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GROUP ALGEBRAS[1]

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Dedicated to the memory of Indar Singh Luthar

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Given a group G and a commutative ring k with identity, one can define an k-algebra $k[G]$ called the group algebra of G over k. An element $\alpha \in k[G]$ is said to be algebraic if $f(\alpha) = 0$ for some non-zero polynomial $f(X) \in k[X]$. We will discuss some of the developments in the study of algebraic elements in group algebras.

Key words : Group algebras, augmentation ideal, dimension sub-groups, algebraic elements, partial augmentation, Jordan decomposition, idempotents, Bass conjecture.

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1. Introduction

Let k be a commutative ring with identity. An algebra $\mathcal{A}$ over k is a ring with identity which is also a unital k-module with the following property:

$$(\lambda a)b = a(\lambda b) = \lambda(ab), \quad \text{for all } \lambda \in k, \text{and } a,\ b \in \mathcal{A}.$$

It is further required that the zero element of the module is the same element as the zero element of the ring. For example, if k is a field, say, the field $\mathbb{C}$ of complex numbers, then (i) the set $k[X]$ of polynomials over k and (ii) the set $M_n(k)$ of $n \times n$ matrices over the field k are algebras over k with the usual addition, multiplication and scalar multiplication.

Given a multiplicative group G and a commutative ring k with identity, the set $k[G]$ consisting of all the finite formal sums $\sum_{g\in G, \alpha(g)\in k} \alpha(g)g$ with addition defined coefficient-wise and multiplication induced by the multiplication in G together with distributivity is an algebra over k called the *group algebra* of the group G over the commutative ring k. Thus, for $\alpha = \sum_{g\in G} \alpha(g)g,\ \beta = \sum_{h\in G} \beta(h)h \in k[G]$,

$$\alpha + \beta = \sum_{g\in G} (\alpha(g) + \beta(g))g, \quad \alpha\beta = \sum_{g\in G} \left(\sum_{xy=g} \alpha(x)\beta(y) \right) g.$$

The element $1_k e_G$, where 1_k is the identity element of the ring k and e_G is the identity element of G, is the identity element of the algebra $k[G]$.

Observe that the map $g \mapsto 1_k g$ is a 1-1 group homomorphism from G into the group of units of $k[G]$, and the map $\lambda \mapsto \lambda e_G$ is a 1-1 ring homomorphism $k \to k[G]$. We can thus identify both G and k with their respective images in $k[G]$ under the above maps. In particular, we then have $1_k = e_G = 1_k e_G$ and this element is the identity element of $k[G]$ which we will denote by 1.

For a ring R with identity, let $\mathcal{U}(R)$ denote its group of units. If $\theta : G \to \mathcal{U}(\mathcal{A})$ is a homomorphism from a group G into the group of units of a k-algebra $\mathcal{A}$, then it extends uniquely to an algebra homomorphism

$$\overline{\theta} : k[G] \to \mathcal{A}, \quad \sum \alpha(g)g \mapsto \sum \alpha(g)\theta(g).$$

The trivial homomorphism $G \to \mathcal{U}(k),\ g \mapsto 1$, induces an algebra homomorphism, called the *augmentation homomorphism*,

$$\epsilon : k[G] \to k, \quad \sum \alpha(g)g \mapsto \sum \alpha(g).$$

The ideal $\mathfrak{g}_k := \ker \epsilon$ of $k[G]$ is called the *augmentation ideal*. In case k is the ring $\mathbb{Z}$ of integers, we will drop the suffix k.

