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GENERALIZED BROWDER-TYPE FIXED POINT THEOREM WITH STRONGLY GEODESIC CONVEXITY ON HADAMARD MANIFOLDS WITH APPLICATIONS[1]

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In this paper, a generalized Browder-type fixed point theorem on Hadamard manifolds is introduced, which can be regarded as a generalization of the Browder-type fixed point theorem for the set-valued mapping on an Euclidean space to a Hadamard manifold. As applications, a maximal element theorem, a section theorem, a Ky Fan-type Minimax Inequality and an existence theorem of Nash equilibrium for non-cooperative games on Hadamard manifolds are established.

Key words : Generalized Browder-type fixed point theorem, Maximal element theorem, Hadamard manifolds, Ky Fan Minimax Inequality, Section theorem, Nash equilibrium.

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1. INTRODUCTION

In 1961, [7] established an elementary but very basic geometric lemma for multi-valued mappings by the mean of his own generalization of the classical Knaster-Kuratowski-Mazurkievicz theorem [2, 3] used this fact to prove the Fan-Browder fixed-point theorem, which is a more convenient form of Fan's lemma. Later, by establishing the existence of selection functions for set-valued mappings with open fibers in product spaces, Deguire and Lassonde gave some fixed-point theorems in product spaces for both compact and non-compact domains (see [6]). Readers may consult [1, 5, 8, 9, 11, 13].

On the other hand, in the last few years, several important concepts of nonlinear analysis have been extended from an Euclidean space to a Riemannian manifold setting in order to go further in the study of the convexity theory, the fixed point theory, the variational inequality and related topics. In fact, a manifold is not a linear space. In this setting the linear space is replaced by a Hadamard manifold and the line segment by a geodesic in [14,16,17]. In 2003, [12] introduced and studied the variational inequality on Hadamard manifolds. Some existence theorems of solutions for the variational inequality are proved in [10].

Motivated and inspired by the research works mentioned above, in this paper, we are interested in investigating generalized Browder-type fixed point theorems on Hadamard manifolds. As applications, some existence results of solutions for maximal element theorems, section theorems, a Ky Fan-type Minimax Inequality and non-cooperative games on Hadamard manifolds are obtained.

2. PRELIMINARIES

First we recall some definitions in [4,12]. Let (M, g) be a complete finite-dimensional Riemannian manifold with the Levi-Civita connection $\bigtriangledown$ on M. Let $x \in M$ and let T_xM denote the tangent space at x to M. We denote by $\langle \cdot, \cdot \rangle_x$ the scalar product on T_xM with the associated norm $\| \cdot \|_x$, where the subscript x is sometimes omitted. For $x, y \in M$, let $c : [0, 1] \longrightarrow M$ be a piecewise smooth curve joining x to y. Then the arc-length of c is defined by $l(c) = \int_0^1 \|\dot{c}(t)\| dt$, while the Riemannian distance from x to y is defined by $d(x,y) := \inf_c l(c)$, where the infimum is taken over all piecewise smooth curves $c : [0,1] \longrightarrow M$ joining x to y. Recall that a curve $c : [0,1] \longrightarrow M$ joining x to y is a geodesic if $c(0) = x$, $c(1) = y$ and $\nabla_{\dot{c}}\dot{c} = 0$, $\forall t \in [0,1]$. A geodesic $c : [0,1] \longrightarrow M$ joining x to y is minimal if its arc-length equals its Riemannian distance between x and y. By the Hopf-Rinow Theorem (see [4]), if M is additionally connected, then (M,d) is a complete metric space, and there is at least one minimal geodesic joining x to y. Moreover, the exponential map at x, $\exp_x : T_xM \longrightarrow M$ is well-defined on T_xM. Clearly, a curve $c : [0,1] \longrightarrow M$ is a minimal geodesic joining x to y if and only if there exists a vector $v \in T_xM$ such that $\|v\| = d(x,y)$ and $c(t) = \exp_x(tv)$ for each $t \in [0,1]$.

Definition 2.1 — A Hadamard manifold M is a simply-connected complete Riemannian manifold of non-positive sectional curvature.

Throughout the remainder of the paper, we always assume that M is a m-dimensional Hadamard manifold. The following result is well-known and will be useful.

