniformly bounded if $\phi \in \overline{H^\infty \cap \mathcal{D}}$ where $\mathcal{D}$ is the Dirichlet space consisting of all functions $f(z) = \sum_{n=0}^\infty a_n z^n$ analytic in the disc for which $\|f\|_{\mathcal{D}}^2 = \sum_{n=0}^\infty (n+1)|a_n|^2 < \infty$. It is not difficult to verify that if $\phi \in \overline{H^\infty \cap \mathcal{D}}$, then the sequence $\{R^nT_\phi S^n\}$ converges strongly to $T_\phi + C$ where C is a bounded operator on $L_a^2(\mathbb{D})$. The following theorem shows that if the sequence $\{W^{*n}D_2TD_1W^n\}$ converges in (weak, strong, uniform sense) the limit is always an operator on $L_a^2(\mathbb{D})$ with a classical Toeplitz matrix with respect to the standard orthonormal basis of $L_a^2(\mathbb{D})$.

***Theorem* 2.3** — *If $T \in \mathcal{L}(L_a^2(\mathbb{D}))$ is asymptotic Toeplitz then the limit M is an operator of the form $L_{\tilde{\psi}}, \tilde{\psi} \in \mathbf{L}^\infty(\mathbb{T})$.*

PROOF: Suppose $T \in \mathcal{L}(L_a^2(\mathbb{D}))$ such that $\{W^{*n}D_2TD_1W^n\}$ converges strongly to M in $\mathcal{L}(L_a^2(\mathbb{D}))$. Then $\langle Me_{j+1}, e_{i+1}\rangle = \lim_{n\to\infty}\langle W^{*n+1}D_2TD_1W^{n+1}e_j, e_i\rangle = \langle Me_j, e_i\rangle, i, j \geq 0$. Thus $M = L_{\tilde{\psi}}$ where $\tilde{\psi} \in \mathbf{L}^\infty(\mathbb{T})$. □

The following theorem shows that every compact operator K in $\mathcal{L}(L_a^2(\mathbb{D}))$ for which D_2KD_1 is a bounded operator on $L_a^2(\mathbb{D})$ is an asymptotic Toeplitz operator with zero as its symbol.

***Theorem* 2.4** — *If K is a compact operator in $\mathcal{L}(L_a^2(\mathbb{D}))$ such that D_2KD_1 is a bounded operator on $L_a^2(\mathbb{D})$ then $\{W^{*n}D_2KD_1W^n\}$ converges strongly to* 0.

PROOF: The sequence of operators $\{W^{*n}D_2KD_1W^n\}$ is uniformly bounded since D_2KD_1 is a bounded operator on $L_a^2(\mathbb{D})$. Since $\{e_n\} = \{\sqrt{n+1}z^n\}$ form an orthonormal basis for $L_a^2(\mathbb{D})$ hence $e_n \to 0$ weakly and for $j \geq 0$, $Ke_{n+j} \to 0$. Therefore $\frac{\sqrt{j+n+1}}{\sqrt{j+1}}KS^ne_j \to 0$. Thus $(\frac{1}{\sqrt{n+1}}R^n)\frac{\sqrt{j+n+1}}{\sqrt{j+1}}KS^ne_j \to 0$ because $\{A_n\} = \{\frac{1}{\sqrt{n+1}}R^n\}$ is uniformly bounded. It follows therefore that $\frac{\sqrt{j+n+1}}{\sqrt{n+1}}R^nKS^nD_1e_j \to 0$. Therefore $W^{*n}D_2KD_1W^n \to 0$ in the strong operator topology. □

We shall now give a necessary and sufficient condition for the operator T in $\mathcal{L}(L^2_a(\mathbb{D}))$ to be asymptotic Toeplitz in the weak sense.

***Theorem* 2.5** — *Let $[t_{ij}]_{i,j=0}^{\infty}$ be the matrix representation of the bounded linear operator T on $L^2_a(\mathbb{D})$ with respect to the standard orthonormal basis. Suppose further that $D_2TD_1 \in \mathcal{L}(L^2_a)$. Then $\{W^{*n}D_2TD_1W^n\}$ is uniformly bounded and T is weakly asymptotic Toeplitz if and only if for each $-\infty < j < \infty$ the sequence $\{t_{i,i+j}\}$ converges and the sequence of limits $\{t_{-j}\}_{j=-\infty}^{\infty}$ is the sequence of Fourier coefficients of some $\tilde{\phi} \in \mathbf{L}^{\infty}(\mathbb{T})$.*

PROOF: The necessity is obvious. For the sufficiency, since $D_2TD_1 \in \mathcal{L}(L^2_a)$, the sequence $\{W^{*n}D_2TD_1W^n\}$ is uniformly bounded and

$$\langle W^{*n}D_2TD_1W^n e_j, e_i\rangle$$

$$= \frac{\sqrt{i+n+1}}{\sqrt{j+n+1}}\langle Te_{j+n}, e_{i+n}\rangle = \frac{\sqrt{i+n+1}}{\sqrt{j+n+1}} t_{i+n,j+n}.$$

By a change of index this can be written as $\frac{\sqrt{n+1}}{\sqrt{n+(j-i)+1}} t_{n,n+j-i}$ which by hypothesis converges to $t_{-(j-i)}$ where $t_{-(j-i)}$ are the Fourier coefficients of a function in $L^{\infty}(\mathbb{T})$. □

On the Hardy space a similar characterisation exists for describing weak asymptotic Toeplitz operators.

3. Relation with Asymptotic Hardy Toeplitz Operators

Let U be a mapping from the Hardy space $H^2(\mathbb{D})$(for definition see [11]) into $L^2_a(\mathbb{D})$ defined by $Uz^n = \sqrt{n+1}z^n$. Then the operator U is unitary. In [4], Barria and Halmos defined asymptotic Toeplitz operators as follows. An operator $T \in \mathcal{L}(H^2(\mathbb{D}))$ is asymptotic Hardy Toeplitz if and only if $\{(\tilde{\mathbf{S}}^*)^n T\tilde{\mathbf{S}}^n\}$ converges in the strong operator topology in $\mathcal{L}(H^2(\mathbb{D}))$ where $\tilde{\mathbf{S}}$ is the unilateral shift on $H^2(\mathbb{D})$. In this work we have defined asymptotic Bergman Toeplitz operators (to mark the difference we renamed them) on the Bergman space $L^2_a(\mathbb{D})$. Since the operator U is a unitary operator from $H^2(\mathbb{D})$ into $L^2_a(\mathbb{D})$, the operator $T \in \mathcal{L}(L^2_a(\mathbb{D}))$ if and only if $U^*TU \in \mathcal{L}(H^2(\mathbb{D}))$. Therefore the question that arises at this point is whether U^*TU is an asymptotic Hardy Toeplitz if $T \in \mathcal{L}(L^2_a(\mathbb{D}))$ is an asymptotic Bergman

Toeplitz operator. The following theorem gives an affirmative answer in the case the operator T is lower triangular. Notice that $U^*T_zU - \widetilde{S} = \widetilde{S}\ \text{diag}\left(\sqrt{\frac{n+1}{n+2}} - 1\right)$ is a compact operator in $\mathcal{L}(H^2(\mathbb{D}))$.