Example 1 : Let $G = \langle a \rangle$ be a finite cyclic group of order n, say. Then the group algebra $k[G]$ consists of the elements $\sum_{i=0}^{n-1} \alpha_i a^i$, where $\alpha_i \in k$. Observe that the 'evaluation' map

$$\theta : k[X] \to k[G], \quad f(X) \to f(a)$$

is an algebra homomorphism with kernel the ideal generated by $X^n - 1$. Thus we have

$$k[G] \simeq k[X]/\langle X^n - 1\rangle.$$

Example 2 : Let $G = \langle a, b \,|\, a^4 = 1,\ a^2 = b^2,\ b^{-1}ab = a^3\rangle$ be the quaternion group of order 8. Let $\mathbb{H}_{\mathbb{Q}}$ be the rational quaternion algebra. We have a homomorphism $\theta : \mathbb{Q}[G] \to \mathbb{H}_{\mathbb{Q}}$ defined by $a \mapsto i,\ b \mapsto j$ which is clearly an epimorphism. Also, we have four epimorphisms $\theta_i : \mathbb{Q}[G] \to \mathbb{Q}$ $(i = 1, 2, 3, 4)$ induced by $a \mapsto \pm 1,\ b \mapsto \pm 1$. Consequently we have an algebra homomorphism

$$\varphi : \mathbb{Q}[G] \to \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{H}_{\mathbb{Q}}.$$

It is easy to check that φ is an isomorphism.

Example 3 : Let $D_4 = \langle a, b \,|\, a^4 = 1 = b^2,\ b^{-1}ab = a^3\rangle$ be the dihedral group of order 8. Then

$$\mathbb{Q}[D_4] \simeq \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q} \oplus M_2(\mathbb{Q}).$$

We have four homomorphisms $\mathbb{Q}[D_4] \to \mathbb{Q}$ induced by $a \mapsto \pm 1,\ b \mapsto \pm 1$ and a fifth homomorphism $\rho : \mathbb{Q}[D_4]$ defined by

$$a \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ b \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

My doctoral thesis [46] was written under the guidance of Professor David Rees. Before taking up the main topic of this lecture, viz algebraic elements in group algebras, I would like to mention one of the problems I investigated for my thesis. If G is a group, then we have a descending central series $\{D_n(G)\}_{n\geq 1}$ of subgroups of G, called the *dimension series*, with

$$D_n(G) = G \cap (1 + \mathfrak{g}^n),\ n \geq 1,$$

and there arises the following:

Problem 1 : Determine the subgroups $D_n(G)$, $n \geq 1$.

The subgroup $D_n(G)$ is called the nth *dimension subgroup* of G. For subsets $H,\ K$ of a group G, let $[H,\ K]$ denote the subgroup generated by all elements of the type $h^{-1}k^{-1}hk$ with $h \in H,\ k \in K$. The *lower central series* of G is defined inductively by setting

$$\gamma_1(G) = G,\ \gamma_{n+1}(G) = [G,\ \gamma_n(G)],\ n \geq 1.$$

The dimension series of any group G is closely related to its *lower central series*. It is easily seen that $D_n(G) \supseteq \gamma_n(G)$ for all groups G and all natural numbers, $n \geq 1$, and equality holds for $n = 1,\ 2$.

***Theorem* 1** (G. Higman, D. Rees; see [47]) — *For evey group G, $D_3(G) = \gamma_3(G)$.*

As a contribution to the above problem, I proved the following result [47].

***Theorem* 2** — *If G is a finite p-group of odd order, then $D_4(G) = \gamma_4(G)$.*

The popular conjecture of the time was that the dimension and lower central series of any group are identical. This conjecture was later refuted by Rips.

***Theorem* 3** [51] — *There exists a group of order 2^{38} for which $D_4(G) \neq \gamma_4(G)$.*

The study of dimension subgroups has continued to be one of my interests all along. Interested readers may refer to the monographs [23], [40], [48].

I will now take up the discussion of algebraic elements. For an earlier survey of this topic see [45].

Let $\mathcal{A}$ be an algebra over a a commutative ring k. An element $\alpha \in \mathcal{A}$ is called an *algebraic element* over k if there exists a non-constant polynomial $f(X) \in k[X]$ such that $f(\alpha) = 0$. For example, if k is a field and $\mathcal{A}$ is a finite-dimensional algebra over k, then clearly every element of $\mathcal{A}$ is an algebraic element. The algebraic elements of most interest to algebraists are the *nilpotent elements*, *idempotents* and *torsion units*; i.e., the elements which satisfy one of the following equations:

$$X^n = 0, \quad X^2 - X = 0, \quad X^n - 1 = 0.$$

The term *idempotent* was introduced in 1870 by Benjamin Peirce in the context of elements of an algebra that remain invariant when raised to a positive integer power. The word idempotent literally means "(the quality of having) the same power", from idem + potence (same + power).

