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# IDEALS OF ANALYTIC DEVIATION ONE WITH RESPECT TO A COHEN-MACAULAY MODULE

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Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring, $M$ a Cohen-Macaulay $A$-module of dimension $d \geq 1$ and $I$ a proper ideal of analytic deviation one with respect to $M$. In this paper we study the Cohen-Macaulayness of associated graded module of a Cohen-Macaulay module. We show that if $I$ is generically a complete intersection of analytic deviation one and reduction number at most one with respect to $M$ then $G_I(M)$ is Cohen-Macaulay. When analytic spread of $I$ with respect to $M$ equals $d$ we prove a similar result when reduction number of an ideal is atmost two.

**Key words** : Blow-up algebra, analytic deviation, analytic spread, reductions, Cohen-Macaulay modules.

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## Introduction

Let $(A, \mathfrak{m})$ be a Cohen-Macaulay(CM) local ring with infinite residue field $k = A/\mathfrak{m}$. Let $M$ be a CM $A$-module. Let $I$ be proper ideal of height $d \geq 1$. Let $G_I(A) = \bigoplus_{n\geq 0} I^n/I^{n+1}$ be the associated graded ring of $A$ with respect to $I$ and let $G_I(M) = \bigoplus_{n\geq 0} I^nM/I^{n+1}M$ be the associated graded module of $M$ with respect to $I$, considered as $G_I(A)$ module. Let $F(I) = \bigoplus_{n\geq 0} I^n/\mathfrak{m}I^n$ be the fiber cone and let $F_I(M) = \bigoplus_{n\geq 0} I^nM/\mathfrak{m}I^nM$ be the fiber cone of $I$ with respect to $M$, considered as a graded $F(I)$ module. We define analytic spread of an ideal to be the Krull dimension of $F(I)$ and denote it by $l(I)$. Analogously the analytic spread of an ideal with respect to $M$ is defined to be the dimension of $F_I(M)$ and denote this by $l_M(I)$.

We say that an ideal $J \subseteq I$ is reduction of $I$ with respect to $M$ if there exists a natural number $m$ such that $JI^nM = I^{n+1}M$ for all $n \geq m$. We define $r_J(I, M)$ to be the least such $m$. A reduction $J$ of $I$ is called minimal if it is minimal with respect to inclusion. Reduction number of $I$ with respect to $M$ is defined as follows,

$$r(I, M) = \min\{r_J(I, M) \mid J \; is \; minimal \; reduction \; of \; I \; with \; respect \; to \; M\}.$$

It is well known that if $J$ is a minimal reduction of $I$ with respect to $M$ then $\mu(J) = l_M(I)$. The *height* of an ideal $I$ with respect to $M$, denoted by $\operatorname{ht}_M(I)$ is defined to be the number $\inf\{\dim M_P \mid P \in \operatorname{Supp}(M/IM)\}$. *Analytic deviation* of an ideal $I$ with respect to $M$ is the difference $l_M(I) - \operatorname{ht}_M(I)$ and we denote it by $\operatorname{ad}_M(I)$. We prove in Lemma 1.6 that $\operatorname{ht}_M(I) \leq l_M(I) \leq \dim M$. Thus we always have $\operatorname{ad}_M(I) \geq 0$. An ideal $I$ in $A$ is said to be a *complete intersection* with respect to $M$ if $I$ is minimally generated in $A$ by an $M-$regular sequence. An ideal $I$ is said to be *generically a complete intersection* with respect to $M$ if $I_P$ is complete intersection for all minimal primes $P \in \operatorname{Supp}(M/IM)$.

When $M = A$ and $l(I) = \operatorname{ht}(I)$ i.e. when $I$ has analytic deviation zero there has been a lot of research regarding the CM property of $G_I(A)$. When analytic deviation is positive not much is known. However considerable efforts have been made to understand the case when analytic deviation is one or two (See [7, 4, 9]. When analytic deviation is either one or two then Huckaba and Huneke show that if $I$ is generically complete intersection of analytic deviation one and reduction number at most one then the associated graded ring and Rees ring are CM (see 7, Theorem 2.9] and [8, Theorem 2.1]). These conditions were generalized by many authors (see [5 and 9]).

It is of some interest to extend results in case of rings and ideals to the case of modules. A systematic account of this transition from rings to the modules is given in the notes of Rossi and Valla, see [11]. In the notes of Rossi-Valla, to do the transition from rings to modules, techniques of Hilbert functions, superficial sequences, Valabrega-Valla criterion etc. are used as the basic tools. Unfortunately in the case of ideals of analytic deviation one, theory of Hilbert functions doesn't work as plainly as it works in the case of $m$-primary ideals. Assume $\mathrm{ht}(I) = d \geq 1$ and $l(I) = d+1$. Let $I$ be generically a complete intersection and $r(I) \leq 1$. Huckaba and Huneke in their paper (7, [Theorem 2.9]) prove that $G_I(A)$ is CM if $\mathrm{depth}(A/I) \geq \dim(A/I) - 1$. In this paper we extend this result of Huckaba and Huneke to modules.

***Theorem* 1** — *Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring and $M$ a Cohen-Macaulay $A-$module. Let $I$ be an equidimensional ideal of analytic deviation one with respect to $M$. Assume $\mathrm{ht}_M(I) = s \geq 1$, $r(I, M) \leq 1$, and $I$ is generically a complete intersection with respect to $M$. If $\mathrm{depth}(M/IM) \geq \dim(M/IM) - 1$ then $G_I(M)$ is Cohen-Macaulay.*

We next extend to the module case the result of Hübl and Huneke ([6, Proposition 2.2]) on Cohen-Macaulayness of $G_I(A)$. The condition $r(I, M) \leq 1$ of the above theorem is now replaced by the weaker condition $JM \cap I^{n+1}M = JI^nM$ for all $n \geq 2$. But we make an extra assumption that $l_M(I) = \dim M$.

***Theorem* 2** — *Let $(A, \mathfrak{m})$ be a Cohen-Macaulay ring and $M$ be a Cohen-Macaulay $A-$module of dimension $d > 1$. Let I be an unmixed ideal with respect to $M$ such that $\mathrm{ht}_M(I) = d - 1 = s$ and $l_M(I) = d$. Assume $I$ is generically a complete intersection with respect to $M$. Let $J$ be a minimal reduction of $I$ satisfying the condition $JM \cap I^{n+1}M = JI^nM$ for all $n \geq 2$. Then $G_I(M)$ is Cohen-Macaulay.*

In section 5 we extend Burch's inequality to modules.

***Theorem* 3** — *Let $(A, \mathfrak{m})$ be local, $I$ an ideal in $A$. Let $M$ be an $A-$module. Then*

$$l_M(I) \leq \dim M - \inf\{\mathrm{depth}(M/I^nM) \mid n \geq 1\}.$$

It is of some interest to know when the bound is attained. We generalize the result of Eisenbud and Huneke to show that the bound is attained when associated graded module of the ideal is CM, see ([3, Proposition 3.3]).

***Theorem* 4** — *Let $(A, \mathfrak{m})$ be local, $I$ an ideal in $A$. Let $M$ be an $A-$module.*

*Suppose $G_I(M)$ is CM. Then*

$$l_M(I) = \dim M - \inf\{\operatorname{depth}(M/I^nM) \mid n \geq 1\}.$$

In section 1 we introduce notation and state some preliminary results. In section 2 we extend the techniques of Huckaba and Huneke to the module case i.e. when $M$ is a CM module. As in the ring case the condition $r(I, M) \leq 1$ plays an important role. In section 3 we prove Theorem 1 mentioned above. In section 4 we prove Theorem 2. In section 5 we prove Theorem 3 where Burch's inequality is proved in the module case. We then prove Theorem 4 wherein the bound in the Theorem 3 is attained under the assuption that $G_I(M)$ is CM. In the last section we provide examples of ideals of analytic deviation one with respect to modules. We make the following observations for maximal CM modules:

(i) If $I$ is generically complete intersection with respect to $A$ then it is so with respect to $M$.

(ii) $\operatorname{ht}(I) \leq \operatorname{ht}_M(I) \leq l_M(I) \leq l(I)$.

(iii) If $\operatorname{ad}(I) = 1$ then $\operatorname{ad}_M(I) \leq 1$.

We thus see that for maximal CM modules our results can be applied to get the Cohen-Macaulayness of $G_I(M)$.

## 1. Preliminaries

In this paper all rings are commutative Noetherian and all modules are assumed to be finitely generated. Let $(A, \mathfrak{m})$ be a CM local ring with infinite residue field $k = A/\mathfrak{m}$. Let $M$ be a C-M $A$-module. Let $I$ be a proper ideal with $\operatorname{ht}_M(I) = d \geq 1$. We say an ideal $I$ in $A$ is *unmixed* with respect to $M$ if all associated primes of $M/IM$ have same height. We shall denote the sequence $a_1, ..., a_d$ by $\underline{a}$. An ideal $I$ in $A$ is said to be equidimensional with respect to $M$ if $\dim(M/PM) = \dim(M/IM)$ for all minimal primes $P$ of $\operatorname{Supp}(M/IM)$.