***Theorem* 3.1** — *If the operator $T \in \mathcal{L}(L_a^2(\mathbb{D}))$ is an asymptotic Bergman Toeplitz operator in the weak sense and the operator T has a lower triangular matrix representation with respect to the standard orthonormal basis of $L_a^2(\mathbb{D})$ then U^*TU is an asymptotic Hardy Toeplitz operator in the weak sense.*

PROOF: Suppose $T \in \mathcal{L}(L_a^2(\mathbb{D}))$ is an asymptotic Bergman Toeplitz operator in the weak sense. Then $\{W^{*n}D_2TD_1W^n\}$ converges weakly. It is to observe that $\langle W^{*n}D_2TD_1W^n e_j, e_i\rangle = \sqrt{\frac{i+n+1}{j+n+1}}\langle Te_{j+n}, e_{i+n}\rangle$ for all $i, j \geq 0$. Again $\{(\tilde{\mathbf{S}}^*)^n U^*TU\tilde{\mathbf{S}}^n\}$ is uniformly bounded and $\langle(\tilde{\mathbf{S}}^*)^n U^*TU\tilde{\mathbf{S}}^n z^j, z^i\rangle = \langle U^*TUz^{j+n}, z^{i+n}\rangle = \langle T(\sqrt{j+n+1}z^{j+n}), \sqrt{i+n+1}z^{i+n}\rangle = \langle Te_{j+n}, e_{i+n}\rangle$ for all $i, j \geq 0$. Hence the result follows. □

If the operator T belong to $\mathcal{L}(L_a^2(\mathbb{D}))$ and $T = T_\phi$ then $U^*T_\phi U$ belong to the essential commutant [5] of the unilateral shift on $H^2(\mathbb{T})$. But whether $U^*T_\phi U$ belong to the Toeplitz algebra in $\mathcal{L}(H^2)$ is not known.

4. ASYMPTOTIC HANKEL OPERATORS

We now define for an operator T acting on the Hilbert space $L_a^2(\mathbb{D})$, a sequence of operators on $L_a^2(\mathbb{D})$ which when it converges (strong, weak, uniform sense),converges to an operator $H(T)$ in $\mathcal{L}(L_a^2(\mathbb{D}))$ which has a classical Hankel matrix with respect to the standard orthonormal basis of $L_a^2(\mathbb{D})$. In case when T is a Toeplitz operator T_ϕ with symbol $\phi \in \mathcal{A}$ on the space $L_a^2(\mathbb{D})$, we have shown the operator $H(T)$ on $L_a^2(\mathbb{D})$ has a classical Hankel matrix associated with the symbol $\tilde{\phi} \in \mathbf{L}^\infty(\mathbb{T})$. We give a necessary and sufficient condition for the existence of $H(T)$.

The truncation projection on $L_a^2(\mathbb{D})$ will be denoted by $P_i, 0 \leq i \leq \infty$. If the function $f = \sum_{k=0}^\infty c_k z^k$ belongs to $L_a^2(\mathbb{D})$ then $P_i f = \sum_{k=0}^i c_k z^k$. These are orthogonal projections on $L_a^2(\mathbb{D})$ which converges strongly to the identity I on $L_a^2(\mathbb{D})$. On the subspace $P_n L_a^2(\mathbb{D})$ spanned by $\{e_0, e_1, e_2, \cdots, e_n\}$, define the operator J_n by $J_n e_i = \frac{\sqrt{i+1}}{\sqrt{n-i+1}} e_{n-i}, 0 \leq i \leq n$. We extend J_n to $L_a^2(\mathbb{D})$ by defining it to be zero on $(I - P_n)L_a^2(\mathbb{D})$ and we will denote the operator on $L_a^2(\mathbb{D})$ by J_n as well. The operator J_n is neither self adjoint nor unitary. It can also easily be checked that $J_n = J_nP_n = P_nJ_n, J_n^* = J_n^*P_n = P_nJ_n^*$ and $S^{n+1}R^{n+1} = I - P_n$. Moreover, $J_n^2e_i = J_n^{*2}e_i = P_ne_i$ for all i. It is to be noted that $J_ne_i = D_1M_nD_2e_i$, for all i where $D_1e_i = \frac{1}{\sqrt{i+1}}e_i$, $M_ne_i = e_{n-i}, 0 \leq i \leq n(= 0, otherwise)$ and $D_2e_i = \sqrt{i+1}e_i$. Further $\|J_n\| = \sqrt{n+1}, \|S^{n+1}\| = 1$. Let $\mathcal{C}$ be the weakly closed algebra of bounded linear operators in $\mathcal{L}(L_a^2(\mathbb{D}))$ which have a lower triangular matrix representation with respect to the standard orthonormal basis. The set $\mathcal{K}$ will denote the uniformly closed two-sided ideal of compact operators on $L_a^2(\mathbb{D})$. The algebra $\mathcal{C} + \mathcal{K}$ is a uniformly closed algebra [1]. It is characterized as $\{T \in \mathcal{L}(L_a^2) : \|P_iT(I - P_i)\| \to 0\}$ and is also known as the algebra of quasi-triangular operators with respect to the sequence $\{P_i\}$.

Let $\mathcal{H}$ be a Hilbert space with $\{y_1, y_2, \cdots y_n, \cdots\}$ as the standard orthonormal basis. We denote by $\mathcal{H} \bigotimes \mathcal{H}$ their tensor product. The vectors $\{y_i \otimes y_j\}$ forms an orthonormal basis for $\mathcal{H} \bigotimes \mathcal{H}$. The norm of $x \otimes y = \sum_i \sum_j c_{ij}(y_i \otimes y_j)$ is equal to $(\sum_i \sum_j |c_{ij}|^2)^{\frac{1}{2}}$. Let A and B be two elements of $\mathcal{L}(\mathcal{H})$ with matrices as $[a_{jk}]$ and $[b_{jk}]$ respectively with respect to the standard orthonormal basis. Then $[a_{jk}b_{jk}]$ is the matrix of a bounded linear operator. Let P' denote the orthogonal projection of $\mathcal{H} \bigotimes \mathcal{H}$ onto $\mathcal{E}$, the space spanned by the orthonormal set of vectors $\{y_1 \otimes y_1, y_2 \otimes y_2, \cdots\}$. Then $P'(A \otimes B)P'|_{\mathcal{E}}$ is the operator having the matrix $[a_{jk}b_{jk}]$ (for reference see [9]). The operator C whose (j,k)th matrix entry is $a_{jk}b_{jk}$ is referred to as the Schur product of the operators A and B. We shall denote C by $C = A \hat{\bigotimes} B$ where $\hat{\bigotimes}$ denotes the Schur product. Let A be an operator from $L_a^2(\mathbb{D})$ into $L_a^2(\mathbb{D})$ defined by $\langle Ae_j, e_i \rangle = \frac{i+1}{i+j+2}$. Thus $Ae_j = \sum_{i=0}^{\infty} \langle Ae_j, e_i \rangle e_i = \sum_{i=0}^{\infty} \frac{i+1}{i+j+2} e_i, j \geq 0$. The operator A is a bounded linear operator on $L_a^2(\mathbb{D})$.