Our interest here is in the study of algebraic elements in group algebras. Recall that, in a group G, an element $x \in G$ is said to be *conjugate* to an element $y \in G$ if there exists $z \in G$ such that

$$x = z^{-1}yz.$$

Conjugacy is an equivalence relation; the corresponding equivalence classes are called the *conjugacy classes* of G. For each conjugacy class κ of G, the map $\epsilon_\kappa : k[G] \longrightarrow k$ defined by

$$\epsilon_\kappa(\alpha) = \sum_{g \in \kappa} \alpha(g)$$

is called the *partial augmentation* corresponding to the conjugacy class κ.

In the investigation of an algebraic element $\alpha = \sum \alpha(g)g \in k[G]$, the partial augmentations $\epsilon_\kappa(\alpha)$ corresponding to the various conjugacy classes κ of G provide a useful tool.

Let G be a finite group. A complex representation of G of degree $n \geq 1$ is, by definition, a homomorphism

$$\rho : G \to \mathrm{GL}_n(\mathbb{C}).$$

Let us recall some of the basic facts about complex representations of finite groups. Two representations $\rho_i : G \to \mathrm{GL}_n(\mathbb{C})$ $(i = 1,\ 2)$ are said to be *equivalent* if there exists a matrix $P \in \mathrm{GL}_n(\mathbb{C})$ such that

$$\rho_1(g) = P^{-1}\rho_2(g)P \quad \text{for all } g \in G.$$

A representation $\rho : G \to \mathrm{GL}_n(\mathbb{C})$ is said to be *reducible* if there exists a matrix $P \in \mathrm{GL}_n(\mathbb{C})$ such that

$$P^{-1}\rho(g)P = \begin{pmatrix} \lambda(g) & 0 \\ \mu(g) & \nu(g) \end{pmatrix} \quad \text{for all } g \in G,$$

where $\lambda(G),\ \mu(G),\ \nu(G)$ are $r \times r,\ (n-r) \times r,\ (n-r) \times (n-r)$ matrices respectively, and $1 \leq r < n$. *Every finite group has exactly as many inequivalent irreducible complex representations as the number of its distinct conjugacy classes.* The map $\chi : G \to \mathbb{C}$ given by

$$\chi(g) = \mathrm{Trace}(\rho(\mathrm{g})) \quad (g \in G)$$

is called the *character* of the representation ρ. Note that if $g,\ h \in G$ are conjugate elements, then $\chi(g) = \chi(h)$. *Two representations of a finite group G are equivalent if and only if the corresponding characters are equal.*

Let G be a finite group, $\kappa_1,\ \ldots\ ,\ \kappa_k$ the conjugacy classes of G and $\chi^{(1)},\ \ldots\ ,\ \chi^{(k)}$ its full set of irreducible complex characters. Let $\chi_j^{(i)}$ denote the value that the character $\chi^{(i)}$ takes on the elements of the conjugacy class κ_j If $\kappa_r,\ \kappa_s$ are two conjugacy classes, then

$$\sum_{i=1}^{k} \chi_r^{(i)}\overline{\chi_s^{(i)}} = \frac{|G|}{|\kappa_i|}\delta_{rs},$$

where δ_{ij} is the Kronecker delta and $|S|$ denotes the cardinality of the set S. Let $\chi : G \to \mathbb{C}$ be a character. Extend χ to the group algebra $\mathbb{C}[\mathbb{G}]$ by linearity:

$$\chi(\alpha) = \sum_{g\in G} \alpha(g)\chi(g) \quad (\alpha = \sum_{g\in G} \alpha(g)g).$$

Observe that we then have $\chi(\alpha) = \sum_{\kappa} \epsilon_\kappa(\alpha)\chi_\kappa$, where $\epsilon_\kappa : \mathbb{C}[\mathbb{G}] \to \mathbb{C}$ is the partial augmentation corresponding to the conjugacy class κ and χ_κ is the value that the character χ takes on the elements of κ.