The following result is well known. We include it here for the lack of suitable reference.

*Lemma* 1.1 — Let $A$ be a ring, $M$ be an $A-$module Let $L$ and $N$ be $A-$submodules of $M$ such that $L_P \subseteq N_P$ for all $P \in \operatorname{Ass}(M/N)$ then $L \subseteq N$.

PROOF : Consider the submodule $D = (L+N)/N$ of $M/N$. If $P$ is an associated prime of $M/N$ then by assumption $D_P = 0$. Now since $\operatorname{Ass}(D) \subseteq \operatorname{Ass}(M/N)$ it follows that $\operatorname{Ass}(D)$ is empty set. So $D = 0$. Therefore $L \subseteq N$.

*Lemma* 1.2 — Let $A$ be Cohen-Macaulay and $M$ be a Cohen-Macaulay $A-$module. Assume that $\underline{a}$ is an $M-$regular sequence. Then $M/(\underline{a})^i M$ is Cohen-Macaulay $A/(\underline{a})^i-$module for all $i \geq 1$.

PROOF : Since $\operatorname{Supp}(M/(\underline{a})^i M) = \operatorname{Supp}(M/(\underline{a})M)$ we have the following

$$\dim\left(\frac{M}{(\underline{a})^i M}\right) = \dim\left(\frac{M}{(\underline{a})M}\right).$$

Consider the following exact sequence

$$0 \longrightarrow (\underline{a})^{i-1}M/(\underline{a})^i M \longrightarrow M/(\underline{a})^i M \longrightarrow M/(\underline{a})^{i-1}M \longrightarrow 0$$

By [2, Theorem 1.1.8] we see that $(\underline{a})^{i-1}M/(\underline{a})^i M \cong \bigoplus_{j=1}^{k} M/(\underline{a})M$ for some $k$. The result now follows using induction on $i$. □

When $A$ is Cohen-Macaulay we know that $\operatorname{grade}(I) = \operatorname{ht}(I)$. Similarly we have,

*Lemma* 1.3 — Let $(A, \mathfrak{m})$ be a local ring and let $M$ be a Cohen-Macaulay $A-$module. Then

$$\operatorname{grade}(I, M) = \operatorname{ht}_M(I).$$

PROOF : If $\operatorname{ht}_M(I) = 0$ then it is esay to see that $\operatorname{grade}(I, M) = 0$. So assume that $\operatorname{ht}_M(I) = s \geq 1$. So

$$\begin{aligned}\operatorname{ht}_M(I) &= \inf\{\dim M_P \mid P \in \operatorname{Supp}(M/IM)\}\\ &= \inf\{\operatorname{depth} M_P \mid P \in \operatorname{Supp}(M/IM)\}.\end{aligned}$$

Observe that if $P \in V(I) \setminus \operatorname{Supp}(M)$ then $\operatorname{depth} M_P = \infty$. So we have the following

$$\begin{aligned}\operatorname{ht}_M(I) &= \inf\{\operatorname{depth} M_P \mid P \in V(I)\}\\ &= \operatorname{grade}(I, M).\end{aligned}$$

□

*Remark* 1.4 : Let $J$ be reduction of $I$ with respect to $M$ then we see that

$$\operatorname{Supp}(M/IM) = \operatorname{Supp}(M/JM)$$

To see this let $P \in \operatorname{Supp}(M/JM)$. Suppose $I \nsubseteq P$ then since for $n \gg 0$ $I^{n+1}M = JI^nM$ we have $M_P = J_P M_P$. This gives us $M_P = 0$ contradicting the fact that $P \in \operatorname{Supp}(M)$. So $I \subseteq P$ and hence $P \in \operatorname{Supp}(M/IM)$. The other inclusion is obvious.

*Remark* 1.5 : Let $I$ be an ideal in $A$ and $M$ an $A-$module then $\operatorname{grade}(I, M) \leq \mu(I)$.

The following result is well known in the case of rings.

*Lemma* 1.6 — Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring, $M$ be a Cohen-Macaulay $A-$module. Then

$$\operatorname{ht}_M(I) \leq l_M(I) \leq \dim M.$$

PROOF : Let $l_M(I) = s$. Let $J$ be minimal reduction of $I$ with respect to $M$. So $\mu(J) = s$. From Remark 1.4 it follows that $\dim M/IM = \dim M/JM$. Hence $\operatorname{grade}(I, M) = \operatorname{grade}(J, M)$. Since $M$ is Cohen-Macaulay it follows from Lemma 1.3 that $\operatorname{ht}_M(I) = \operatorname{grade}(I, M)$. By Remark 1.5 we have $\operatorname{grade}(J, M) \leq \mu(J)$. So $\operatorname{ht}_M(I) \leq l_M(I)$. For the other inequality we observe that $F_I(M)$ is quotient of $G_I(M)$. Therefore $l_I(M) \leq \dim G_I(M) = \dim M$.

## 2. BASIC TECHNIQUES

In this section we prove some basic results which we shall often use in subsequent sections.

*Proposition* 2.1 — Let $A$ be a local ring, $M$ be an $A-$module. Let $I$ be an ideal of $A$ with $\operatorname{ht}_M(I) = s \geq 1$.

(i) If $\underline{a}$ is an $M-$regular sequence with $a_1, ..., a_s \in I$ then

$$(\underline{a})M/(\underline{a})IM \cong \bigoplus_{i=1}^{i=d}(M/IM).$$

(ii) If $c \in A$ is an $M-$regular element, then

$$\operatorname{depth}(M/cIM) \geq \min\{\operatorname{depth}(M/IM), \operatorname{depth}(M/(c)M)\}.$$

(iii) Suppose $M$ is Cohen-Macaulay, $\underline{a}$ is an $M-$regular sequence in $I$ that generates $I$ generically with respect to $M$, and $c \in I$ such that

$$\mathrm{Supp}(M/IM) = \mathrm{Supp}(M/(\underline{a}, c)M)$$

then $((\underline{a})^m M :_M c^n) \cap I^m M = (\underline{a})^m M$ for all positive integers $m, n$.

PROOF : (i) Let $J = (a_1, ..., a_s)$. By [2, 1.1.8] we have

$$M/JM[X_1, ..., X_s] \cong \bigoplus_{n \geq 0} J^n M/J^{n+1}M.$$

Therefore $\bigoplus_{i=1}^s M/JM \cong JM/J^2M$. Now tensoring with $R/I$ gives the result.

(ii) Consider the exact sequence

$$0 \longrightarrow \frac{(c)M}{cIM} \longrightarrow \frac{M}{cIM} \longrightarrow \frac{M}{(c)M} \longrightarrow 0.$$

Since $c$ is $M-$regular we have $(c)M/cIM \cong M/IM$. So by [2, 1.2.9]

$$\mathrm{depth}(M/cIM) \geq \min\{\mathrm{depth}(M/IM), \mathrm{depth}(M/(c)M)\}$$

(iii) Fix $m \geq 1$. Let $\mathrm{Ass}(M/(\underline{a})^m M) = \{P_1, ..., P_k, Q_1, ..., Q_l\}$, where $I \subseteq P_i$ for $i = 1, ..., k$ and $I \nsubseteq Q_i$ for $i = 1, ..., l$. We write $(\underline{a})^m M = L \cap N$ where $L$ is the intersection of primary components of $(\underline{a})^m M$ belonging to $\{P_1, ..., P_k\}$ and $N$ is the intersection of the primary components of $(\underline{a})^m M$ belonging to $\{Q_1, ..., Q_l\}$.

Using Lemma 1.2 we have that $M/(\underline{a})^m M$ is Cohen-Macaulay. So we have $\mathrm{ht}(P_i) = \mathrm{ht}(Q_j) = s$ for all $i, j$.

*Claim*: $I^m M \subseteq L$.

Note that $\mathrm{Ass}(M/L) = \{P_1, ..., P_k\}$. Since $\{a_1, ..., a_s\}$ generates $I$ generically with respect to $M$ and $\{P_1, ..., P_k\} \subseteq \mathrm{Min}(M/IM)$ we have $(I^m M)_{P_i} = (\underline{a})^m_{P_i} M_{P_i} \subseteq (L)_{P_i}$ for all $P_i \in \mathrm{Ass}(M/L)$. It follows from Lemma 1.1 that $I^m M \subseteq L$. Thus the claim is true.

It is clear that $(\underline{a})^m M \subseteq ((\underline{a})^m M :_M c^n) \cap I^m M$. For reverse inclusion let $c^n t \in (\underline{a})^m M$ where $t \in I^m M$. Thus $c^n t \in L \cap N = (\underline{a})^m M$.

Since $I \nsubseteq Q_i$ for $i = 1, ..., l$ and $\mathrm{Supp}(M/IM) = \mathrm{Supp}(M/(\underline{a}, c)M)$, we must have $c^n \notin Q_i$ for $i = 1, ..., l$. For if $c^n \in Q_i$ then $c \in Q_i$ which gives

$(\underline{a}, c)^m \subseteq Q_i$. But then $I \subseteq Q_i$ because $\operatorname{Supp}(M/IM) = \operatorname{Supp}(M/(\underline{a}, c)M)$. This is not possible since $I \nsubseteq Q_i$.