Let $T \in \mathcal{L}(L_a^2(\mathbb{D}))$, define a sequence $\{H_n(T)\}$ of operators on $L_a^2(\mathbb{D})$ by

$$H_n(T) = D_2J_nTS^{n+1}D_1 = M_nD_2TD_1W^{n+1}, n = 1, 2, 3, \cdots.$$

Thus $\text{rank}H_n(T) \leq n + 1$ since $\text{rank}M_n = n + 1$. The sequence $\{H_n(T)\}$ will be referred to as the Hankel sequence associated with the operator T. We shall now compute $H_n(T)$ when T is a Toeplitz operator T_ϕ with the symbol ϕ in $H^\infty(\mathbb{D}) + \overline{\mathbb{H}^\infty(\mathbb{D})}$. Let $\phi = \chi + \psi$ where $\chi \in H^\infty$ and $\psi \in \overline{H^\infty}$. Then $H_n(T_\phi) = M_nD_2T_\phi D_1W^{n+1} = M_nD_2T_\chi D_1W^{n+1} + M_nD_2T_\psi D_1W^{n+1}$. It is to note that $M_nD_2T_\chi D_1W^{n+1} = 0$ for all n since $\chi \in H^\infty$, the operator M_n is self adjoint and $M_ne_i = 0$ for all $i > n$. Now let $\psi(z) = \sum_{k=0}^{\infty} \hat{\psi}(-k)\bar{z}^k$. We see that $|\langle M_nD_2T_\psi D_1W^{n+1}e_j, e_i \rangle| = |\frac{\sqrt{n-i+1}}{\sqrt{j+n+2}}\hat{\psi}(-(j+i+1))|, 0 \leq i \leq n$(=0, otherwise). Further $|\langle D_2 T_\psi D_1 e_j, e_i\rangle| \leq |\langle T_\psi e_j, e_i\rangle|$ since the matrix of T_ψ is upper triangular. Therefore the sequence $\{M_n D_2 T_\psi D_1 W^{n+1}\}$ is uniformly bounded. It is easy to check that $\{M_n D_2 T_\psi D_1 W^{n+1} e_j\}$ converges to $B_{\tilde{\psi}}$ where $B_{\tilde{\psi}}$ is the operator on $L^2_a(\mathbb{D})$ having a classical Hankel matrix with respect to the standard orthonormal basis of $L^2_a(\mathbb{D})$ with symbol $\tilde{\psi}$ in $L^\infty(\mathbb{T})$. Thus the sequence $\{M_n D_2 T_\phi D_1 W^{n+1}\}$ converges strongly to $B_{\tilde{\phi}}$ since $B_{\tilde{\chi}} = 0$.

Definition 4.1 — An operator $T \in \mathcal{L}(L^2_a(\mathbb{D}))$ is strongly asymptotic Hankel if the sequence $\{M_n D_2 T D_1 W^{n+1}\}$ converges strongly. Similarly we define $T \in \mathcal{L}(L^2_a(\mathbb{D}))$ to be weakly (uniformly) asymptotic Hankel if $\{M_n D_2 T D_1 W^{n+1}\}$ converges weakly (uniformly).

When $\{H_n(T)\}$ converges weakly we shall denote its limit by $H(T)$. We shall show later that when $H(T)$ exists, it is an operator $B_{\tilde{\phi}}$ on the space $L^2_a(\mathbb{D})$ whose matrix with respect to the standard orthonormal basis of $L^2_a(\mathbb{D})$ is a classical Hankel matrix with symbol $\tilde{\phi} \in \mathbf{L}^\infty(\mathbb{T})$. If $\{H_n(T)\}$ converges weakly to $B_{\tilde{\phi}}$ we shall define the asymptotic Hankel symbol of T as $\tilde{\phi}$ and we shall denote it as $\sigma^*(T) = \tilde{\phi}$. Except where otherwise stated whenever we say asymptotic Hankel we shall mean it in the strong sense.

***Theorem* 4.2** — *If $\phi \in \mathcal{A}$, then T_ϕ is a strongly asymptotic Hankel operator with symbol $\tilde{\phi}$.*

PROOF: If $\phi \in H^\infty$ then $M_n D_2 T_\phi D_1 W^{n+1} = 0$ for all n. If $\overline{\phi} \in \overline{\mathbf{H}^\infty}$ then $\{M_n D_2 T_{\overline{\phi}} D_1 W^{n+1}\}$ converges to $B_{\tilde{\overline{\phi}}}$ in the strong operator topology (follows from the above discussion). If $\phi \in \mathcal{A}$ and $\phi = \sum_{k=1}^m \phi_k \overline{\psi_{\mathbf{k}}}$ where ϕ_k and ψ_k belong to H^∞ then $M_n D_2 T_\phi D_1 W^{n+1} = \sum_{k=1}^m M_n D_2 T_{\overline{\psi_{\mathbf{k}}}} T_{\phi_k} D_1 W^{n+1} = \sum_{k=1}^m M_n D_2 T_{\overline{\psi_{\mathbf{k}}}} D_1 W^{n+1} L_{\tilde{\phi}_{\mathbf{k}}}$ which converges to $\sum_{k=1}^m B_{\tilde{\overline{\psi_{\mathbf{k}}}}} L_{\tilde{\phi}_{\mathbf{k}}} = B_{\tilde{\phi}}$ in the strong operator topology. □

The following theorem tells us when the sequence of operators $\{J_n T S^{n+1} \hat{\bigotimes} A\}$ is uniformly bounded which will be useful in relating Toeplitz operators on $L^2_a(\mathbb{D})$ to little Hankel operators.

***Theorem* 4.3** — *If (t_{ij}) is the (i,j)th entry of the matrix of the operator $T \in L^2_a(\mathbb{D})$ and $\sup_n \sum_i \sum_j (n+1)|t_{n-i,j+n+1}|^2 < \infty$ then $\{J_n T S^{n+1}\}$ is uniformly bounded and if $\phi = \chi + \psi$ where $\chi \in H^\infty$ and $\psi \in \overline{H^\infty \cap \mathcal{D}}$ then $\{(J_n T_\phi S^{n+1}) \hat{\bigotimes} A\}_{n=1}^\infty$ converges to $S_{z\phi}$ in the strong operator topology.*

PROOF: The square of the Hilbert-Schmidt norm of J_nTS^{n+1} is given by

$$\sum_{i=0}^{n}\sum_{j=0}^{\infty}\left(\frac{j+1}{i+1}\right)\left(\frac{n-i+1}{j+n+2}\right)|t_{n-i,j+n+1}|^2$$

which is finite if $\sum_i\sum_j(n+1)|t_{n-i,j+n+1}|^2 < \infty$. Now if $\phi \in H^\infty$ then $J_nT_\phi S^{n+1}\hat{\bigotimes}A = 0$ for all n. Hence $\{(J_nT_\phi S^{n+1})\hat{\bigotimes}A\}$ is uniformly bounded. If $\phi \in \overline{H^\infty \cap \mathcal{D}}$ and $\phi(z) = \sum_{k=0}^{\infty}\hat{\phi}(-k)\bar{z}^k$ then it is easy to see that $\{J_nT_\phi S^{n+1}\hat{\bigotimes}A\}$ is uniformly bounded, converges in the weak operator topology to $S_{z\phi}$ because $\langle S_{z\phi}e_j, e_i\rangle = \dfrac{\sqrt{i+1}\sqrt{j+1}}{i+j+2}\hat{\phi}(-(j+i+1)), i,j \geq 0$ and $\langle J_nT_\phi S^{n+1}\hat{\bigotimes}Ae_j, e_i\rangle = \dfrac{\sqrt{i+1}\sqrt{j+1}}{i+j+2}(\dfrac{\sqrt{n-i+1}}{\sqrt{j+n+2}})\hat{\phi}(-(j+i+1)), 0 \leq i \leq n$(=0, otherwise). Again $|\langle(J_nT_\phi S^{n+1}\hat{\bigotimes}A)e_j, e_i\rangle| \leq |\langle S_{z\phi}e_j, e_i\rangle|$. Hence $\{J_nT_\phi S^{n+1}\hat{\bigotimes}A\}$ converges to $S_{z\phi}$ in the strong operator topology since it converges weakly to the operator $S_{z\phi}$. If $\phi = \chi+\psi$ where $\chi \in H^\infty$ and $\psi \in \overline{H^\infty \cap \mathcal{D}}$ then $S_{z\chi} = 0$ and $J_nT_\chi S^{n+1} = 0$ for all n, hence $\{(J_nT_\phi S^{n+1})\hat{\bigotimes}A\}_{n=1}^{\infty} = \{(J_nT_\chi S^{n+1})\hat{\bigotimes}A+(J_nT_\psi S^{n+1})\hat{\bigotimes}A\}_{n=1}^{\infty}$ converges to $S_{z\phi}$ in the strong operator topology. □

In this way we can link a Toeplitz operator to a little Hankel operator. There exists many operators $T \in \mathcal{L}(L_a^2(\mathbb{D}))$ for which $\{(J_nTS^{n+1})\hat{\bigotimes}A\}$ does not converge even weakly as the following example shows.