Let $\{s_i : s_i \in \kappa_i\}$ be a system of representatives of the conjugacy classes κ_i of a finite group G and let $\epsilon_i : \mathbb{C}[\mathbb{G}] \to \mathbb{C}$ denote the partial augmentation corresponding to the conjugacy class κ_i. Let $\alpha \in \mathbb{C}[\mathbb{G}]$ be an element satisfying a polynomial $f(X) \in \mathbb{C}[\mathbb{X}]$. Suppose that λ is the maximum of the absolute values of the roots of the polynomial $f(X)$. If ρ is a representation with character χ, then $\chi(\alpha)$, being the trace of the matrix $\rho(\alpha)$, is a sum of $\chi(1)$ eigenvalues of $\rho(\alpha)$. Consider the sum

$$S := \frac{1}{|G|} \sum_{\chi} \chi(\alpha)\overline{\chi(\alpha)}$$

with χ running over the distinct irreducible characters of G. Because of the assumption on λ, we have $|\chi(\alpha)| \leq \chi(1)|\lambda|$. Thus we conclude that

$$S \leq \frac{1}{|G|} \sum_{\chi} \chi(1)^2\lambda^2 = \lambda^2.$$

On the other hand, note that

$$S = \frac{1}{|G|} \sum_{\chi\,,\,i\,,\,j} \epsilon_i(\alpha)\overline{\epsilon_j(\alpha)}\chi(s_i)\overline{\chi(s_j)} = \sum_i |\epsilon_i(\alpha)|^2/|\kappa_i|.$$

We thus have the following result.

***Theorem* 4** [24] — *If G is a finite group, $\alpha \in \mathbb{C}[\mathbb{G}]$ is an element satisfying a polynomial $f(X) \in \mathbb{C}[\mathbb{X}]$ and λ is the maximum of the absolute values of the roots of $f(X)$, then*

$$\sum_i |\epsilon_i(\alpha)|^2/|\kappa_i| \leq \lambda^2.$$

Let $\mathcal{A}$ be a k-algebra. A map $\tau : \mathcal{A} \to k$ is called a *trace* if $\tau(ab) = \tau(ba)$ for all $a,\, b \in \mathcal{A}$. Note that for every conjugacy class κ of a group G, the partial augmentation $\epsilon_\kappa : k[G] \to k$ is a trace map. In particular, arising from the conjugacy class of the identity element of G, the map $\epsilon_1 : k[G] \to k, \alpha \mapsto \alpha(1)$ is a trace map; this map is called the *Kaplansky trace*.

***Theorem* 5** [31, 38] — *If e is an idempotent in the complex group algebra $\mathbb{C}[G]$, then $\epsilon_1(e)$ is a totally real number, $0 \leq \epsilon_1(e) \leq 1$, and $\epsilon_1(e) = 0$ if and only if $e = 0$.*

Kaplansky conjectured that if $e \in \mathbb{C}[G]$ *is an idempotent, then $\epsilon_1(e)$ is a rational number*. An affirmative answer to this conjecture was provided by A. E. Zalesskii [58].

The above assertions are easily seen for finite groups. If G is a finite group and $e \in \mathbb{C}[G]$ is such that $e^2 = e$, then we have a decomposition $\mathbb{C}[G] = e\mathbb{C}[G] \oplus (1-e)\mathbb{C}[G]$. Left multiplication by e defines a linear transformation $\tau : \mathbb{C}[G] \to \mathbb{C}[G]$ which is identity on the first component and zero on the second component. Therefore the trace of the linear transformation τ equals the dimension of $e\mathbb{C}[G]$ over $\mathbb{C}$. On the other hand, if we compute the trace of τ using G as a basis, then the trace works out to be $\epsilon_1(e)|G|$. Consequently, we have

$$\epsilon_1(e) = \frac{\dim_{\mathbb{C}}(e\mathbb{C}[G]).}{|G|}$$

Thus, for finite groups, we have a proof of Kaplansky's theorem and the validity of his conjecture.