Now let $N = N_1 \cap N_2 \cap ... \cap N_l$ be a primary decomposition of $N$. Since $c^n \notin Q_i$ for $i = 1, ..., l$ and $Q_i = \sqrt{\operatorname{ann}(M/N_i)}$ it follows that $c^n M \nsubseteq N_i$ for $i = 1, ..., l$. Since $c^n t \in N_i$ and $N_i$ is $Q_i-$primary submodule we get that $t \in N_i$ for $i = 1, ..., l$. Thus $t \in N$. So $t \in N \cap I^m M \subseteq N \cap L = (\underline{a})^m M$ and the result follows.

The following result allows us to choose good generating set for minimal reductions of ideals having analytic deviation one. This is known in the ring case. We present a proof of it in the module case. This proof is different from that available in the literature.

*Lemma* 2.2 — Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring, $M$ be Cohen-Macaulay $A-$ module. Let $I$ be an ideal in $A$ and $J$ be a minimal reduction of $I$ w.r.t. $M$. Let $\operatorname{ht}_M(I) = s \geq 1$ and $l_M(I) = \operatorname{ht}_M(I) + 1$. Assume $I$ is generically a complete intersection w.r.t. $M$. Then there exists $\{a_1, ..., a_s\}$ an $M-$regular sequence and $c$ an $M-$regular element such that $J = (a_1, ..., a_s, c)$ with $\{a_1, ..., a_s\}$ generating $I$ generically w.r.t $M$.

PROOF : Let $A = \operatorname{Min}(\operatorname{Supp}(M/IM)) = \{P_1, ..., P_n\}$ and $\operatorname{Min}(M) = \{Q_1, ..., Q_m\}$. Since $I$ is generically a complete intersection we have for $P \in A$, $I_P = (x_1, ..., x_s)$ where $\{x_1, ..., x_s\}$ is $M_P-$regular sequence in $R_P$.

Since $J_P$ is a reduction of $I_P$ we have that $J_P = I_P$ for all $P \in A$. We therefore have $\dim_{k(P_i)} J_{P_i}/P_i R_{P_i} J_{P_i} = s$. We also have $\dim_k J/\mathfrak{m}\mathfrak{J} = \mathfrak{s} + \mathfrak{1}$ since $J$ is minimal reduction of $I$ and $l_M(I) = s + 1$.

Consider the following natural maps

$$\phi : J \rightarrow \frac{J}{\mathfrak{m}\mathfrak{J}}$$

$$\psi_i : J \rightarrow \frac{J_{P_i}}{P_i R_{P_i} J_{P_i}}$$

Now let $T = \{J \cap Q_1, J \cap Q_2, ..., J \cap Q_m\}$. Consider the images of $J \cap Q_i$ in $J/\mathfrak{m}\mathfrak{J}$. Set $V_i = (J \cap Q_i + \mathfrak{m}\mathfrak{J})/\mathfrak{m}\mathfrak{J} \subseteq \mathfrak{J}/\mathfrak{m}\mathfrak{J}$. Since $\operatorname{grade}(I, M) = \operatorname{grade}(J, M)$ we get that $J \nsubseteq Q_i$ for each $i$. So by Nakayama's lemma $J \cap Q_i + \mathfrak{m}\mathfrak{J} \neq \mathfrak{J}$. Thus $V_i \subset J/\mathfrak{m}\mathfrak{J}$ are proper subspaces of $J/\mathfrak{m}\mathfrak{J}$. Set $K_i = \psi_i^{-1}(0)$. Note that since $I$ is generically a complete intersection, $K_i$ are ideals properly contained in $J$. Let $W_i$

be the image of $K_i$ in $J/\mathfrak{m}\mathfrak{J}$. Since $K_i$ are proper in $J$ so by Nakayama's lemma it follows that $W_i$ are proper subspaces in $J/\mathfrak{m}\mathfrak{J}$. Now since $J/\mathfrak{m}\mathfrak{J}$ is a vector space over an infinite field there exists $\bar{a}_1 \in J/\mathfrak{m}\mathfrak{J} \setminus \{\bigcup_{\mathfrak{i}=\mathfrak{1}}^{\mathfrak{m}} \mathfrak{V}_{\mathfrak{i}} \bigcup_{\mathfrak{i}=\mathfrak{1}}^{\mathfrak{n}} \mathfrak{W}_{\mathfrak{i}}\}$. Note first that $a_1$ is one of the minimal generators of $J$. Clearly $a_1 \notin J \cap Q_i$ for $1 \leq i \leq m$ so that $a_1$ is $M-$regular element. Also since $a_1 \notin K_i$ for $1 \leq i \leq n$ we have $a_1/1 \notin P_i R_{P_i} J_{P_i}$. Hence $a_1/1$ is one of the minimal generators of $J_{P_i}$.

Now going modulo $(a_1)$ and repeating the same argument we can choose $a_2$ such that $\{a_1, a_2\}$ is an $M-$regular sequence. Again as above $a_2/1$ is one of the minimal generators of $J_{P_i}$. We continue doing so and arrive at the set $\{a_1, ..., a_s\}$. By our choice this is an $M-$regular sequence which forms a part of minimal generating set of $J$. Also $\{a_1/1, ..., a_s/1\}$ is part of minimal generating set of $J_{P_i}$.

Since $\dim_{k(P_i)} J_{P_i}/P_i R_{P_i} J_{P_i} = s$ we get $J_{P_i} = (a_1/1, ..., a_s/1)$. Since $(\bar{a}_1, ..., \bar{a}_s)$ is a proper subspace of $J/\mathfrak{m}\mathfrak{J}$ we can choose $\bar{c} \in J/\mathfrak{m}\mathfrak{J} \setminus \{\bigcup_{\mathfrak{i}=\mathfrak{1}}^{\mathfrak{m}} \mathfrak{V}_{\mathfrak{i}} \bigcup (\bar{\mathfrak{a}}_{\mathfrak{1}}, ..., \bar{\mathfrak{a}}_{\mathfrak{s}})\}$. Clearly $c$ is $M-$regular. Since $\dim_k J/\mathfrak{m}\mathfrak{J} = \mathfrak{s}+\mathfrak{1}$ we get that $J/\mathfrak{m}\mathfrak{J} = (\bar{\mathfrak{a}}_{\mathfrak{1}}, ..., \bar{\mathfrak{a}}_{\mathfrak{s}}, \bar{\mathfrak{c}})$. Hence it generates $I$ generically.

*Lemma* 2.3 — Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring, $M$ a Cohen-Macaulay $A-$ module, $I$ an ideal in $R$ with $\mathrm{ht}_M(I) = s \geq 1$. Suppose $\dim A/I \geq 1$. Let $\{a_1, ..., a_s\}$ be an $M-$regular sequence in $I$ that generates $I$ generically. Let $c \in I$ be an $M-$regular element such that $(\underline{a}, c)IM = I^2M$. Then for $1 \leq i \leq n$ we have

(i) $I^nM = (\underline{a})^i I^{n-i} M + c^{n-i} I^i M.$

(ii) $(\underline{a})^i M \cap c^{n-i} I^i M = (\underline{a})^i c^{n-i} M.$

(iii) $(\underline{a})^i M \cap I^n M = (\underline{a})^i I^{n-i} M.$

(iv) $a_1^*, ..., a_s^*$ is a $G_I(M)-$regular sequence.

PROOF : (i) Fix $n \geq 1$.

For $i = n$. We have $(\underline{a})^n M \subseteq I^n M$. Therefore $(\underline{a})^n M + I^n M = I^n M$. Also note that when $n = 1$ we have $i = 1$. Hence $(\underline{a})M + IM = IM$.

So let $n \geq 2$ and let $1 \leq i \leq n-1$. Since $I^2M = (\underline{a}, c)IM$ we have

$$\begin{aligned} I^nM &= ((\underline{a}) + (c))^{n-1}IM \\ &= ((\underline{a})^{n-1} + (\underline{a})^{n-1}c + ... + (c)^{n-1})IM \\ &= ((\underline{a})^{n-1} + ... + (\underline{a})^i c^{n-i-1})IM + ((\underline{a})^{i-1}c^{n-i} + ... + (c)^{n-1})IM \\ &= (\underline{a})^i((\underline{a})^{n-i-1} + ... + c^{n-i-1})IM + c^{n-i}((\underline{a})^{i-1} + ... + (c)^{i-1})IM \\ &= (\underline{a})^i((\underline{a}) + (c))^{n-i-1}IM + c^{n-i}((\underline{a}) + (c))^{i-1}IM \\ &= (\underline{a})^i I^{n-i}M + c^{n-i}I^iM. \end{aligned}$$

(ii) Clearly $(\underline{a})^iM \cap c^{n-i}I^iM \supseteq (\underline{a})^i c^{n-i}M$. For the other inclusion let $c^{n-i}t \in (\underline{a})^iM$ where $t \in I^iM$. This gives us $t \in ((\underline{a})^iM :_M c^{n-i}) = (\underline{a})^iM$ (by Proposition 2.1(ii)). Therefore $c^{n-i}t \in (\underline{a})^i c^{n-i}M$ giving us the other inclusion. So (ii) follows.