Example 4.4 — Let $T \in \mathcal{L}(L_a^2(\mathbb{D}))$ be such that $Te_0 = 0, Te_i = e_{i-1}$ if i is even and $Te_i = 2e_{i-1}$ if i is odd. Then

$$J_nTS^{n+1}e_i = J_nT(\sqrt{i+1}z^{i+n+1})$$

$$= \frac{\sqrt{i+1}}{\sqrt{i+n+2}}J_nTe_{i+n+1} = \begin{cases} \dfrac{\sqrt{i+1}}{\sqrt{i+n+2}}J_n.0 = 0, & \text{if } i+n+1 = 0; \\ \dfrac{\sqrt{i+1}}{\sqrt{i+n+2}}J_ne_{i+n}, & \text{if } i+n+1 \text{ is even;} \\ 2\dfrac{\sqrt{i+1}}{\sqrt{i+n+2}}J_ne_{i+n}, & \text{if } i+n+1 \text{ is odd.} \end{cases}$$

Hence $J_nTS^{n+1}e_i = 0$ if $i > 0$. Therefore

$$J_nTS^{n+1} = \begin{cases} \dfrac{\sqrt{n+1}}{\sqrt{n+2}}P_0, & \text{if } n \text{ is odd;} \\ 2\dfrac{\sqrt{n+1}}{\sqrt{n+2}}P_0, & \text{if } n \text{ is even.} \end{cases}$$

This implies $\{(J_nTS^{n+1})\hat{\bigotimes}A\}$ does not converge. □

It is also not difficult to verify that the operator defined in Example 4.4 is not an asymptotic Hankel operator.

***Theorem* 4.5** — *If $T \in \mathcal{L}(L_a^2)$ has matrix representation $[t_{ij}]_{i,j=0}^{\infty}$ with respect to the standard orthonormal basis and $\{H_n(T)\}$ converge weakly then its limit is an operator $B_{\tilde{\phi}}$ in $\mathcal{L}(L_a^2)$ whose matrix with respect to the standard orthonormal basis of $L_a^2(\mathbb{D})$ is a classical Hankel matrix with symbol $\tilde{\phi}$ in $L^\infty(\mathbb{T})$.*

PROOF: Suppose $\{H_n(T)\}$ converges weakly to $H(T)$. Then for $i, j \geq 0$, $\langle H_n(T)e_j, e_i\rangle \to \langle H(T)e_j, e_i\rangle$ as $n \to \infty$. But

$$\langle H_n(T)e_j, e_i\rangle = \frac{\sqrt{n-i+1}}{\sqrt{j+n+2}} t_{n-i,j+n+1}, 0 \leq i \leq n (= 0, \text{otherwise}).$$

For n sufficiently large this is just the sequence $\{t_{n,n+i+j+1}\}$. Since $\{t_{n,n+i+j+1}\}$ by hypothesis converges to a number which depends only on the sum $i + j + 1$ we shall denote its limit by $t_{-(i+j+1)}$. Thus the matrix representation of $H(T)$ is $\{t_{-(i+j+1)}\}_{i,j\geq 0}$ which is of course a classical Hankel matrix with respect to the standard orthonormal basis of $L_a^2(\mathbb{D})$ with symbol in $L^\infty(\mathbb{T})$. □

The following theorem characterises the asymptotic Bergman Hankel operators in the weak sense.

***Theorem* 4.6** — *For $T \in \mathcal{L}(L_a^2(\mathbb{D}))$, let $[t_{ij}]_{i,j=0}^{\infty}$ denote the matrix representation of the operator T with respect to the standard orthonormal basis. Suppose further that $D_2TD_1 \in \mathcal{L}(L_a^2)$. Then $\{H_n(T)\}$ converges weakly if and only if for each $j \geq 1$, the sequence $\{t_{i,i+j}\}$ converges and the sequence of limits $\{t_{-j}\}_{j=1}^{\infty}$ is the sequence of Fourier coefficient of some $\tilde{\phi} \in \mathbf{L}^\infty(\mathbb{T})$.*

PROOF: The proof is similar to that in [7]. □

5. RELATION WITH ASYMPTOTIC HARDY HANKEL OPERATORS

Feintuch [7] defined an asymptotic Hardy Hankel operator as an operator $T \in \mathcal{L}(H^2(\mathbb{D}))$ for which the sequence $\{\widetilde{\mathbf{M}}_\mathbf{n}\mathbf{T}\widetilde{\mathbf{S}}^{\mathbf{n+1}}\}$ converges in the strong operator topology in $\mathcal{L}(H^2(\mathbb{D}))$ to an operator $\widetilde{H}(T)$ where $\widetilde{\mathbf{M}}_\mathbf{n}\mathbf{z}^\mathbf{j} = \mathbf{z}^{\mathbf{n-j}}$ for $0 \leq j \leq n$ and is equal to 0, otherwise. Here we have defined asymptotic Bergman Hankel operator as the operator $T \in \mathcal{L}(L_a^2(\mathbb{D}))$ for which the sequence $\{M_nD_2TD_1W^{n+1}\}$

converges in the strong operator topology in $\mathcal{L}(L^2_a(\mathbb{D}))$. Next we shall show that if T is an asymptotic Bergman Hankel operator in the weak sense and T has a lower triangular matrix representation with respect to the standard orthonormal basis of $L^2_a(\mathbb{D})$ then U^*TU is an asymptotic Hardy Hankel operator in the weak sense.

***Theorem* 5.1** — *If the operator $T \in \mathcal{L}(L^2_a(\mathbb{D}))$ is an asymptotic Bergman Hankel operator in the weak sense and the operator T has a lower triangular matrix representation with respect to the standard orthonormal basis then U^*TU is an asymptotic Hardy Hankel operator in the weak sense.*

PROOF: Suppose $T \in \mathcal{L}(L^2_a(\mathbb{D}))$ is an asymptotic Bergman Hankel operator in the weak sense. Then $\{M_n D_2 T D_1 W^{n+1}\}$ converges weakly and it is to note that $\langle M_n D_2 T D_1 W^{n+1} e_j, e_i\rangle = \sqrt{\dfrac{n-i+1}{j+n+2}}\langle T e_{j+n+1}, e_{n-i}\rangle$ if $0 \leq i \leq n$(=0, otherwise). The sequence $\{\widetilde{\mathbf{M}}_\mathbf{n}\mathbf{U}^*\mathbf{T}\mathbf{U}\widetilde{\mathbf{S}}^{\mathbf{n+1}}\}$ is uniformly bounded and $\langle\widetilde{\mathbf{M}}_\mathbf{n}\mathbf{U}^*\mathbf{T}\mathbf{U}\widetilde{\mathbf{S}}^{\mathbf{n+1}}\mathbf{z}^\mathbf{j}, \mathbf{z}^\mathbf{i}\rangle = \langle \mathbf{U}^*\mathbf{T}\mathbf{U}(\mathbf{z}^{\mathbf{j+n+1}}), \mathbf{z}^{\mathbf{n-i}}\rangle$ if $0 \leq i \leq n$ (=0, otherwise) $= \langle T(\sqrt{j+n+2}z^{j+n+1}), \sqrt{n-i+1}z^{n-i}\rangle$ if $0 \leq i \leq n$(=0, otherwise) $= \langle Te_{j+n+1}, e_{n-i}\rangle$, if $0 \leq i \leq n$(=0, otherwise). The result follows. □

Similarly one can show that if T is strongly asymptotic Bergman Hankel and T has a lower triangular matrix representation with respect to the standard orthonormal basis then U^*TU is strongly asymptotic Hardy Hankel. Let $\mathcal{R}_n$ denote the set of rational functions with at most n poles (counting multiplicities) all of which are in the interior of $\mathbb{T}$ and $C(\mathbb{T})$ denote the algebra of continuous functions on $\mathbb{T}$. Let V_ϕ be the Toeplitz operator with symbol ϕ on $H^2(\mathbb{T})$.