An immediate consequence of Theorem 5 is that if α and β are in $\mathbb{C}[G]$ and $\alpha\beta = 1$, then $\beta\alpha = 1$. For, if $\alpha\beta = 1$, then $\beta\alpha$ is an idempotent. We thus have $\epsilon_1(\beta\alpha) = \epsilon_1(\alpha\beta) = 1$. Hence, by Theorem 5, $\beta\alpha = 1$.

An extension of the above result of Kaplansky, the subsequent bound given by Weiss [57], and Theorem 4 is provided by the following result.

***Theorem* 6** [49] — *If α is an element of $\mathbb{C}[G]$ satisfying the equation $f(\alpha) = 0$*

for some non-zero polynomial $f(X) \in \mathbb{C}[X]$, *and if* λ *denotes the maximum of the absolute values of the complex roots of* $f(X)$, *then*

$$\sum_{\kappa} |\epsilon_\kappa(\alpha)|^2/|\kappa| \leq \lambda^2,$$

where $|\kappa|$ *is the cardinality of the conjugacy class* κ*,* $\epsilon_\kappa(\alpha)$*is the partial augmentation of* α *with respect to* κ*.*

For infinite groups we clearly cannot follow the approach of [24]. and so a different method is needed. The above result has been extended to matrices over $\mathbb{C}[G]$, by Luthar-Passi [33] (when G is finite) and to semisimple elements in $(\tau, *)$-algebras by Alexander [3] in a thesis written under the supervision of D. S. Passman.

We next consider nilpotent elements in group algebras.

***Theorem* 7** ([5], Theorem 8.5; [49]) — *If* $\alpha \in \mathbb{C}[G]$ *is nilpotent, then* $\epsilon_\kappa(\alpha) = 0$ *if either* $|\kappa| < \infty$ *or* κ *is a torsion conjugacy class.*

Problem 2 — If $\alpha \in \mathbb{C}[G]$ is nilpotent, must $\epsilon_\kappa(\alpha)$ be zero for every conjugacy class κ of G?

Let us consider one source of nilpotent elements in group algebras. Let $x, y \in G$ with x of finite order n, say. Then the element

$$\alpha := (1-x)y(1+x+x^2+\cdots+x^{n-1})$$

is clearly nilpotent: $\alpha^2 = 0$. Thus, if the group algebra $k[G]$ has no non-zero nilpotent elements, then $\alpha = 0$, and therefore $y = xyx^i$ for some $i,\ 0 \leq i \leq n-1$. It follows that if $k[G]$ does not have non-zero nilpotent elements, then every finite cyclic subgroup $\langle x \rangle$, and therefore every subgroup of G if G is finite, must be normal. Recall that a group all of whose subgroups are normal is called a *Dedekind group*, and a non-abelian Dedekind group is called *Hamiltonian*. For the following characterization of such groups, see [52].

***Theorem* 8** (Dedekind, Baer) — *All the subgroups of a group* G *are normal if and only if* G *is abelian or the direct product of a quaternion group of order 8,* *an elementary abelian 2-group and an abelian group with all its elements of odd order.*

The following result gives a characterization of the rational group algebras without nonzero nilpotent elements.

***Theorem* 9** [53] — *The rational group algebra $\mathbb{Q}[G]$ of a finite group G has no nonzero nilpotent element if and only if either the group G is abelian or it is a Hamiltonian group of order $2^n m$ with order of 2 mod m odd.*

For algebraic number fields, in general, a complete answer has been given by Arora [4]. If the group algebra $k[G]$ of a finite group G over an algebraic number field k has no nonzero nilpotent element, then, as seen above, every subgroup of G must be normal and so G must either be abelian or a group of the type $Q_8 \oplus E \oplus A$, where Q_8 is the quaternion group of order 8, E is an elementary abelian 2-group, and A is an abelian group of odd order. For the characterization of finite group algebras without nonzero nilpotent elements one thus needs to analyse the structure of the group algebras of groups of the above kind over algebraic number fields, and this is essentially a number-theoretic question. It is easily seen that in this investigation one needs information about the stufe of fields, in particular, the following result.