(iii) By (i) we have $I^nM = (\underline{a})^i I^{n-i}M + c^{n-i}I^iM$. So

$$\begin{aligned} (\underline{a})^iM \cap I^nM &= (\underline{a})^iM \cap ((\underline{a})^i)I^{n-i}M + c^{n-i}I^iM) \\ &= (\underline{a})^i I^{n-i}M + (\underline{a})^iM \cap c^{n-i}I^iM \\ &= (\underline{a})^i I^{n-i}M + (\underline{a})^i c^{n-i}M \quad \text{(by (ii) above).} \\ &= (\underline{a})^i(I^{n-i}M + (c)^{n-i}M) \\ &= (\underline{a})^i I^{n-i}M \quad \text{for } n \geq 1 \text{ and } 1 \leq i \leq n. \end{aligned}$$

(iv) Taking $i = 1$ in (iii) above we get $(\underline{a})M \cap I^nM = (\underline{a})I^{n-1}M$ for all $n \geq 1$. So from Valabrega-Valla Theorem [12, 2.6] it follows that $a_1^*, ..., a_s^*$ is a $G_I(M)-$regular sequence.

***Theorem* 2.4** — *Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring, $M$ be a Cohen-Macaulay $A-$module and $I$ an ideal in $A$ with $\mathrm{ht}_M(I) = s \geq 1$. Suppose $\dim A/I \geq 1$. Let $\{a_1, ..., a_s\}$ be an $M-$regular sequence in $I$ that generates $I$ generically and let $c \in I$ be an $M-$regular element such that $(\underline{a}, c)IM = I^2M$. Then for $1 \leq i \leq n$ we have*

$$\mathrm{depth}\left(\frac{M}{((\underline{a})^i + I^n)M}\right) \geq \min\{\mathrm{depth}(M/I^iM), \dim(M/IM) - 1\}.$$

PROOF : Fix $n \geq 1$.

For $i = n$. We have $(\underline{a})^n M \subseteq I^n M$. Therefore $(\underline{a})^n M + I^n M = I^n M$. So $\operatorname{depth}(M/((\underline{a})^n + I^n)M) = \operatorname{depth}(M/I^n M)$. The result follows in this case.

So let $1 \leq i \leq n-1$. By Lemma 2.3(i)we write $I^n M = (\underline{a})^i I^{n-i} M + c^{n-i} I^i M$.

So $(\underline{a})^i M + I^n M = (\underline{a})^i M + c^{n-i} I^i M$. Consider the usual exact sequence

$$0 \longrightarrow \frac{M}{(\underline{a})^i M \cap c^{n-i} I^i M} \longrightarrow \frac{M}{(\underline{a})^i M} \bigoplus \frac{M}{c^{n-i} I^i M} \longrightarrow \frac{M}{(\underline{a})^i M + I^n M} \longrightarrow 0.$$

Using Lemma 1.2 and 1.3 we get the following

$$\begin{aligned} \dim(M/IM) &= \dim(M) - \operatorname{grade}(I, M) \\ &= \dim(M) - s \\ &= \dim(M/(\underline{a})^i M) \end{aligned}$$

It follows from [2, 1.2.9] that

$$\operatorname{depth}(\frac{M}{((\underline{a})^i + I^n)M}) \geq \operatorname{depth}(\frac{M}{(\underline{a})^i M} \bigoplus \frac{M}{c^{n-i} I^i M})$$

or

$$\operatorname{depth}(M/((\underline{a})^i + I^n)M) = \operatorname{depth}(M/((\underline{a})^i M \cap c^{n-i} I^i M)) - 1.$$

Case (i): $$\begin{aligned} \operatorname{depth}(\frac{M}{((\underline{a})^i + I^n)M}) &\geq \operatorname{depth}(\frac{M}{(\underline{a})^i M} \bigoplus \frac{M}{c^{n-i} I^i M}) \\ &\geq \min\{\operatorname{depth}(\frac{M}{(\underline{a})^i M}), \operatorname{depth}(\frac{M}{c^{n-i} I^i M})\} \end{aligned}$$

By Lemma 1.2 we have $\operatorname{depth}(M/(\underline{a})^i M) = \dim M/IM$. Note also that since $\operatorname{ht}_M(I) \geq 1$ we have from Proposition 2.1 (ii) that

$$\operatorname{depth}(\frac{M}{c^{n-i} I^i M}) \geq \min\{\operatorname{depth}(\frac{M}{I^i M}), \operatorname{depth}(\frac{M}{cM})\} = \operatorname{depth}(\frac{M}{I^i M}).$$

This takes care of the first case.

Case(ii) : $$\operatorname{depth}(\frac{M}{((\underline{a})^i + I^n)M}) = \operatorname{depth}(\frac{M}{((\underline{a})^i M \cap c^{n-i} I^i M)}) - 1.$$

By Lemma 2.3 (ii) we have $(\underline{a})^i M \cap c^{n-i} I^i M = (\underline{a})^i c^{n-i} M$. Now by the Proposition 2.1(i) we get

$$\begin{aligned} \operatorname{depth}(\frac{M}{c^{n-i}(\underline{a})^i M}) &\geq \min\{\operatorname{depth}(\frac{M}{(\underline{a})^i M}), \operatorname{depth}(\frac{M}{c^{n-i} M})\} \\ &= \operatorname{depth}(\frac{M}{(\underline{a})^i M}) \\ &= \dim(\frac{M}{IM}). \end{aligned}$$

So $\operatorname{depth}(M/((\underline{a})^i M \cap c^{n-i} I^i M)) - 1 \geq \dim(M/IM) - 1$ and the result follows.

## 3. Cohen-Macaulayness of Associated Graded Module

Let $(A, \mathfrak{m})$ be a CM local ring. Let $I$ be an ideal in $A$ with $\operatorname{ht}(I) = d \geq 1$ and $l(I) = d + 1$. Let $I$ be generically a complete intersection and $r(I) \leq 1$. Huckaba and Huneke in their paper ([7, Theorem 2.9]) prove that $G_A(I)$ is CM if $\operatorname{depth}(A/I) \geq \dim(A/I) - 1$. In Theorem 3.6 we extend this result to CM modules.

For the purpose of induction we need the following result.

*Lemma* 3.1 — Let $(A, \mathfrak{m})$ be local ring, $I$ an ideal in $A$ and $M$ an $A-$module. If $x \in \mathfrak{m} \setminus \mathfrak{I}$ and $x^*$ is $G_I(M)$-regular element then

$$l_{\bar{M}}(\frac{(I, x)}{(x)}) = l_M(I)$$

where $\bar{M} = M/xM$.

Proof : Set $T = A/(x)$ and $K = (I, x)/(x)$. Let $R(I, M) = \bigoplus_{i \geq 0} I^i M t^i$ and $T(K, \bar{M}) = \bigoplus_{i \geq 0} K^i \bar{M} t^i$. We first show that $T(K, \bar{M}) \cong R(I, M)/(x)R(I, M)$. We observe that

$$K^i \bar{M} = \frac{(I^i, x)}{(x)} \bar{M} = \frac{I^i M + xM}{xM} \cong \frac{I^i M}{xM \cap I^i M}.$$

Since $x^*$ is $G_I(M)$-regular we have $xM \cap I^i M = x I^i M$ for all $i$. Hence

$$\frac{I^i M}{xM \cap I^i M} \cong \frac{I^i M}{x I^i M}$$

and we derive the isomorphisms

$$T(K,\bar{M}) = \bigoplus_{i\geq 0} K^i \bar{M} t^i \cong \bigoplus_{i\geq 0} \frac{I^i M}{xI^i M} t^i \cong R(I,M)/(x)R(I,M).$$

This isomorphism implies that

$$\frac{R(I,M)}{\mathfrak{m}\mathfrak{R}(\mathfrak{I},\mathfrak{M})} \cong \frac{T(K,\bar{M})}{\mathfrak{m}/(\mathfrak{x})\mathfrak{T}(\mathfrak{K},\bar{\mathfrak{M}})}.$$

Hence $l_M(I) = l_{\bar{M}}((I,x)/(x))$. □

Our main focus in this thesis is to look at maximal Cohen-Macaulay modules $M$ and generically a complete intersection ideals $I$ with $l_M(I) = d = \dim A$ and $\mathrm{ht}_M(I) = d-1$. In this case if $r(I,M) \leq 1$ then the following result shows that $G_I(M)$ is Cohen-Macaulay. This result is an essential ingredient in the proof of Theorem 3.6.

***Theorem* 3.2** — *Let* $(A,\mathfrak{m})$ *be a Cohen-Macaulay local ring and* $M$ *be a Cohen-Macaulay* $A-$*module. Let* $I$ *be an ideal in* $A$ *of analytic deviation one with respect to* $M$. *Assume* $\mathrm{ht}_M(I) = s \geq 1$, $r(I,M) \leq 1$, *and* $I$ *is generically a complete intersection with respect to* $M$. *If* $\dim(M/IM) = 1$ *then* $G_I(M)$ *is Cohen-Macaulay.*

PROOF : Let $J$ be a minimal reduction of $I$ such that $JIM = I^2M$. Choose $\{a_1,...,a_s,c\}$ as in Lemma 2.2. By Lemma 2.3(iii) we have $(\underline{a}) \cap I^n M = (\underline{a}) I^{n-1} M$. Now by the Proposition 2.3(iv) we get that $\{a_1^*,...,a_s^*\}$ is $G_I(M)-$regular.