Let $\widehat{\mathcal{T}}$ denotes the family of operators $T \in \mathcal{L}(H^2(\mathbb{D}))$ for which $\{\widetilde{H}_n(T)\} = \{\widetilde{\mathbf{M}}_\mathbf{n}\mathbf{T}\widetilde{\mathbf{S}}^{\mathbf{n+1}}\}$ converges strongly and

$$\begin{aligned}
\mathcal{L}_0 &= \{T \in \widehat{\mathcal{T}} : \widetilde{H}_n(T) \text{ converges strongly to } 0\} \\
\mathcal{L}_k &= \mathcal{L}_0 + \{V_\phi : \phi \in H^\infty(\mathbb{T}) + \mathcal{R}_k)\}; \\
\mathcal{L}_\infty &= \mathcal{L}_0 + \{V_\phi : \phi \in H^\infty(\mathbb{T}) + \mathbb{C}(\mathbb{T}))\}.
\end{aligned}$$

Let $s_k(T)$ be the kth singular value of T. Then it is shown by Feintuch [7] that (i) $d(T, \mathcal{L}_0) = \|\widetilde{H}(T)\|$; (ii) $d(T, \mathcal{L}_k) = s_k(\widetilde{H}(T))$; (iii) $d(T, \mathcal{L}_\infty) = \|\widetilde{H}(T)\|_e$. Now let $E \in \mathcal{L}(L^2_a(\mathbb{D}))$ is lower triangular and is strongly asymptotic Bergman Hankel. Then U^*EU is strongly asymptotic Hardy Hankel. Thus the following distance formulae holds for the operator $E \in \mathcal{L}(L^2_a(\mathbb{D}))$:

$(i)\, \inf_{I\in\mathcal{L}_0} \|E - UIU^*\| = \inf_{I\in\mathcal{L}_0} \|U^*EU - I\| = \|\widetilde{H}(U^*EU)\|$;
$(ii)\, \inf_{I\in\mathcal{L}_k} \|E - UIU^*\| = \inf_{I\in\mathcal{L}_k} \|U^*EU - I\| = s_k(\widetilde{H}(U^*EU))$;
$(iii)\, \inf_{I\in\mathcal{L}_\infty} \|E - UIU^*\| = \inf_{I\in\mathcal{L}_\infty} \|U^*EU - I\| = \|\widetilde{H}(U^*EU)\|_e$.

6. The Algebra Generated by Toeplitz Operators

In this section we show that the algebra generated by Toeplitz operators T_ϕ on $L^2_a(\mathbb{D})$ with $\phi \in \mathcal{A}$ consist of asymptotic Toeplitz operators.

Lemma 6.1 — If $\phi \in \mathcal{A}$ then $\{W^{*n}D_2T_\phi D_1 M_{n-1}\}$ converges strongly to $B_{\widetilde{\phi}^+}$ where $\widetilde{\phi}^+(e^{i\theta}) = \widetilde{\phi}(e^{-i\theta})$.

Proof: We shall first show that if $\psi \in H^\infty$ then $R^nT_{\bar{\psi}}J_{n-1} = 0$ for all n. We see that $R^nT_{\bar{\psi}}J_{n-1}e_i = \dfrac{\sqrt{i+1}}{\sqrt{n-i}}R^nT_{\bar{\psi}}e_{n-1-i}, 0 \leq i \leq n-1$(=0, otherwise). Since the highest power of z in e_{n-1-i} for $0 \leq i \leq n-1$ is $n-1$ and $\bar{\psi} \in \overline{H^\infty}$, hence $R^nT_{\bar{\psi}}J_{n-1} = 0$ for all n. Therefore $W^{*n}D_2T_{\bar{\psi}}D_1M_{n-1} = 0 = B_{\widetilde{\bar{\psi}}^+}$.

Next we shall show that if $\phi \in H^\infty$ then $\{W^{*n}D_2T_\phi D_1 M_{n-1}\}$ converges strongly to $B_{\widetilde{\phi}^+}$ where $\widetilde{\phi}^+(e^{i\theta}) = \widetilde{\phi}(\mathbf{e}^{-\mathbf{i}\theta})$. Since

$$\begin{aligned} W^{*n}D_2T_\phi D_1M_{n-1} &= W^{*n}L_{\widetilde{\phi}}M_{n-1} \\ &= B_{\widetilde{\phi}^+}P_{n-1}, \end{aligned}$$

hence the sequence $\{W^{*n}D_2T_\phi D_1M_{n-1}\}$ converges strongly to $B_{\widetilde{\phi}^+}$ as $\{P_n\}$ converges to I, the identity operator in the strong operator topology (see [7]). The result follows. Now let $\phi \in \mathcal{A}$ and $\phi = \sum_{k=1}^m \bar{\psi}_{\mathbf{k}}\phi_{\mathbf{k}}$ where ψ_k and ϕ_k belong to H^∞. Again we see that

$$\begin{aligned} W^{*n}D_2T_\phi D_1M_{n-1} &= \textstyle\sum_{k=1}^m W^{*n}D_2T_{\bar{\psi}_{\mathbf{k}}\phi_{\mathbf{k}}}D_1M_{n-1} \\ &= \textstyle\sum_{k=1}^m W^{*n}D_2T_{\bar{\psi}_{\mathbf{k}}}T_{\phi_k}D_1M_{n-1}. \end{aligned}$$

But

$$\begin{aligned} W^{*n}D_2T_{\bar{\psi}_{\mathbf{k}}}T_{\phi_k}D_1M_{n-1} &= (W^{*n}D_2T_{\bar{\psi}_{\mathbf{k}}}D_1W^n)(W^{*n}D_2T_{\phi_k}D_1M_{n-1}) \\ &+ (W^{*n}D_2T_{\bar{\psi}_{\mathbf{k}}}D_1M_{n-1})(M_{n-1}D_2T_{\phi_k}D_1M_{n-1}). \end{aligned}$$

Since $W^{*n}D_2T_{\bar{\psi}_{\mathbf{k}}}D_1W^n \to L_{\widetilde{\bar{\psi}_{\mathbf{k}}}}$ strongly, $W^{*n}D_2T_{\phi_k}D_1M_{n-1} \to B_{\widetilde{\phi_{\mathbf{k}}}^+}$ in the strong operator topology, $W^{*n}D_2T_{\bar{\psi}_{\mathbf{k}}}D_1M_{n-1} = 0$ for all n and $\{M_{n-1}D_2T_{\phi_k}D_1M_{n-1}\}$

converges strongly to $L^t_{\widetilde{\phi_{\mathbf{k}}}}$, the transpose of $L_{\widetilde{\phi_{\mathbf{k}}}}$ (for reference see [6] and [7]), hence $\{\sum_{k=1}^m W^{*n} D_2 T_{\bar{\psi}_{\mathbf{k}}} T_{\phi_k} D_1 M_{n-1}\}$ converges strongly to $B_{\widetilde{\phi}^+}$. □