***Theorem* 10** [39] — *Let $\mathbb{Q}(\zeta_m)$ be a cyclotomic field, where ζ_m is a primitive $m\text{th}$ root of unity, m is odd and ≥ 3 . Then the equation $-1 = x^2 + y^2$ has a solution $x, y \in \mathbb{Q}(\zeta_m)$ if and only if the multiplicative order of 2 modulo m is even.*

Turning to idempotents, I would like to discuss a conjecture of H. Bass. To state Bass' conjecture, let us first recall the definition of the rank of a finitely generated projective module.

Let R be a ring with identity and P a finitely generated projective right R-module. The dual module $P^* = \mathrm{Hom}_R(P, R)$ of P then carries a left R-module structure and there exists an isomorphism

$$\alpha : P \otimes_R P^* \simeq \mathrm{End}_R(P)$$

which is defined as follows. For $x \in P$ and $f \in P^*$, $\alpha(x \otimes f) : P \to P$ is the map $y \mapsto xf(y),\ y \in P$. Let $[R,\ R]$ be the additive subgroup of R spanned by the elements $rs - sr\ (r,\ s \in R)$ and define

$$\beta : P \otimes_R P^* \to R/[R,\ R]$$

by setting $\beta(x \otimes f) = f(x) + [R,\ R]$.

The Hattori-Stallings rank (Hattori [25], Stallings [54]) of P, denoted r_P is defined to be $\beta \circ \alpha^{-1}(1_P)$, where 1_P is the identity endomorphism of P. Let $T(G)$ denote the set of conjugacy classes of the group G. Note that if R is the group algebra $k[G]$ of G over a commutative ring k with identity, then $R/[R,\ R]$ is the free k-module on the set $T(G)$. Thus, we can write the rank of a finitely generated projective $k[G]$-module P as

$$r_P = \sum_{\kappa \in T(G)} r_P(\kappa)\kappa.$$

Conjucture 1 ([5], Remark 8.2) — If P is a finitely generated projective $\mathbb{C}[G]$-module, then $r_P(\kappa) = 0$ for the conjugacy classes of elements of infinite order.

This conjecture is still open.

For recent advancements in this direction see [6], [15], [16], [18], [19], [20], [21], [37], [43], [55], [56].

A fundamental question in the theory of group algebras is the determination of their structure and the computation of primitive central idempotents. Of particular interest is the case of $k[G]$ when k is a prime field, i.e., the field $\mathbb{Q}$ of rationals or the finite field $\mathbb{F}_p$ of p elements for some prime p. My recent work with Gurmeet K. Bakshi, Shalini Gupta and Ravi S. Kulkarni has been directed to this problem ([13], [7], [8], [12]).

Another problem concerning group rings that I have been interested in for some time now is that of Jordan decomposition.

Let V be a finite-dimensional vector space over a perfect field k and $\alpha : V \to V$ a linear transformation. Then α can be written in one and only one way as $\alpha = \beta + \gamma$, β diagonalizable, γ nilpotent and $\beta\gamma = \gamma\beta$; furthermore, $\beta,\ \gamma$ are polynomials in α with coefficients from k. This decomposition of α is called its *Jordan decomposition*. If G is a finite group, then every element α of the group algebra $k[G]$ induces a linear transformation from the k-vector space $k[G]$ to itself by left multiplication and hence has a Jordan decomposition as above, i.e., α can be written in the form $\alpha = \alpha_s + \alpha_n$, where α_s and α_n both lie in $k[G]$ (and are polynomials in α over k), α_s is semisimple (i.e., satisfies a polynomial in $k[X]$ with no repeated roots), α_n is nilpotent, and $\alpha_s\alpha_n = \alpha_n\alpha_s$. Thus, in particular, if G is a finite group, then every element α of the rational group algebra $\mathbb{Q}[G]$ has a unique Jordan decomposition. We say that the integral group ring $\mathbb{Z}[\mathbb{G}]$ has *additive Jordan decomposition* (AJD) if, for every element $\alpha \in \mathbb{Z}[\mathbb{G}]$, both the semisimple and the nilpotent components $\alpha_s,\ \alpha_n$ also lie in $\mathbb{Z}[\mathbb{G}]$. If $\alpha \in \mathbb{Q}[G]$ is an invertible element, then the semisimple component α_s is also invertible and so $\alpha = \alpha_s\alpha_u$ with $\alpha_u\,(= 1 + \alpha_s^{-1}\alpha_n)$ unipotent and $\alpha_s\alpha_u = \alpha_u\alpha_s$. Furthermore, such a decomposition is again unique. We say that the integral group ring $\mathbb{Z}[\mathbb{G}]$ has *multiplicative Jordan decomposition* (MJD) if both the semisimple component α_s and the unipotent component α_u of every invertible element $\alpha \in \mathbb{Z}[\mathbb{G}]$ lie in $\mathbb{Z}[\mathbb{G}]$. To characterize finite groups whose integral group rings have additive (resp. multiplicative) Jordan decomposition is an interesting problem (see [29]).