Since $\dim(M/IM) = 1$ we have $\dim(M) = s+1$. To show that $G_I(M)$ is Cohen-Macaulay it is enough to find a regular sequence of length $s+1$ in the graded maximal ideal $\mathfrak{m}/\mathfrak{I}\bigoplus_{\mathfrak{n}\geq 1}\mathfrak{I}^{\mathfrak{n}}/\mathfrak{I}^{\mathfrak{n}+1}$. Choose $y \in \mathfrak{m}$ such that $\{a_1,...,a_s,y\}$ is an $M-$regular sequence. Note that $y^* \in \mathfrak{m}/\mathfrak{I}$ and $c^* \in I/I^2$.

*Claim* 1: $\{a_1^*,...,a_s^*,y^*+c^*\}$ is $G_I(M)$-regular sequence.

So let $(y^*+c^*)(u(0)^*+...+u(n)^*) = a_1^*(r_1(0)^*+...+r_1(n)^*)+...+a_s^*(r_s(0)^*+...+r_s(n)^*)$ where $deg(u(i)^*) = i$ and $deg(r_j(i)^*) = i$ for each $i$. We show that $u(0)^*+...+u(n)^* \in (a_1^*,...,a_s^*)G_I(M)$. This can be achieved by reducing $n$ and using induction. This we do in the following claims.

*Claim* 1.1 : $u(n)^* \in (a_1^*,...,a_s^*)G_I(M)$ *for* $n \geq 1$.

Let $u(i)$ and $r_j(i)$ be liftings of $u(i)^*$ and $r_j(i)^*$ respectively for $0 \leq i \leq n$ and $1 \leq j \leq s$. From the relation above we have $u(n)c - a_1r_1(n) - ... - a_sr_s(n) \in I^{n+2}M$ where $r_i(n) \in I^nM$. Hence $u(n)c \in (\underline{a})I^nM + I^{n+2}M$. Writing $I^{n+2}M = (\underline{a})I^{n+1}M + cI^{n+1}M$ it follows that $u(n)c \in (\underline{a})I^nM + cI^{n+1}M$. Hence there exists $b \in I^{n+1}M$ such that $u(n)c - bc \in (\underline{a})M$ i.e. $(u(n)-b)c \in (\underline{a})M$. Since $n \geq 1$, $u(n)-b \in ((\underline{a})M :_M c)\bigcap IM$. By Lemma 2.1(ii) $((\underline{a})M :_M c)\bigcap IM = (\underline{a})M$. Thus $u(n) - b \in (\underline{a})M$. So $u(n) - b \in (\underline{a})M \bigcap I^nM$. Since $(\underline{a})M \bigcap I^nM = (\underline{a})I^{n-1}M$ we see that $u(n) \in (\underline{a})I^{n-1}M + I^{n+1}M$. It follows that $u(n)^* \in (a_1^*, ..., a_s^*)G_I(M)$.

Thus we can reduce $n$ if $n \geq 1$. So suppose now that $n = 0$.
*Claim* 1.2 : $u(0)^* = 0$.

Since $n = 0$ we must have $yu(0) \in IM$ and $cu(0) - a_1r_1(0) - ... - a_sr_s(0) \in I^2M$. Writing $I^2M = (\underline{a})IM + cIM$ we see that $cu(0) \in (\underline{a})M + cIM$. Thus $cyu(0) \in y(\underline{a})M + cyIM$. Hence there exists $b_1 \in IM$ such that $c(yu(0) - yb_1) \in (\underline{a})M$. But since $yu(0) - yb_1 \in IM$ we obtain $yu(0) - yb_1 \in ((\underline{a})M :_M c) \cap IM$. So $yu(0) - yb_1 \in (\underline{a})M$ by Lemma 2.1. Also since $\{a_1, ..., a_s, y\}$ is $M-$regular sequence it follows that $u(0) - b_1 \in (\underline{a})M$. Hence $u(0) \in IM$ which gives $u(0)^* = 0$. This proves our claim. Hence $G_I(M)$ is Cohen-Macaulay.

Following lemma shows that a certain set of primes is a Zariski closed set. This fact is used in the proof of the main theorem. This is similar to a result in [?].

*Lemma* 3.3 — Let $(A, \mathfrak{m})$ be Cohen-Macaulay local ring and $M$ be a Cohen-Macaulay $A-$module. Let $I$ an ideal in $A$ with $\mathrm{ht}_M(I) = s \geq 1$. Then the set

$$N = \{Q \in Spec(A) \mid Q \supset I \text{ and } I_Q \text{ is not a complete intersection w.r.t. } M_Q\}$$

is Zariski closed.

PROOF : We show that $N^c$ is Zariski open. Since $\mathrm{grade}(I, M) = \mathrm{ht}_M(I)$ it follows from [2, 1.6.19] that $N^c = \{Q \in Spec(A) \mid I_Q$ is generated by $s$ elements $\}$. Let $P \in N^c$ and so $I_P = A_Px_1 + ... + A_Px_s$. We may assume that $x_i \in A$. Consider the linear map $\phi : A^s \longrightarrow I$ defined by $\phi(a_1, ...a_d) = a_1x_1 + ... + a_dx_s$. Let $C$ be the cokernel of this map. Localizing the exact sequence $A^s \longrightarrow I \longrightarrow C \longrightarrow 0$ at a prime ideal $Q$ we get an exact sequence

$$A_Q^s \longrightarrow I_Q \longrightarrow C_Q \longrightarrow 0$$

and when $Q = P$ we get $C_Q = 0$. Clearly $\mathrm{Supp}(C) = V(\mathrm{ann}\, C)$ is closed set in $Spec(A)$ and $P \notin \mathrm{Supp}(C)$. Set $V = SpecA \setminus \mathrm{Supp}(C)$. Clearly $V$ is open. Notice that $C_Q = 0$ for $Q \in V$. Thus $P \in V \subseteq N^c$. Hence $N^c$ is an open set. □

In the proof of the main theorem we need an element which is $G_I(M)-$regular for the purpose of induction. Following proposition guarantees the existence of such an element.

*Proposition* 3.4 — Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring and $M$ a Cohen-Macaulay $A-$module. Let $I$ be an ideal of analytic deviation one with respect to $M$. Assume $\operatorname{ht}_M(I) = s \geq 1$, $r(I, M) \leq 1$, and $I$ is generically a complete intersection with respect to $M$. If $x$ is regular on $M/((\underline{a}) + I^n)M$ for all $n \geq 1$ then $x^*$ is $G_I(M)-$regular.

PROOF : Choose $J = (a_1, ..., a_s, c)$ a minimal reduction of $I$ as in Theorem 3.2. Notice $\{a_1^*, ..., a_s^*\}$ is $G_I(M)-$regular sequence by Proposition 2.3(iv). It suffices to show that $\{a_1^*, ..., a_s^*, x^*\}$ is a regular sequence because under any permutation it would again form a regular sequence. So let $x^*u(n)^* = a_1^*r_1(n-1)^* + ... + a_s^*r_s(n-1)^*$ be homogeneous relation of degree $n$. Let $u(i)$ and $r_j(i)$ be liftings of $u(i)^*$ and $r_j(i)^*$ respectively for $0 \leq i \leq n$ and $1 \leq j \leq s$. Thus $xu(n) - a_1r_1(n-1) - ... - a_sr_s(n-1) \in I^{n+1}M$. Therefore $xu(n) \in (\underline{a})M + I^{n+1}M$. Since $x$ is $M/((\underline{a}) + I^n)M-$regular it follows that $u(n) \in ((\underline{a}) + I^{n+1})M \cap I^nM$. Since $((\underline{a}) + I^{n+1})M \cap I^nM = (\underline{a})I^{n-1}M + I^nM$ by 2.3(iii) we have $u(n) \in (\underline{a})I^{n-1}M + I^nM$. Hence $u(n)^* \in (a_1^*, ..., a_s^*)G_I(M)$. So $x^*$ is $G_I(M)-$regular.

*Lemma* 3.5 — Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring and $M$ a Cohen-Macaulay $A-$module. Let $I$ be an equidimensional ideal with respect to $M$ with $\operatorname{ht}_M(I) = s \geq 1$.

(i) If $x \in \mathfrak{m}$ and $P \in \operatorname{Min}(M/xM)$ then $\operatorname{ht}_M(P) \leq 1$.

If $Q \in \operatorname{Supp}(M/IM)$ then $\operatorname{ht}_M(I) = \operatorname{ht}_{M_Q}(I_Q)$.

If $x$ is $M/IM-$regular and $Q \in \operatorname{Min}(M/(I, x)M)$ then $(I, x)$ is equidimensional with respect to $M$. In particular $\operatorname{ht}_M(I, x) = s + 1$.