Proposition 2.1 [15] — Let $x \in M$. Then, $\exp_x : T_xM \longrightarrow M$ is a diffeomorphism, and for any two points $x, y \in M$, there exists a unique normal geodesic joining x to y, which is in fact a minimizing geodesic.

A curve $c : [0,1] \longrightarrow M$ is a minimal geodesic joining x to y, then $c(1) = \exp_x v = y$. By Proposition 2.1, we have $\exp_x^{-1}(y) = v$. Thus $c(t) = \exp_x(t \exp_x^{-1} y)$. The geodesic connecting two points is unique on Hadamard manifold, so just say it geodesic but not necessarily minimal geodesic.

Remark 2.1 : (i) The exponential mapping and its inverse are continuous on Hadamard manifolds; (ii) For any $p, q \in M$, the geodesic joining p to q is $\exp_p(t \exp_p^{-1} q)$ for $t \in [0,1]$.

In [12], geodesic convexity is introduced by Nemeth.

Definition 2.2 [12] — A set $K \subset M$ is said to be geodesic convex if, for any

$p, q \in K$, the geodesic joining p to q is contained in K, i.e., for any $p, q \in K$, $\exp_p(t \exp_p^{-1} q) \in K$ for all $t \in [0, 1]$.

In this paper, we introduce strongly geodesic convexity on Hadamard manifolds.

Definition 2.3 — A set $K \subset M$ is said to be strongly geodesic convex if, for any given $o \in M$ and for any $p, q \in K$, $\exp_o((1-t)\exp_o^{-1} p + t \exp_o^{-1} q) \in K$ for all $t \in [0, 1]$.

Remark 2.2 : When $o = p \in K$ and $q \in K$,

$$\exp_o((1-t)\exp_o^{-1} p + t \exp_o^{-1} q) = \exp_p(t \exp_p^{-1} q).$$

Hence we have

$$\text{strongly geodesic convexity} \ \Rightarrow \ \text{geodesic convexity}.$$

Lemma 2.1 — $K \subset M$ is strongly geodesic convex if and only if, for any given $o \in M$ and for any finite $\{x_1, \cdots, x_n\} \subset K$, $\lambda_1, \cdots, \lambda_n \in [0, 1]$ with $\sum_{i=1}^n \lambda_i = 1$, we have $\exp_o\left(\sum_{i=1}^n \lambda_i \exp_o^{-1} x_i\right) \in K$.

PROOF : It is enough to prove $\Rightarrow$, because $\Leftarrow$ is trivial. We use mathematical induction. When $n = 1, 2$, the statement is true. Suppose that it is true for some n, we only prove that $\exp_o\left(\sum_{i=1}^{n+1} \lambda_i \exp_o^{-1} x_i\right) \in K$ for any finite $\{x_1, \cdots, x_{n+1}\} \subset K$, $\lambda_1, \cdots, \lambda_{n+1} \in [0, 1]$ with $\sum_{i=1}^{n+1} \lambda_i = 1$. Without loss of generality, we can assume that $0 < \lambda_{n+1} < 1$. Thus,

$$\sum_{i=1}^{n} \frac{\lambda_i}{1 - \lambda_{n+1}} = 1.$$

Since it is true for n, then

$$y := \exp_o\left(\sum_{i=1}^{n} \frac{\lambda_i}{1 - \lambda_{n+1}} \exp_o^{-1} x_i\right) \in K.$$

Hence, we have

$$\exp_o(\lambda_{n+1} \exp_o^{-1} x_{n+1} + (1 - \lambda_{n+1}) \exp_o^{-1} y) \in K.$$

Since

$$\begin{aligned} \exp_o(\lambda_{n+1}\exp_o^{-1}x_{n+1} &+ (1-\lambda_{n+1})\exp_o^{-1}y) \\ &= \exp_o\left(\lambda_{n+1}\exp_o^{-1}x_{n+1}+\sum_{i=1}^{n}\exp_o^{-1}x_i\right), \end{aligned}$$

then

$$\exp_o\left(\sum_{i=1}^{n+1}\lambda_i\exp_o^{-1}x_i\right)\in K.$$

This completes the proof.

Definition 2.4 — A set $S \subset M$, the smallest strongly geodesic convex set containing S is defined by $Gco(S)$, called strongly geodesic convex hull of S.

Next, the representation theorem of strongly geodesic convex hull is given.