***Theorem* 6.2** — *If $\phi \in \mathcal{A}$ then T_ϕ is an asymptotic Toeplitz operator with symbol $\widetilde{\phi}$.*

PROOF: Suppose $\phi \in H^\infty$ and $\bar{\psi} \in \overline{\mathbf{H}^\infty}$. We shall show first that $T_{\bar{\psi}\phi}$ is strongly asymptotic Toeplitz with symbol $\widetilde{\bar{\psi}\phi}$. It is easy to see that $W^{*n} D_2 T_{\bar{\psi}} T_\phi D_1 W^n = (W^{*n} D_2 T_{\bar{\psi}} D_1 W^n) L_{\widetilde{\phi}}$ since $\phi \in H^\infty$. Hence $\{W^{*n} D_2 T_{\bar{\psi}} T_\phi D_1 W^n\}$ converges strongly to $L_{\widetilde{\bar{\psi}}} L_{\widetilde{\phi}} = L_{\widetilde{\bar{\psi}\phi}}$. Let $\phi \in \mathcal{A}$ and $\phi = \sum_{k=1}^m \bar{\psi}_{\mathbf{k}} \phi_{\mathbf{k}}$ where ψ_k and ϕ_k belong to H^∞. Then $W^{*n} D_2 T_\phi D_1 W^n = \sum_{k=1}^m W^{*n} D_2 T_{\bar{\psi}_{\mathbf{k}}} T_{\phi_k} D_1 W^n$. The assertion of the theorem follows. □

The following lemma will be useful in obtaining more asymptotic Bergman Hankel operators.

Lemma 6.3 — If $\phi \in \mathcal{A}$ then $\{M_n D_2 T_\phi D_1 M_n\}$ is uniformly bounded and converges to $L^t_{\widetilde{\phi}}$ in the strong operator topology where $L^t_{\widetilde{\phi}}$ is the transpose of the operator $L_{\widetilde{\phi}}$.

PROOF: If $\phi \in H^\infty$ then the sequence $\{M_n D_2 T_\phi D_1 M_n\}$ is uniformly bounded. Again $M_n D_2 T_\phi D_1 M_n = M_n L_{\widetilde{\phi}} M_n$ and converge to $L^t_{\widetilde{\phi}}$ in the strong operator topology where $L^t_{\widetilde{\phi}}$ is the transpose of the operator $L_{\widetilde{\phi}}$ (for reference see [6] and [7]). If $\bar{\phi} \in \overline{\mathbf{H}^\infty}$, the sequence $\{M_n D_2 T_{\bar{\phi}} D_1 M_n\}$ is uniformly bounded and $M_n D_2 T_{\bar{\phi}} D_1 M_n = M_n D_2^2 L_{\widetilde{\bar{\phi}}} D_1^2 M_n$ which converges strongly to $L^t_{\widetilde{\bar{\phi}}}$ in the strong operator topology where $L^t_{\widetilde{\bar{\phi}}}$ is the transpose of the operator $L_{\widetilde{\bar{\phi}}}$. Now we shall show if ϕ and ψ belong to H^∞ then $\{M_n D_2 T_{\bar{\psi}\phi} D_1 M_n\}$ is uniformly bounded and converges to $L^t_{\widetilde{\bar{\psi}\phi}}$ in the strong operator topology. Since $W^{n+1} W^{*n+1} = I - P_n$ and $M_n^2 = P_n$ we have

$$\begin{aligned} M_n D_2 T_{\bar{\psi}\phi} D_1 M_n &= M_n D_2 T_{\bar{\psi}} T_\phi D_1 M_n \\ &= (M_n D_2 T_{\bar{\psi}} D_1 W^{n+1})(W^{*n+1} D_2 T_\phi D_1 M_n) \\ &+ (M_n D_2 T_{\bar{\psi}} D_1 M_n)(M_n D_2 T_\phi D_1 M_n). \end{aligned}$$

The sequence $\{M_n D_2 T_{\bar{\psi}} D_1 W^{n+1}\}$ converges strongly to $B_{\widetilde{\bar{\psi}}}$ and $\{W^{*n+1} D_2 T_\phi D_1 M_n\}$ converges strongly to $B_{\widetilde{\phi}^+}$ by Lemma 6.1. Moreover, $\{M_n D_2 T_{\bar{\psi}} D_1 M_n\}$ converges strongly to $L^t_{\widetilde{\bar{\psi}}}$ and $\{M_n D_2 T_\phi D_1 M_n\}$ converges to $L^t_{\widetilde{\phi}}$. Hence $\{M_n D_2 T_{\bar{\psi}\phi} D_1 M_n\}$ is uniformly bounded and converges to $L^t_{\widetilde{\bar{\psi}\phi}}$ in the strong operator topology. □

***Theorem* 6.4** — *The algebra generated by Toeplitz operators with symbol in $\mathcal{A}$ consists of asymptotic Hankel operators.*

PROOF: Let ϕ and ψ belong to $\mathcal{A}$. Since

$$\begin{aligned} M_n D_2 T_\phi T_\psi D_1 W^{n+1} &= (M_n D_2 T_\phi D_1 W^{n+1})(W^{*n+1} D_2 T_\psi D_1 W^{n+1}) \\ &+ (M_n D_2 T_\phi D_1 M_n)(M_n D_2 T_\psi D_1 W^{n+1}), \end{aligned}$$

hence by Theorem 4.2, Theorem 6.2 and Lemma 6.3, the sequence $\{M_n D_2 T_\phi T_\psi D_1 W^{n+1}\}$ converges strongly to $B_{\tilde{\phi}} L_{\tilde{\psi}} + L^t_{\tilde{\phi}} B_{\tilde{\psi}} = B_{\widetilde{\phi\psi}}$ [cf. [4]]. The result follows. □

***Theorem* 6.5** — *If ϕ and ψ belong to $\mathcal{A}$ then $T_\phi T_\psi$ is asymptotic Toeplitz and the symbol is $\widetilde{\phi\psi}$.*

PROOF: We can write $W^{*n} D_2 T_\phi T_\psi D_1 W^n$ as follows: $W^{*n} D_2 T_\phi T_\psi D_1 W^n = (W^{*n} D_2 T_\phi D_1 W^n)(W^{*n} D_2 T_\psi D_1 W^n) + (W^{*n} D_2 T_\phi D_1 M_{n-1})(M_{n-1} D_2 T_\psi D_1 W^n)$. Hence by Lemma 6.1, Theorem 6.4 and Theorem 6.2, the sequence $\{W^{*n} D_2 T_\phi T_\psi D_1 W^n\}$ converges to $L_{\tilde{\phi}} L_{\tilde{\psi}} + B_{\tilde{\phi}^+} B_{\tilde{\psi}} = L_{\widetilde{\phi\psi}}$ (see [4]). □

***Theorem* 6.6** — *If $\phi \in \overline{H^\infty}$ then S_ϕ is an asymptotic Toeplitz operator with zero as its symbol.*

PROOF: It is easy to see that $W^{*n} D_2 S_\phi D_1 W^n = W^{*n} D_2^2 B_{\tilde{\phi} * \tilde{\phi}_{\mathbf{1}}} W^n$ where $\tilde{\phi}_{\mathbf{1}}(\mathbf{e}^{\mathbf{i}\theta}) = \sum_{k=0}^{\infty} \dfrac{1}{k+1} e^{-ik\theta}$. The sequence $\{W^{*n} D_2^2 B_{\tilde{\phi} * \tilde{\phi}_{\mathbf{1}}} W^n\}$ is uniformly bounded and converges to 0 in the strong operator topology since $\{D_2^2 B_{\tilde{\phi} * \tilde{\phi}_{\mathbf{1}}} W^n\}$ converges to 0 in the strong operator topology [4] and W^{*n} converges to 0 in the strong operator topology. Hence the result follows [8]. □

Lemma 6.7 — If $\phi \in \mathcal{A}$ then $\{W^{*n} D_2 S_\phi D_1 M_{n-1}\}$ converges strongly to 0.