PROOF : (i) Let $P \in \operatorname{Min}(M/xM)$. Clearly $x(A/\operatorname{ann} M) = (x, \operatorname{ann} M)/\operatorname{ann} M) \subseteq P/\operatorname{ann} M$. By Krull's Principal Ideal theorem we have $\operatorname{ht}_{A/\operatorname{ann} M}(P/\operatorname{ann} M) \leq 1$. Therefore $\dim M_P = \dim A_P/\operatorname{ann} M_P \leq 1$. Thus $\operatorname{ht}_M(P) \leq 1$.

(ii) First observe that $s = \operatorname{grade}(I, M) \leq \operatorname{grade}(I_Q, M_Q) = \operatorname{ht}_{M_Q}(I_Q)$. Since $I$ is equidimensional w.r.t. $M$ we have $\operatorname{ht}_M(I) = \operatorname{ht}_M(P)$ for $P \in \operatorname{Min}(M/IM)$. Let $P \in \operatorname{Min}(M/IM)$ be such that $P \subseteq Q$. Clearly $I_Q \subseteq PA_Q$ and $\operatorname{ann} M_Q \subseteq$

$PA_Q$. Hence

$$\mathrm{ht}_{M_Q}(I_Q) \leq \dim(M_Q)_{PA_Q} = \dim M_P = \mathrm{ht}_M(P) = s.$$

Therefore $\mathrm{ht}_M(I) = \mathrm{ht}_{M_Q}(I_Q)$.

(iii) Since $Q \in \mathrm{Min}(M/(I,x)M)$ we get $Q/I \in \mathrm{Min}(\frac{M/IM}{x(M/IM)})$. Note that we have

$$\begin{aligned}\mathrm{ht}_{M/IM}(Q/I) &= \dim(M/IM)_{Q/I}\\ &= \dim M_Q/I_Q M_Q\\ &= \dim M_Q - \mathrm{grade}(I_Q, M_Q)\\ &= \mathrm{ht}_M(Q) - \mathrm{ht}_M(I) \ \text{ by (ii) above.}\end{aligned}$$

By (i) above we get that $\mathrm{ht}_{M/IM}(Q/I) \leq 1$. When $\mathrm{ht}_{M/IM}(Q/I) = 0$ it follows that $\mathrm{ht}_M(Q) = \mathrm{ht}_M(I)$. This is not possible since $x$ is $M/IM-$regular. So $\mathrm{ht}_{M/IM}(Q/I) = 1$. Hence $\mathrm{ht}_M(Q) = s+1$. Thus $(I,x)$ is equidimensional with respect to $M$ and $\mathrm{ht}_M(I,x) = s+1$. □

We now prove one of the main theorems of this paper. In the case of interest i.e. when $\dim M = d$ and $\mathrm{ht}_M(I) = d-1$ the condition $\mathrm{depth}(M/IM) \geq \dim(M/IM)-1$ of the theorem below is trivially satisfied. The condition $r(I,M) \leq 1$ however is an important condition. This is crucially used in the proof of the theorem.

***Theorem* 3.6** — *Let $(A,\mathfrak{m})$ be a Cohen-Macaulay local ring and $M$ a Cohen-Macaulay $A-$module. Let $I$ be an equidimensional ideal of analytic deviation one with respect to $M$. Assume $\mathrm{ht}_M(I) = s \geq 1$, $r(I,M) \leq 1$, and $I$ is generically a complete intersection with respect to $M$. If $\mathrm{depth}(M/IM) \geq \dim(M/IM) - 1$ then $G_I(M)$ is Cohen-Macaulay.*

PROOF : Assume $\mathrm{depth}(M/IM) \geq \dim(M/IM) - 1$. Let $J$ be a minimal reduction of $I$ such that $JIM = I^2M$. Choose $\{a_1, ..., a_s, c\}$ as in Lemma 2.2. By Proposition 2.3(iv) we get that $\{a_1^*, ..., a_s^*\}$ is $G_I(M)-$regular. To show $G_I(M)$ is Cohen-Macaulay it is enough to find a regular sequence of length $\dim M$ in the graded maximal ideal $\mathfrak{m}/\mathfrak{I} \bigoplus_{\mathfrak{n}\geq 1} \mathfrak{I}^{\mathfrak{n}}/\mathfrak{I}^{\mathfrak{n}+1}$. We have found first $s$ elements of this regular sequence namely $\{a_1^*, ..., a_s^*\}$. We now want to find $\dim M - s = \dim(M/IM)$ many elements. We do this by induction on $\dim(M/IM)$. When $\dim(M/IM) = 1$ then $G_I(M)$ is Cohen-Macaulay by Theorem 3.2. Suppose

$\dim(M/IM) = m > 1$ and assume the claim is true if $\dim(M/IM) < m$. Now consider the following set

$$N = \{Q \in Spec(A) \mid Q \supset I \text{ and } I_Q \text{ is not a complete intersection w.r.t. } M_Q\}$$

By the Lemma 3.3 the set $N$ is Zariski closed and hence $N \cap \operatorname{Supp}(M) = V(B)$ for some ideal $B$ containing $I$. Since $I$ is generically a complete intersection it is easily seen that $\operatorname{ht}_M(B) > s$. Thus the set

$$\begin{aligned} \mathfrak{A} &= N \cap \operatorname{Supp}(M) \cap \{Q \in Spec(A) \mid Q \supset I \ and \ \operatorname{ht}_M(Q) = s+1\} \\ &\subseteq \operatorname{Min}\{\operatorname{Supp}(M/BM)\} \end{aligned}$$

and so $\mathfrak{A}$ is a finite set.

Consider the following set

$$\mathfrak{B} = \{Q \in Spec(A) \mid Q \in \operatorname{Ass}(M/((\underline{a}) + I^n)M) \ for \ some \ n \geq 1\} \cup \operatorname{Min}(A)$$

Observe that

$$\frac{M}{((\underline{a}) + I^n)M} = \frac{M/(\underline{a})M}{I^n(M/(\underline{a})M)}$$

It follows that $\mathfrak{B}$ is a finite set(see [1]). By Theorem 2.4 we have for all $n \geq 1$

$$\operatorname{depth}\left(\frac{M}{((\underline{a}) + I^n)M}\right) \geq \min\{\operatorname{depth}(M/IM), \dim(M/IM) - 1\}$$

By assumption $\operatorname{depth}(M/IM) \geq \dim(M/IM) - 1$ which gives the following

$$\operatorname{depth}(M/((\underline{a}) + I^n)M) \geq \dim(M/IM) - 1 \geq 1.$$

Since $\dim(A/I) > 1$ and $\operatorname{depth}(M/((\underline{a}) + I^n)M) \geq 1$ we see that $\mathfrak{m} \notin \mathfrak{A} \cup \mathfrak{B}$. Hence we can choose an element $x \in \mathfrak{m} \setminus \bigcup_{\mathfrak{Q} \in \mathfrak{A} \cup \mathfrak{B}} \mathfrak{Q}$. Then $x$ satisfies the following conditions:

(i) $x$ is regular on $M/((\underline{a}) + I^n)M$ for all $n \geq 1$ as $x \notin \mathfrak{B}$.

(ii) $\operatorname{ht}_M(I, x) = s + 1$ by Lemma 3.5(iii).

(iii) $(I, x)$ is generically a complete intersection: Let $Q \in \operatorname{Min}\{\operatorname{Supp}(M/(I, x)M)\}$.

So by Lemma 3.5(iii) we get $\operatorname{ht}_M(Q) = s + 1$. Also $x \notin P$ for $P \in \mathfrak{A}$ it follows that $Q \notin N$. So $I_Q$ is complete intersection. Also since $x$ is $M/IM$-regular it follows that $(I, x)_Q$ is complete intersection.

Set $K = \frac{(I,x)}{(x)}$, $T = A/(x)$ and $\bar{M} = M/xM$. By our choice of $x$, $x$ is $A-$regular. So $T$ is Cohen-Macaulay. We now have

(a) $\mathrm{ht}_M(I) = \mathrm{ht}_{\bar{M}}(K)$.

(b) $\dim(\bar{M}/K\bar{M}) = m - 1$.

(c) $\mathrm{depth}(\bar{M}/K\bar{M}) \geq \dim(\bar{M}/K\bar{M}) - 1$.

(a) and (b) are easy to see. For (c) we argue as follows:

Since $x$ is $M/IM$-regular we have $\mathrm{depth}(\bar{M}/K\bar{M}) = \mathrm{depth}(M/IM) - 1$. So by assumption $\mathrm{depth}(\bar{M}/K\bar{M}) \geq \dim(M/IM) - 2$. By (b) above it follows that $\mathrm{depth}(\bar{M}/K\bar{M}) \geq \dim(\bar{M}/K\bar{M}) - 1$.