Lemma 2.2 — Let $S \subset M$, and o be any given point in M.

$$\begin{aligned} Gco(S) = & \left\{\exp_o\left(\sum_{i=1}^{n}\lambda_i\exp_o^{-1}x_i\right):\right. \\ & \left.\forall x_1,\cdots,x_n\in S;\lambda_1,\cdots,\lambda_n\in[0,1],\sum_{i=1}^{n}\lambda_i=1\right\}. \end{aligned}$$

PROOF : Denote

$$\begin{aligned} D = & \left\{\exp_o\left(\sum_{i=1}^{n}\lambda_i\exp_o^{-1}x_i\right):\right. \\ & \left.\forall x_1,\cdots,x_n\in S;\lambda_1,\cdots,\lambda_n\in[0,1],\sum_{i=1}^{n}\lambda_i=1\right\}. \end{aligned}$$

(I) We prove $D \subset Gco(S)$.

For any finite $\{x_1,\cdots,x_n\}\subset S\subset Gco(S)$, $\lambda_1,\cdots,\lambda_n\in[0,1]$ with $\sum_{i=1}^{n+1}\lambda_i = 1$, since $Gco(S)$ is strongly geodesic convex by the definition 2.4, then, by lemma 2.1,

$$\exp_o\left(\sum_{i=1}^{n}\lambda_i\exp_o^{-1}x_i\right)\in Gco(S).$$

Thus $D \subset Gco(S)$.

(II) We prove $Gco(S) \subset D$.

For any $x, y \in D$, there exist m_1, m_2, finite $\{x_1, \cdots, x_{m_1}\} \subset S$, $\lambda_1, \cdots, \lambda_{m_1} \in [0,1]$ with $\sum_{i=1}^{m_1} \lambda_i = 1$, and finite $\{y_1, \cdots, y_{m_2}\} \subset S$, $\mu_1, \cdots, \mu_{m_2} \in [0,1]$ with $\sum_{i=1}^{m_2} \mu_i = 1$ such that

$$x = \exp_o \left(\sum_{i=1}^{m_1} \lambda_i \exp_o^{-1} x_i \right), \quad y = \exp_o \left(\sum_{i=1}^{m_2} \mu_i \exp_o^{-1} y_i \right).$$

Hence, for any $t \in [0,1]$, we have

$$\begin{aligned} \exp_o(t \exp_o^{-1} x \quad &+ \quad (1-t) \exp_o^{-1} y) \\ &= \quad exp_o \left(\sum_{i=1}^{m_1} t\lambda_i \exp_o^{-1} x_i + \sum_{i=1}^{m_2} (1-t)\mu_i \exp_o^{-1} y_i \right). \end{aligned}$$

Since

$$\sum_{i=1}^{m_1} t\lambda_i + \sum_{i=1}^{m_2} (1-t)\mu_i = 1,$$

then

$$exp_o(t \exp_o^{-1} x + (1-t) \exp_o^{-1} y) \in D.$$

Thus D is strongly geodesic convex. Since $S \subset D$, then $Gco(S) \subset D$ by the definition 2.3.

This completes the proof.

We also need the following result, which are due to Nemeth [12].

Lemma 2.3 [12] — If $K \subset M$ is nonempty, compact and geodesic convex, then every continuous function $f : K \longrightarrow K$ has a fixed point.

By Remark 2.2, we have:

Lemma 2.4 — If $K \subset M$ is nonempty, compact and strongly geodesic convex, then every continuous function $f : K \longrightarrow K$ has a fixed point.

Definition 2.5 — Let X and Y be two Hausdorff topological spaces, and $F : X \rightrightarrows Y$ be a set-valued mapping,

(1) If, for any open subset O of Y with $O \supset F(x)$, there exists an open neighborhood $U(x)$ of x such that $O \supset F(x')$ for any $x' \in U(x)$, F is said to be upper semicontinuous at $x \in X$;

(2) If F is upper semicontinuous on each $x \in X$, F is said to be upper semicontinuous on X;

(3) If, for any open subset O of Y with $O \bigcap F(x) \neq \emptyset$, there exists an open neighborhood $U(x)$ of x such that $O \bigcap F(x') \neq \emptyset$ for any $x' \in U(x)$, F is said to be lower semicontinuous at $x \in X$;

(4) If F is lower semicontinuous on each $x \in X$, F is said to be lower semicontinuous on X;

(5) for any $y \in Y$, $F^{-1}(y) := \{x \in X | y \in F(x)\}$.