PROOF: We shall first show that if $\phi \in H^\infty$ then $\{W^{*n} D_2 S_{\bar{\phi}} D_1 M_{n-1}\}$ converges strongly to 0. It is to observe that

$$\begin{aligned} W^{*n} D_2 S_{\bar{\phi}} D_1 M_{n-1} &= W^{*n} D_2^2 B_{\tilde{\bar{\phi}} * \tilde{\phi}_{\mathbf{1}}} M_{n-1} \\ &= (M_{n-1} B^*_{\tilde{\bar{\phi}} * \tilde{\phi}_{\mathbf{1}}} D_2^2 W^n)^* \end{aligned}$$

where $\widetilde{\phi}_{\mathbf{1}}(\mathbf{e}^{\mathbf{i}\theta}) = \sum_{\mathbf{k}=\mathbf{0}}^{\infty} \dfrac{\mathbf{1}}{\mathbf{k}+\mathbf{1}} \mathbf{e}^{-\mathbf{ik}\theta}$ and $M_{n-1} B^*_{\tilde{\bar{\phi}} * \tilde{\phi}_{\mathbf{1}}} D_2^2 W^n \to 0$ in the strong operator topology [6],[7]. Hence the result follows. Let $\phi \in \mathcal{A}$ and $\phi = \sum_{k=1}^{m} \bar{\psi}_{\mathbf{k}} \phi_{\mathbf{k}}$

where ψ_k and ϕ_k belong to H^∞. Then

$$\begin{aligned} W^{*n}D_2S_{\bar{\psi}_{\mathbf{k}}\phi_{\mathbf{k}}}D_1M_{n-1} &= (W^{*n}D_2S_{\bar{\psi}_{\mathbf{k}}}D_1W^n)(W^{*n}D_2T_{\phi_k}D_1M_{n-1}) \\ &+ (W^{*n}D_2S_{\bar{\psi}_{\mathbf{k}}}D_1M_{n-1})(M_{n-1}D_2T_{\phi_k}D_1M_{n-1}). \end{aligned}$$

Since each of the factors converges strongly (see Theorem 6.6, Lemma 6.1 and Lemma 6.3) the result follows. □

***Theorem* 6.8** — *If $\phi \in \mathcal{A}$ then S_ϕ is an asymptotic Toeplitz operator with zero as its symbol.*

PROOF: It is sufficient to show that if ϕ and ψ belong to H^∞ then $S_{\bar{\phi}\psi}$ is asymptotic Toeplitz with zero as its symbol. Note that we can factorize $W^{*n}D_2S_{\bar{\phi}\psi}D_1W^n$ as follows:

$$\begin{aligned} W^{*n}D_2S_{\bar{\phi}\psi}D_1W^n &= W^{*n}D_2S_{\bar{\phi}}T_\psi D_1W^n \\ &= (W^{*n}D_2S_{\bar{\phi}}D_1W^n)(W^{*n}D_2T_\psi D_1W^n) \\ &+ (W^{*n}D_2S_{\bar{\phi}}D_1M_{n-1})(M_{n-1}D_2T_\psi D_1W^n). \end{aligned}$$

The result follows from Theorem 6.6, Theorem 6.2, Lemma 6.7 and Theorem 4.2. □

***Theorem* 6.9** — *If ϕ and ψ belong to H^∞ then $T_\phi S_{\bar{\psi}}$ is asymptotic Toeplitz.*

PROOF: It is easy to see that $W^{*n}D_2T_\phi S_{\bar{\psi}}D_1W^n = W^{*n}L_{\tilde{\phi}}D_2^2B_{\tilde{\bar{\psi}}*\tilde{\phi}_{\mathbf{1}}}W^n$ where $\tilde{\phi}_{\mathbf{1}}(\mathbf{e^{i\theta}}) = \sum_{\mathbf{k=0}}^\infty \dfrac{\mathbf{1}}{\mathbf{k+1}}\mathbf{e^{-ik\theta}}$. Since $W^{*n}L_{\tilde{\phi}}D_2^2B_{\tilde{\bar{\psi}}*\tilde{\phi}_{\mathbf{1}}}W^n \to 0$ in the strong operator topology [6], hence the result follows. □

Lemma 6.10 — If $\phi \in \mathcal{A}$ then $M_{n-1}D_2S_\phi D_1W^n \to 0$ in the strong operator topology.

PROOF: If $\phi \in H^\infty$ then $M_{n-1}D_2S_\phi D_1W^n = 0$ for all n since $S_\phi = 0$. If $\bar{\phi} \in \overline{\mathbf{H}^\infty}$ then $M_{n-1}D_2S_{\bar{\phi}}D_1W^n = M_{n-1}D_2^2B_{\tilde{\bar{\phi}}*\tilde{\phi}_{\mathbf{1}}}W^n$ where $\widetilde{\phi_{\mathbf{1}}}(\mathbf{e^{i\theta}}) = \sum_{\mathbf{k=0}}^\infty \dfrac{\mathbf{1}}{\mathbf{k+1}}\mathbf{e^{-ik\theta}}$. Since $\{M_{n-1}D_2^2B_{\tilde{\bar{\phi}}*\tilde{\phi}_{\mathbf{1}}}W^n\}$ converges to 0 in the strong operator topology [6], hence $M_{n-1}D_2S_{\bar{\phi}}D_1W^n \to 0$ strongly. If $\phi \in \mathcal{A}$ and $\phi = \sum_{k=1}^m \phi_k\bar{\psi}_{\mathbf{k}}$ where ϕ_k and ψ_k belong to H^∞ then $M_{n-1}D_2S_\phi D_1W^n = \sum_{k=1}^m M_{n-1}D_2S_{\bar{\psi}_{\mathbf{k}}}T_{\phi_k}D_1W^n = \sum_{k=1}^m M_{n-1}D_2S_{\bar{\psi}_{\mathbf{k}}}D_1W^nL_{\tilde{\phi}_{\mathbf{k}}}$ since ϕ_k belong to H^∞. Hence the result follows. □

***Theorem* 6.11** — *If ϕ and ψ belong to $\mathcal{A}$ then $T_\phi S_\psi$ is an asymptotic Toeplitz operator with zero as its symbol.*

PROOF: If ϕ and ψ belong to $\mathcal{A}$ then we can factorize $W^{*n}D_2T_\phi S_\psi D_1W^n$ as follows:

$$\begin{aligned} W^{*n}D_2T_\phi S_\psi D_1W^n &= (W^{*n}D_2T_\phi D_1W^n)(W^{*n}D_2S_\psi D_1W^n) \\ &+ (W^{*n}D_2T_\phi D_1M_{n-1})(M_{n-1}D_2S_\psi D_1W^n). \end{aligned}$$

The result follows from Theorem 6.2, Theorem 6.8, Lemma 6.1 and Lemma 6.10.