Since $x$ is regular on $M/((\underline{a}) + I^n)M$ for all $n \geq 1$ it follows from Proposition 3.4 that $x^*$ is $G_I(M)-$regular. So by Lemma 3.1 we get $l_M(I) = l_{\bar{M}}(K)$. Thus $r(K, \bar{M}) \leq r(I, M) \leq 1$. Now if $Q \in \mathrm{Min}\{\mathrm{Supp}(M/(I,x)M)\}$ then $I_Q$ is a complete intersection since $x \notin Q$ if $Q \in N$ (by our choice of $x$). Hence $K$ is generically a complete intersection. Therefore the induction hypotheses are all satisfied for the pair $\{\bar{M}, K\}$ so we conclude that $G_K(\bar{M}) \cong G_I(M)/(x^*)G_I(M)$ is Cohen-Macaulay. Hence $G_I(M)$ is Cohen-Macaulay. This completes the proof of the theorem. □

## 4. IDEALS OF REDUCTION NUMBER AT MOST TWO

We now weaken one of the assumptions of the above theorem. Namely we replace the condition $r(I, M) \leq 1$ by the condition $JM \cap I^{n+1}M = JI^nM$ for all $n \geq 2$. Note that when $r(I, M) \leq 2$ this intersection condition is satisfied. We assume that $l_M(I) = \dim M$. Under this condition we get the Cohen-Macaulayness of $G_I(M)$. This we do in the following theorem where we generalize the theorem of Hübl R. and Huneke C. ([6, Proposition 2.2])to the module case.

***Theorem* 4.1** — *Let $(A, \mathfrak{m})$ be a Cohen-Macaulay ring and $M$ be a Cohen-Macaulay $A-$module of dimension $d > 1$. Let I be an unmixed ideal with respect to $M$ such that $\mathrm{ht}_M(I) = d - 1 = s$ and $l_M(I) = d$. Assume $I$ is generically a complete intersection with respect to $M$. Let $J$ be a minimal reduction of $I$ satisfying the condition $JM \cap I^{n+1}M = JI^nM$ for all $n \geq 2$. Then $G_I(M)$ is Cohen-Macaulay.*

PROOF : We first note that by Lemma 2.2 there exists $\{a_1, ..., a_s\}$ an $M-$regular sequence and $c$ an $M-$regular element such that $J = (a_1, ..., a_s, c)$ such that $\{a_1, ..., a_s\}$ generates $I$ generically with respect to $M$. We claim the following:

*Claim* : $a_1^*, ..., a_s^*$ is $G_I(M)-$regular.

From Valabrega-Valla criterion [12, 2.6] it is enough to show that

$$(\underline{a})M \cap I^{n+1}M = (\underline{a})I^nM \quad for\ all\ n \geq 1.$$

We prove this by induction on $n$. There is nothing to show when $n = 0$. So let $n = 1$. We first observe that $(\underline{a})I$ is unmixed ideal with respect to $M$. This is because by 2.1(iii) we have

$$(\underline{a})M/(\underline{a})IM \cong \bigoplus_{i=1}^{i=s}(M/IM).$$

Now the short exact sequence

$$0 \longrightarrow (\underline{a})M/(\underline{a})IM \longrightarrow M/(\underline{a})IM \longrightarrow M/(\underline{a})M \longrightarrow 0$$

implies that

$$\begin{aligned}\mathrm{Ass}(M/(\underline{a})IM) &\subseteq \mathrm{Ass}((\underline{a})M/(\underline{a})IM) \cup \mathrm{Ass}(M/(\underline{a})M)\\ &= \mathrm{Min}(M/IM) \cup \mathrm{Min}(M/(\underline{a})M).\end{aligned}$$

Since $I$ is unmixed with respect to $M$ and $M/(\underline{a})M$ is Cohen-Macaulay, all these primes have the same height. Hence $(\underline{a})I$ is unmixed. We now have to show $(\underline{a})M \cap I^2M \subseteq (\underline{a})IM$. As $(\underline{a})I$ is unmixed by Lemma 1.1 it is enough to check this at the minimal primes of $(\underline{a})I$. So let $P \in \mathrm{Min}(M/(\underline{a})IM)$. If $I \subseteq P$ then $P \in \mathrm{Min}(M/IM)$ and therefore

$$((\underline{a})M)_P = (IM)_P,$$

since $I$ is generically a complete intersection with respect to $M$.

Hence

$$\begin{aligned}((\underline{a})M \cap I^2M)_P &\subseteq ((\underline{a})M)_P \cap (I^2M)_P\\ &\subseteq (I^2M)_P\\ &= ((\underline{a})IM)_P.\end{aligned}$$

If $I \nsubseteq P$ then $(IM)_P = M_P$ and therefore

$$((\underline{a})M \cap I^2M)_P \subseteq ((\underline{a})M)_P = ((\underline{a})IM)_P.$$

Hence $(\underline{a})M \cap I^2M \subseteq (\underline{a})IM$.

Now let $n \geq 2$. We show the following

$$(\underline{a})M \cap I^{n+1}M \subseteq (\underline{a})I^nM.$$

By assumption we have

$$(\underline{a})M \cap I^{n+1}M \subseteq JM \cap I^{n+1}M = JI^nM.$$

Let $m \in (\underline{a})M \cap I^{n+1}M$ and so we have
(i) $m = \sum_{i=1}^{s} a_i m_i \in (\underline{a})M$.
(ii)$m = \sum_{i=1}^{s} a_i u_i n_i + cn$.
where $m_i, n_i \in M, u_i \in I^n$ and $n \in I^nM$. Hence

$$cn = \sum_{i=1}^{s} a_i(m_i - u_i n_i) \in (\underline{a})M.$$

So we get from 2.1(iii) that

$$n \in ((\underline{a})M :_M c) \cap I^nM = (\underline{a})M \cap I^nM.$$

Now by induction hypothesis it follows that $n \in (\underline{a})I^{n-1}M$ and so the claim is proved since $c \in I$.

Let $S = G_I(A)/(a_1^*, ..., a_s^*),\ N = G_I(M)/(a_1^*, ..., a_s^*)G_I(M)$. We show that $N$ is Cohen-Macaulay. Since $\dim(N) = 1$ and $S$ is graded local it is enough to show that $h\text{–}soc(N) = (0)$ where $h\text{–}soc(N) = (0 :_N \mathfrak{m})$ is the homogeneous socle of $N$. We assume the contrary i.e. let $m^* \in N$ be a non-zero homogeneous element belonging to $h\text{–}soc(N)$. So we have $(\mathfrak{m}_{\mathfrak{A}}, \mathfrak{S}_+)\mathfrak{m}^* = (\mathfrak{o})$.

If $deg(m^*) = 0$, then $m^* = m + IM$ for some $m \in M$. It follows that $\mathfrak{m}_{\mathfrak{A}}\mathfrak{m} \subseteq \mathfrak{I}\mathfrak{M}$, which contradicts to the fact that $M/IM$ is Cohen-Macaulay.

Suppose now that $deg(m^*) \geq 1$. We now replace $A$ by $\bar{A}$, $I$ by $\bar{I}$ and $M$ by $\bar{M}$ where bar denotes reduction by $(a_1, ..., a_s)$. As $a_1^*, ..., a_s^*$ is $G_I(M)-$regular sequence we have

$$G_{\bar{I}}(\bar{M}) = G_I(M)/(a_1^*, ..., a_s^*)G_I(M) = N.$$

Note that $\bar{J} = (\bar{c})$ is a reduction of $\bar{I}$ with respect to $\bar{M}$. Since $(\underline{a}) \subseteq J$ we have the following equality

$$\bar{J}\bar{M} \cap \bar{I}^{n+1}\bar{M} = \bar{J}\bar{I}^n\bar{M} \text{ for each } n \geq 2.$$

Hence we may assume that $s = 0$. Now set $m^* := m + I^{n+1}M$ for some $m \in I^nM \setminus I^{n+1}M$. Since $n \geq 1$ and $(\mathfrak{m}_{\mathfrak{A}}, \mathfrak{G}_+)\mathfrak{m}^* = (\mathfrak{o})$ we get the following

$$cm \in I^{n+2}M \cap (c)M = I^{n+2}M \cap JM = JI^{n+1}M.$$

So there exists $n \in I^{n+1}M$ such that $c(m-n) = 0$.

Hence $m - n \in (0 :_M c) \cap IM = 0$ by Proposition 2.1(iii). Thus $m \in I^{n+1}M$ a contradiction. This completes the proof of the theorem. □

## 5. Generalization to Burch's Inequality

Let $(A, \mathfrak{m})$ be a local ring and $I$ an ideal in $A$. Following inequality known as Burch's inequality is well known.

$$l(I) \leq \dim A - \inf\{\text{depth}(A/I^n) \mid n \geq 1\}.$$

The bound is attained when $G_I(A)$ is CM. This is proved by Eisenbud and Huneke (see [3, Proposition 3.3]). In this section we prove the inequality in the module case and also show that bound is attained when $G_I(M)$ is CM. To prove Burch's inequality for modules we need the following lemma. The proof is analogous to the one given in [10, 2.1]. We put $G = G_I(A)$.