3. Main Results

Throughout this paper, let M be a Hadamard manifold, X be a nonempty, strongly geodesic convex and compact subset of a Hadamard manifold M and o be any given point in M.

***Theorem* 3.1** — *Let X be a nonempty, strongly geodesic convex and compact subset of M. Suppose that $F : X \rightrightarrows X$ and $H : X \rightrightarrows X$ are two set-valued mappings with the following conditions:*

(1) for any $x \in X$, $GcoH(x) \subset F(x)$;

(2) for any $x \in X$, there exists $y \in X$ such that $x \in intH^{-1}(y)$;

Then there exists $x^* \in X$ such that $x^* \in F(x^*)$.

PROOF : By condition (2), we have

$$X = \bigcup_{y \in X} intH^{-1}(y).$$

Since X is nonempty and compact, then there exists a finite number of $intH^{-1}(y_1)$, $\cdots, intH^{-1}(y_n)$ such that

$$X = \bigcup_{i=1}^{n} intH^{-1}(y_i).$$

Let $\{\alpha_i | i = 1, \cdots, n\}$ be the partition of unity subordinate to open covering $\{intH^{-1}(y_i) | i = 1, \cdots, n\}$ of X, i.e.,

$$0 \leq \alpha_i(x) \leq 1, \ \sum_{i=1}^{n} \alpha_i(x) = 1, \ \forall x \in X, \ i = 1, \cdots, n;$$

and if $x \notin intH^{-1}(y_j)$ for some j, then $\alpha_j(x) = 0$.

Now we consider a function $f : X \longrightarrow X$, defined by

$$f(x) = \exp_o(\alpha_1(x) \exp_o^{-1} y_1 + \cdots + \alpha_n(x) \exp_o^{-1} y_n), \ \forall x \in X.$$

Then f is continuous, and by Lemma 2.4, there exists $x^* \in X$ such that $f(x^*) = x^*$.

Let $I = \{i \in \{1, \cdots, n\} | \alpha_i(x^*) > 0\}$, then $\sum_{i \in I} \alpha_i(x^*) = 1$ and $x^* \in intH^{-1}(y_i) \subset H^{-1}(y_i)$ for all $i \in I$, i.e., $y_i \in H(x^*)$ for each $i \in I$. Thus, we have

$$x^* = f(x^*) = \exp_o\left(\sum_{i \in I} \alpha_i(x^*) exp_o^{-1} y_i\right) \in GcoH(x^*) \subset F(x^*).$$

This completes the proof.

Remark 3.1 : If $H^{-1}(y)$ is open in X for each $y \in X$ and $H(x) \neq \emptyset$ for all $x \in X$, then, for any $x \in X$, there exists $y \in X$ such that $y \in H(x)$, i.e., $x \in H^{-1}(y) = intH^{-1}(y)$. The condition (2) of theorem 3.1 is obtained. Thus we have the following corollary:

Corollary 3.1 — Let X be a nonempty, strongly geodesic convex and compact subset of M. Suppose that $F : X \rightrightarrows X$ and $H : X \rightrightarrows X$ are two set-valued mappings with the following conditions:

(1) for any $x \in X$, $GcoH(x) \subset F(x)$;

(2) for any $y \in X$, $H^{-1}(y)$ is open in X;

(3) $H(x) \neq \emptyset$ for all $x \in X$.

Then there exists $x^* \in X$ such that $x^* \in F(x^*)$.

When $H(x) = F(x)$ for all $x \in X$, we have the following corollary, which can be regarded as the Fan-Browder fixed point theorem on Hadamard manifolds.

Corollary 3.2 — Let X be a nonempty, strongly geodesic convex and compact subset of M. Suppose that $F : X \rightrightarrows X$ is a set-valued mapping with the following conditions:

(1) for any $x \in X$, $F(x)$ is nonempty and strongly geodesic convex in X;

(2) for any $y \in X$, $F^{-1}(y)$ is open in X.

Then there exists $x^* \in X$ such that $x^* \in F(x^*)$.

Next we give a maximal element theorem on Hadamard manifolds by theorem 3.1.