***Theorem* 6.12** — *If ϕ and ψ belong to $\mathcal{A}$ then $S_\phi S_\psi$ is asymptotic Toeplitz.*

PROOF: Since $W^{*n}D_2S_\phi S_\psi D_1W^n = (W^{*n}D_2S_\phi D_1W^n)(W^{*n}D_2S_\psi D_1W^n)+ (W^{*n}D_2S_\phi D_1M_{n-1})(M_{n-1}D_2S_\psi D_1W^n)$, the result follows. □

Corollary 6.13 — If a finite sum of finite products of Toeplitz operators with symbols in $\mathcal{A}$ is compact then the corresponding finite sum of finite products of their symbol is zero almost everywhere.

PROOF: The proof follows since the compact operator $K \in \mathcal{L}(L_a^2)$ for which D_2KD_1 is a bounded operator is an asymptotic Toeplitz operator with symbol zero and from the fact that if ϕ and ψ belong to $\mathcal{A}$ then $T_\phi + T_\psi$ and $T_\phi T_\psi$ are asymptotic Toeplitz with symbol $\widetilde{\phi} + \widetilde{\psi}$ and $\widetilde{\phi\psi}$ respectively. □

We shall now give some examples of asymptotic Toeplitz operators on the Bergman space and an example of an operator that is not asymptotic Bergman Toeplitz.

Example 6.14 — Consider the operator T given by the matrix $T = (a_{ij})$ where $a_{ij} = \langle Te_j, e_i\rangle = \sqrt{\frac{j+1}{i+1}}\frac{1}{i+1}$ if $j \leq i$ and is equal to 0 otherwise. Then the operator $T = D_1CD_2$ where C is the Cesaro matrix with $\langle Ce_j, e_i\rangle = \frac{1}{i+1}$ if $j \leq i$ and is equal to 0 otherwise. Since $W^{*n}D_2TD_1W^n = W^{*n}CW^n$, hence the operator T is asymptotic Toeplitz. □

Example 6.15 — Let $T = \text{diag}(\alpha_0, \alpha_1, \alpha_2, \ldots)$, then $\langle W^{*n}D_2TD_1W^ne_j, e_i\rangle = \alpha_{j+n}$ if $i = j$ and is equal to 0 otherwise. Thus

$$W^{*n}D_2TD_1W^n = \text{diag}(\alpha_n, \alpha_{n+1}, \alpha_{n+2}, \ldots).$$

So the operator T is asymptotic Toeplitz if and only if the sequence $\{\alpha_n\}$ is convergent. □

Example 6.16 — Consider the operator $Je_m = (-1)^m e_m$. We see that $\langle W^{*n}D_2JD_1W^ne_j, e_i\rangle = (-1)^{i+n}$ if $i = j$ and is equal to 0 otherwise. Further

$W^{*n}D_2JD_1W^ne_j = (-1)^{j+n}e_j$ which does not converge as n tends to infinity. Hence J is not asymptotic Toeplitz. □

In [6], Feintuch gave an operator theoretic version of Nehari's Theorem, Adamjan, Arov, Krein Theorem and Hartman's Theorem (see [11] for the statement of these theorems) for strongly asymptotic Hankel operators in the Hardy space case. He obtained all these distance formulae by first proving the following result: If T is strongly asymptotic Hankel then $d(T, \mathcal{C}+\mathcal{K}) = \limsup_{n\to\infty} \|P_nT(I-P_n)\| = \|\widetilde{H}(T)\|$. On the Bergman space it is not quite clear how to obtain such type of distance estimates. The following result will give more insight into it.

***Theorem* 6.17** — *If* $\|P_nT(I-P_n)\| = o(\frac{1}{\sqrt{n}})$ *then* $\|J_nTS^{n+1}\| \to 0$. *If* $\|J_nTS^{n+1}\| = o(\frac{1}{n})$ *then* $T \in \mathcal{C}+\mathcal{K}$.

PROOF: Let $f \in L_a^2$ such that $f = \sum_{k=0}^{\infty} a_k e_k$. Then

$$\|J_nP_nf\|^2 = \sum_{k=0}^{n} |a_k|^2 \frac{k+1}{n-k+1}$$

and

$$\|P_nf\|^2 = \sum_{k=0}^{n} |a_k|^2.$$

Hence $\frac{1}{\sqrt{n+1}}\|P_nf\| \leq \|J_nP_nf\|$ for all $f \in L_a^2$ and for all $n \in \mathbb{N}$. Let $T \in \mathcal{L}(L_a^2(\mathbb{D}))$. Then since $J_nP_n = J_n$ and $S^{n+1}R^{n+1} = I - P_n$, we have

$$\begin{aligned} \|J_nTS^{n+1}\| &= \|J_nP_nT(I-P_n)S^{n+1}\| \\ &\leq \|J_n\|\|P_nT(I-P_n)\|\|S^{n+1}\| \\ &\leq \sqrt{n+1}\|P_nT(I-P_n)\|. \end{aligned}$$

On the other hand, since $S^{n+1}R^{n+1} = I - P_n$, we have

$$\begin{aligned} \frac{1}{\sqrt{n+1}}\|P_nT(I-P_n)\| &\leq \|J_nP_nT(I-P_n)S^{n+1}R^{n+1}\| \\ &\leq \sqrt{n+2}\|J_nP_nT(I-P_n)S^{n+1}\| \\ &= \sqrt{n+2}\|J_nP_nTS^{n+1}R^{n+1}S^{n+1}\| \\ &= \sqrt{n+2}\|J_nP_nTS^{n+1}\| \\ &= \sqrt{n+2}\|J_nTS^{n+1}\|. \end{aligned}$$

Since $\|P_nT(I-P_n)\| \to 0$ if and only if $T \in \mathcal{C}+\mathcal{K}$ (by a result of Arveson [1] this holds) hence the assertion of the theorem follows. □

The above result explains that an operator theoretic distance estimate as was obtained in the Hardy space case may not be possible in the case of the Bergman space.

For $T \in \mathcal{L}(L^2_a(\mathbb{D}))$ consider the sequence $\{W^{*n} D_2 T D_1 W^n\}$ and $\{H_n(T)\} = \{M_n D_2 T D_1 W^{n+1}\}$. It is clear that if $\{W^{*n} D_2 T D_1 W^n\}$ converges in the weak (strong, uniform) operator topology then its limit is an operator of the form $L_{\tilde{\phi}}$ where $L_{\tilde{\phi}}$ is the operator in $\mathcal{L}(L^2_a(\mathbb{D}))$ having the classical Toeplitz matrix with symbol $\tilde{\phi}$ in $L^\infty(\mathbb{T})$. When $\{H_n(T)\}$ converges weakly the limit is an operator $B_{\tilde{\psi}}$ in $\mathcal{L}(L^2_a(\mathbb{D}))$ whose matrix with respect to the standard orthonormal basis of $L^2_a(\mathbb{D})$ is a classical Hankel matrix with symbol $\tilde{\psi} \in \mathbf{L}^\infty(\mathbb{T})$.

Two natural questions arise at this stage; (1) Is an asymptotic Toeplitz operator also asymptotic Hankel ? (2) Suppose T is both asymptotic Toeplitz and asymptotic Hankel. Do the symbols correspond ?

The answers to these questions are related. We have shown that in some cases weak asymptotic Toeplitz implies weak asymptotic Hankel and that in this case $M_n D_2 T D_1 W^{n+1} \to B_{\tilde{\phi}}$ weakly and $W^{*n} D_2 T D_1 W^n \to L_{\tilde{\phi}}$ weakly for some $\tilde{\phi} \in \mathbf{L}^\infty(\mathbb{T})$. Thus when T is both asymptotic Toeplitz and asymptotic Hankel in the uniform and strong sense, the symbols must be related in the above mentioned way. In both the uniform and strong topologies we do not know whether asymptotic Toeplitz implies asymptotic Hankel.

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