With the complete answer for the characterization of finite groups whose integral group rings possess AJD having been available [28], the focus in recent years has been on the multiplicative Jordan decomposition, a problem that is still unresolved.

Problem 3 — Characterize the finite groups G whose integral group rings $\mathbb{Z}[\mathbb{G}]$ possess multiplicative Jordan decomposition.

We summarize here the progress on this problem. Both the quaternion group of order 8 and the dihedral group of order 8 have the MJD property [1].

Out of the nine non-abelian groups of order 16, the integral group rings of exactly five of them have the MJD property ([1], [44]).

There are exactly three non-abelian groups of order 32 whose integral group ring have the MJD property but not the AJD property and, if G is a finite 2-group of order at least 64, then $\mathbb{Z}[\mathbb{G}]$ has the MJD property if and only if G is a Hamiltonian group [30]. The case of finite 3-groups is almost settled by the following result.

***Theorem* 11** [34], [35] — *The integral group rings of the two non-abelian groups of order 27 have the multiplicative Jordan decomposition property; and integral group rings of all other non-abelian groups of 3-power order do not have this property except possibly the group of order 81 which is the central product of a cyclic group of order 9 with the non-abelian group of order 27 of exponent 3.*

Another recent progress towards the general solution is the following result.

***Theorem* 12** [35] — *A finite 2, 3-group G with order divisible by 6 satisfies the MJD property if and only if*

(i) $G =$ *Sym*$_3$*, the symmetric group of degree 3,*

or

(ii) $G = \langle x,\, y \,|\, x^3 = 1,\ y^4 = 1,\ y^{-1}xy = x^{-1}\rangle$*, a group of order 12,*

or

(iii) $G = Q_8 \times C_3$*, the direct product of the quaternion group of order 8 with the cyclic group of order 3.*

For another perspective about the notion of Jordan decomposition see [50].

Finally, let us consider torsion units. Let G be a finite abelian group and let $\alpha = \sum_{g\in G} \alpha(g)g \in \mathbb{Z}[\mathbb{G}]$ be such that $\sum_{g\in G} \alpha(g) = 1$ and $\alpha^n = 1$ for some $n \geq 1$, then, by Theorem 4, $\sum_{g\in G} \alpha(g)^2 \leq 1$. Consequently, $\alpha(g)$ is non-zero for exactly one element $g \in G$, and for that element $\alpha(g) = 1$. We thus have the following well-known result.

***Theorem* 13** [27] — *If G is a finite abelian group, then the torsion units in its integral group ring $\mathbb{Z}[\mathbb{G}]$ are exactly the elements $\pm g$ $(g \in G)$.*

For arbitrary groups, the following conjecture is still unresolved.

Conjecture 2 — If G is a finite group and $\alpha \in \mathbb{Z}[\mathbb{G}]$ is a torsion unit of augmentation 1, then there exists an invertible element $\beta \in \mathbb{Q}[G]$ and an element $g \in G$ such that $\beta^{-1}\alpha\beta = g$.

Luthar-Passi [32] verified Zassenhaus conjecture for the alternating group A_5. Their method has since been used successfully by various authors to confirm this conjecture in several other cases (see e.g., [9], [10], [11], [26], [36]).

For an extension of the above result to torsion units in matrix group rings see [14], [33], [41], [42].

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