***Theorem* 3.2** — *Let X be a nonempty, strongly geodesic convex and compact subset of M. Suppose that $A : X \rightrightarrows X$ and $B : X \rightrightarrows X$ are two set-valued mappings with the following conditions:*

(1) For each $x \in X$, $x \notin GcoB(x)$.

(2) If $A(x) \neq \emptyset$, then there exists $y \in X$ such that $x \in intB^{-1}(y)$.

Then there exists $x^* \in X$ such that $A(x^*) = \emptyset$.

PROOF : Suppose the contrary, then $A(x) \neq \emptyset$ for all $x \in X$. By Theorem 3.1, there exists $x^* \in X$ such that $x^* \in GcoB(x^*)$, which contradict the fact that $x \notin GcoB(x)$ for each $x \in X$. This completes the proof.

Remark 3.2 : If $A(x) \subset B(x)$ for each $x \in X$, and $A(x) \neq \emptyset$, then there exists $y \in X$ such that $y \in A(x) \subset B(x)$. Thus $x \in B^{-1}(y)$. We have the following corollary:

Corollary 3.3 — Let X be a nonempty, strongly geodesic convex and compact subset of M. Suppose that $A : X \rightrightarrows X$ and $B : X \rightrightarrows X$ are two set-valued mappings with the following conditions:

(1) For each $x \in X$, $A(x) \subset B(x)$.

(2) For each $x \in X$, $x \notin GcoB(x)$.

(3) For each $y \in X$, $B^{-1}(y)$ is open in X.

Then there exists $x^* \in X$ such that $A(x^*) = \emptyset$.

In the case when $A = B$, Corollary 3.3 become the following result:

Corollary 3.4 — Let X be a nonempty, strongly geodesic convex and compact subset of M. Suppose that $A : X \rightrightarrows X$ is a set-valued mapping with the following conditions:

(1) For each $x \in X$, $x \notin GcoA(x)$.

(2) For each $y \in X$, $A^{-1}(y) = \{x \in X | y \in A(x)\}$ is open in X.

Then there exists $x^* \in X$ such that $A(x^*) = \emptyset$.

Next we prove a new section theorem on Hadamard manifolds by theorem 3.1.

***Theorem* 3.3** — *For each* $i = 1, \cdots, n$, *let* X_i *be a nonempty, strongly geodesic convex and compact subset of a Hadamard manifold* M_i. *Let*

$$X = \prod_{i=1}^{n} X_i,\ X_{-i} = \prod_{1 \leq j \leq n, j \neq i} X_j.$$

Let $A_1, \cdots, A_n$ *and* $B_1, \cdots, B_n$ *be* $2n$ *subsets of* X*,*

$$A_i(x_i) = \{x_{-i} \in X_{-i} | (x_i, x_{-i}) \in A_i\},\ A_i(x_{-i}) = \{x_i \in X_i | (x_i, x_{-i}) \in A_i\},$$
$$B_i(x_i) = \{x_{-i} \in X_{-i} | (x_i, x_{-i}) \in B_i\},\ B_i(x_{-i}) = \{x_i \in X_i | (x_i, x_{-i}) \in B_i\}$$

such that

(1) For each $i = 1, \cdots, n$*, and any* $x_{-i} \in X_{-i}$*, there exists* $y_i \in X_i$ *such that* $x_{-i} \in intB_i(y_i)$*;*

(2) For each $i = 1, \cdots, n$*, and any* $x_{-i} \in X_{-i}$*,* $GcoB_i(x_{-i}) \subset A_i(x_{-i})$*.*

Then

$$\bigcap_{i=1}^{n} A_i \neq \emptyset.$$

PROOF : We define set-valued mappings $F : X \rightrightarrows X$ and $H : X \rightrightarrows X$ by

$$F(x) = \prod_{i=1}^{n} F_i(x_{-i}),\ H(x) = \prod_{i=1}^{n} H_i(x_{-i}),$$

where

$$F_i(x_{-i}) = \{y_i \in X_i | (y_i, x_{-i}) \in A_i\},\ H_i(x_{-i}) = \{y_i \in X_i | (y_i, x_{-i}) \in B_i\}.$$

By condition (2), $GcoH(x) \subset F(x)$ for all $x \in X$. Further, by condition (1), for any $x \in X$, there exists $y \in X$ such that $x \in intH(y)$. Thus, by theorem 3.1, there exists $x^* \in X$ such that $x^* \in F(x^*)$, i.e.,

$$\bigcap_{i=1}^{n} A_i \neq \emptyset.$$

Next, we study a Ky Fan-type Minimax Inequality on Hadamard manifolds. First, we define the strongly geodesic concave functions on Hadamard manifolds.

Definition 3.1 — Let $K \subset M$ be strongly geodesic convex subset and o be any given point in M. A function $f : K \longrightarrow R$ is called strongly geodesic concave if, for any $x_i \in K$, $t_i \geq 0$, $i = 1, \cdots, n$ with $\sum_{i=1}^n t_i = 1$, we have

$$f\left(\exp_o(\sum_{i=1}^n t_i \exp_o^{-1} x_i)\right) \geq \sum_{i=1}^n t_i f(x_i);$$

f is called strongly geodesic quasi-concave if, for any $x_i \in K$, $t_i \geq 0$, $i = 1, \cdots, n$ with $\sum_{i=1}^n t_i = 1$, we have

$$f\left(\exp_o(\sum_{i=1}^n t_i \exp_o^{-1} x_i)\right) \geq \min_{i\in\{1,\cdots,n\}}\{f(x_i)\}.$$

***Theorem* 3.4** — *Let X be a nonempty, compact and strongly geodesic convex subset of a Hadamard manifold M, and let $f : X \times X \longrightarrow R$ be a real-valued function on $X \times X$ such that*

(1) for each $y \in X$, $x \longrightarrow f(x, y)$ is lower semicontinuous on X;

(2) for each $x \in X$, $y \longrightarrow f(x, y)$ is strongly geodesic quasi-concave on X;

(3) $f(x, x) \leq 0$ for all $x \in X$.

Then there exists $x^* \in X$ such that $f(x^*, y) \leq 0$ for all $y \in X$.

PROOF : We define the set-valued mapping $A : X \rightrightarrows X$ by

$$F(x) = \{y \in X | f(x, y) > 0\}.$$

Suppose the contrary, then, for any $x \in X$, there exists $y \in X$ such that $y \in F(x)$, i.e., $F(x) \neq \emptyset$ for any $x \in X$. By condition (1), $F^{-1}(y)$ is open in X for all $y \in X$. By condition (2), $F(x)$ is strongly geodesic convex. Hence, by corollary 3.2, there exists $x^* \in X$ such that $x^* \in F(x^*)$, i.e., $f(x^*, x^*) > 0$, which contradict the fact $f(x, x) \leq 0$ for all $x \in X$. This completes the proof.

4. APPLICATION TO GAMES

Next we shall use the new results to prove some new existence theorems of Nash equilibrium points on Hadamard manifolds.

Consider the $n-$person non-cooperative game $\Gamma\{I, X_i, f_i\}$ on Hadamard manifolds. Assume that

(1) $I = \{1, \cdots, n\}$ is the set of players;

(2) for each $i \in I$, the nonempty set X_i is the strategy set of ith player;

(3) for each $i \in I$, $f_i : \prod_{i\in I} X_i \longrightarrow R$ is the payoff function of ith player.

We shall note $X_{-i} = \prod_{j\in I\setminus\{i\}} X_j$, $x_{-i} = (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n) \in X_{-i}$, $x = (x_i, x_{-i}) \in X$. $x^* = (x_i^*, x_{-i}^*) \in X$ is called a Nash equilibrium point if, for each $i \in I$,

$$f_i(x_i^*, x_{-i}^*) = \max_{u_i\in X_i} f_i(u_i, x_{-i}^*).$$

***Theorem* 4.1** — *Consider the $n-$person non-cooperative game $\Gamma\{I, X_i, f_i\}$ on Hadamard manifolds satisfying the following conditions:*

(1) for each $i \in I$, X_i is a nonempty, compact and strongly geodesic convex subset of a Hadamard manifold M_i;

(2) for each $i \in I$, f_i is upper semicontinuous on X;

(3) for each $i \in I$ and each fixed $u_i \in X_i$, $f_i(u_i, \cdot)$ is lower semicontinuous on X_{-i};

(4) for each fixed $u_{-i} \in X_{-i}$, $f_i(\cdot, u_{-i})$ is strongly geodesic quasi-concave on X_i.

Then there exists at least one Nash equilibrium point x^*.

PROOF : We define the set-valued mapping $A(x) = \prod_{i\in I} A_i(x)$, where

$$A_i(x) = \{y_i \in X_i | f_i(x_i, x_{-i}) < f_i(y_i, x_{-i})\}, \ \forall x \in X.$$

By condition (2,3), $A_i^{-1}(y_i) = \{x \in X | f_i(x_i, x_{-i}) < f_i(y_i, x_{-i})\}$ is open in X for each $i \in I$, then $A^{-1}(y) = \bigcap_{i\in I} A_i^{-1}(y_i)$ is open in X for all $y \in X$.

Suppose that there exists $x \in X$ such that $x \in GcoA(x)$, then there exist $o \in M$, $y^j \in A(x), j = 1, \cdots, m$ and $t_j \geq 0$, $j = 1, \cdots, m$ with $\sum_{j=1}^m t_j = 1$ such that $x = \exp_o \left(\sum_{j=1}^m t_j \exp_o^{-1} y^j \right)$. Since, for each fixed $u_{-i} \in X$, $f_i(\cdot, u_{-i})$ is strongly geodesic quasi-concave on X_i, then

$$f_i(x_i, x_{-i}) = f_i \left(\exp_o(\sum_{j=1}^m t_j exp_o^{-1} y_i^j), x_{-i} \right) \geq \min_{j \in \{1, \cdots, m\}} f_i(y_i^j, x_{-i}),$$

which contradict the fact that $y^j \in A(x)$ for all $j = 1, \cdots, m$, i.e.,

$$f_i(x_i, x_{-i}) < f_i(y_i^j, x_{-i}), \ \forall j = 1, \cdots, m.$$

By corollary 3.4, there exists $x^* \in X$ such that $A(x^*) = \emptyset$, i.e., $f_i(x_i^*, x_{-i}^*) \geq f_i(u_i, x_{-i}^*)$ for all $u_i \in X_i$ and for each $i \in I$.

***Theorem* 4.2** — *Consider the $n-$person non-cooperative game $\Gamma\{I, X_i, f_i\}$ on Hadamard manifolds satisfying the following conditions:*

(1) for each $i \in I$, X_i is a nonempty, compact and strongly geodesic convex subset of a Hadamard manifold M_i;

(2) for each $i \in I$, $\sum_{i=1}^n f_i$ is upper semicontinuous on X;

(3) for each $y \in X$, $x \longrightarrow \sum_{i=1}^n f_i(y_i, x_{-i})$ is lower semicontinuous on X;

(4) for each $x \in X$, $y \longrightarrow \sum_{i=1}^n f_i(y_i, x_{-i})$ is strongly geodesic quasi-concave on X.

Then there exists at least one Nash equilibrium point x^*.

PROOF : We define the function $\varphi : X \times X \longrightarrow R$, where

$$\varphi(x, y) = \sum_{i=1}^n [f_i(y_i, x_{-i}) - f_i(x_i, x_{-i})], \ \forall y = (y_i, y_{-i}), x = (x_i, x_{-i}) \in X.$$

By condition (2–4), we can check that

(i) for each $y \in X$, $x \longrightarrow \varphi(x, y)$ is lower semicontinuous on X;

(ii) for each $x \in X$, $y \longrightarrow \varphi(x, y)$ is strongly geodesic quasi-concave on X;

(iii) $\varphi(x, x) \leq 0$ for all $x \in X$.

By theorem 3.4, there exists $x^* \in X$ such that $\varphi(x^*, y) \leq 0$ for all $y \in X$. For each $i \in I$ and any $u_i \in X_i$, let $y = (u_i, x^*_{-i}) \in X$, then

$$\begin{aligned} \varphi(x^*, y) &= f_i(u_i, x^*_{-i}) - f_i(x^*, x^*_{-i}) \leq 0, \\ f_i(x^*_i, x^*_{-i}) &= \max_{u_i \in X_i} f_i(u_i, x^*_{-i}). \end{aligned}$$

5. Conclusion

In this paper, we study the generalized Browder-type fixed point theorem on Hadamard manifolds, which can be regarded as a generalization of the Browder-type fixed point theorem for the set-valued mapping on an Euclidean space to a Hadamard manifold. As applications, maximal element theorems, section theorems, a Ky Fan-type Minimax Inequality and existence theorems of Nash equilibrium for non-cooperative games on Hadamard manifolds are established.

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