*Lemma* 5.1 — Let $(A, \mathfrak{m})$ be a local ring and $M$ be an $A-$module. Then

$$\text{grade } (\mathfrak{m}\mathfrak{G}, \mathfrak{G}_{\mathfrak{I}}(\mathfrak{M})) = \inf\{\text{depth } \mathfrak{M}/\mathfrak{I}^{\mathfrak{n}}\mathfrak{M} \mid \mathfrak{n} \geq \mathfrak{1}\}.$$

PROOF : Let $r = \text{grade}\,(\mathfrak{m}\mathfrak{G}, \mathfrak{G}_{\mathfrak{I}}(\mathfrak{M}))$ and $s = \inf\{\text{depth}\, M/I^nM \mid n \geq 1\}$. We prove the result by induction on $r$.

Let $r = 0$. Then there exists $P \in \text{Ass}\, G_I(M)$ such that $\mathfrak{m}\mathfrak{G} \subseteq \mathfrak{P}$. Thus $P = (0 :_G f)$ for some homogeneous $f \in G_I(M)$. Let $n = deg(f)$. Since $\mathfrak{m}\mathfrak{f} = \mathfrak{o}$, we get that $\mathfrak{m} \in \text{Ass}_{\mathfrak{R}}\, \mathfrak{G}_{\mathfrak{I}}(\mathfrak{M})_{\mathfrak{n}} \subseteq \text{Ass}\, \mathfrak{M}/\mathfrak{I}^{\mathfrak{n}+1}\mathfrak{M}$. Therefore $\text{depth}(M/I^nM) = 0$.

Now assume that $r > 0$. Given any $a \in \mathfrak{m}$ such that its image in $G_I(R)$ is $G_I(M)-$regular, then it easily follows that $a$ is regular on $M/I^nM$ for any $n \geq 1$. Hence we get $s > 0$. We set $S = A/aA$, $\mathfrak{n} = \mathfrak{m}\mathfrak{S}$, $\bar{\mathfrak{I}} = \mathfrak{I}\mathfrak{S}$ and $N = M/aM$. By Valabrega-Valla theorem [12, 2.6] we have $G_{\bar{I}}(N) = G_I(M)/aG_I(M)$. Therefore $\text{grade}\,(\mathfrak{n}\mathfrak{G}(\bar{\mathfrak{I}}), \mathfrak{G}_{\bar{\mathfrak{I}}}(\mathfrak{N})) = \mathfrak{r} - \mathfrak{1}$. By the induction hypothesis, we have

$$\inf\{\text{depth}\, N/I^nN \mid n \geq 1\} = r - 1.$$

Since $a$ is $M/I^nM-$regular, we have that $\text{depth}(M/I^nM) = \text{depth}(N/I^nN) + 1$ for any $n \geq 1$. The conclusion now follows. □

We now prove Burch's inequality,

***Theorem* 5.2** — *Let $(A, \mathfrak{m})$ be local, $I$ an ideal in $A$. Let $M$ be an $A-$module. Then*

$$l_M(I) \leq \dim M - \inf\{\text{depth}(M/I^nM) \mid n \geq 1\}.$$

PROOF :

$$\begin{aligned} l_M(I) &= \dim\ G_I(M)/\mathfrak{m}\mathfrak{G}_{\mathfrak{I}}(\mathfrak{M}) \\ &\leq \dim\ G_I(M) - \text{grade}(\mathfrak{m}\mathfrak{G}, \mathfrak{G}_{\mathfrak{I}}(\mathfrak{M})) \\ &= \dim M - \inf\{\text{depth}\, M/I^nM \mid n \geq 1\} \ \text{(By Lemma 5.1).} \end{aligned}$$

□

Finally we extend Eisenbud and Huneke's result to modules,

Theorem 5.3 — *Let $(A, \mathfrak{m})$ be local, $I$ an ideal in $A$. Let $M$ be an $A-$module. Suppose $G_I(M)$ is Cohen-Macaulay. Then*

$$l_M(I) = \dim M - \inf\{\text{depth}(M/I^nM) \mid n \geq 1\}.$$

PROOF : Let $s = \inf\{\text{depth}(M/I^nM) \mid n \geq 1\}$. We prove the result by induction on $s$. So suppose first that $s = 0$. Set $J = \mathfrak{m}\mathfrak{G}_{\mathfrak{I}}(\mathfrak{A})$. By definition

$l_M(I) = \dim(G_I(M)/JG_I(M))$. We first show that $\operatorname{grade}(J, G_I(M)) = 0$. Now suppose on the contrary that
$\operatorname{grade}(J, G_I(M)) > 0$. So we can choose an element $x \in \mathfrak{m} \setminus \mathfrak{I}$ such that $x^* \in A/I$ is a nonzero divisor on $G_I(M)$. It then follows that $x$ is nonzero divisor on $M/I^nM$ for all $n \geq 1$. This contradicts to the fact that $s = 0$. Thus $\operatorname{grade}(J, G_I(M))$ $= 0$. Since $G_I(M)$ is Cohen-Macaulay we get the following

$$\begin{aligned}\dim(G_I(M)/JG_I(M)) &= \dim(G_I(M)) - \operatorname{grade}(J, G_I(M)) \\ &= \dim(G_I(M))\end{aligned}$$

Hence $l_M(I) = \dim(M)$. Now let $s > 0$. Since $\bigcup_{n\geq 1} \operatorname{Ass}(M/I^nM)$ is a finite set (see [1]) we can choose an $A-$regular element $x \in \mathfrak{m}$ which is also regular on the modules $M/I^nM$ for all $n \geq 1$. It follows that $x^* \in A/I$ is $G_I(M)-$regular. We therefore have

$$G_I(M)/(x^*)G_I(M) = G_{\bar{I}}(\bar{M})$$

where $\bar{I} = IA/(x)$ and $\bar{M} = M/(x)M$. Since $G_I(M)$ is Cohen-Macaulay and $x^*$ is $G_I(M)-$regular we get that $G_{\bar{I}}(\bar{M})$ is Cohen-Macaulay. Thus $\dim(M) = \dim(\bar{M}) + 1$. Using Lemma 3.1 it follows that $l_{\bar{M}}(\bar{I}) = l_M(I)$. Since $x$ is regular sequence on the modules $M/I^nM$ we have $\operatorname{depth}(\bar{M}/\bar{I}^n\bar{M}) = \operatorname{depth}(M/I^nM)-$ $1$. Hence

$$\inf\{\operatorname{depth}(\bar{M}/\bar{I}^n\bar{M}) \mid n \geq 1\} = s - 1.$$

The result now follows by induction hypothesis.

## 6. EXAMPLES

Let $A$ be a CM ring of dimension $d \geq 2$. Let $I$ be an ideal in $A$ with $\operatorname{ht}(I) = d - 1 \geq 1$ and $l(I) = d$. Suppose $I$ is generically a complete intersection. Let $M$ be a maximal CM $A-$module with $\dim M/IM = 1$. In this section we show that ideals of analytic deviation one in the ring case can be carried over to the case of maximal CM modules. This is the content of the following lemma.

*Lemma* 6.1 — Let $A$ be a CM ring of dimension $d \geq 2$. Let $M$ be a maximal CM $A-$module. Let $I$ be an ideal in $A$ with $\operatorname{ht}(I) = d-1 \geq 1$ and $\dim M/IM =$ $1$. Then

(i) If $I$ is generically a complete intersection with respect to $A$ then it is so with respect to $M$.

(ii) If $\mathrm{ad}(I) = 1$ then $\mathrm{ad}_M(I) \leq 1$.

PROOF : (i) We show that $I_P$ is generated by $M_P-$regular sequence for all primes $P \in \mathrm{Min}\{\mathrm{Supp}(M/IM)\}$. So let $P \in \mathrm{Min}\{\mathrm{Supp}(M/IM)\}$. Since $\dim M/IM = 1$ and $\mathrm{ht}(I) = d-1$ we see that $\mathrm{Min}\{\mathrm{Supp}(M/IM)\} \subseteq \mathrm{Min}(A/I)$. Now since $I$ is generically a complete intersection, $I_P$ is generated by $A_P-$regular sequence. We note that since $P \in \mathrm{Supp}(M/IM)$, $M_P$ is maximal CM $A_P-$module. So generators of $I_P$ form an $M_P-$regular sequence. Thus $I_P$ is a complete intersection with respect to $M_P$. Hence $I$ is generically a complete intersection with respect to $M$.

(ii) Note first that $\mathrm{ht}(I) = \mathrm{ht}_M(I) = d - 1$. From Lemma 1.6 we have $\mathrm{ht}_M(I) \leq l_M(I)$. It is easy to see that for any ideal $I$ one has $l_M(I) \leq l(I)$. Thus

$$\mathrm{ht}(I) = \mathrm{ht}_M(I) \leq l_M(I) \leq l(I).$$

□

The result now follows.

*Remark* 6.2 : Suppose $\mathrm{ad}(I) = 1$ then from the Lemma 6.1 above it follows that $\mathrm{ad}_M(I) \leq 1$. Thus either $\mathrm{ad}_M(I) = 0$ or $\mathrm{ad}_M(I) = 1$.

(i) In the case when $\mathrm{ad}_M(I) = 0$ we can use Valabrega-Valla theorem to study Cohen-Macaulayness of $G_I(M)$.

(ii) When $\mathrm{ad}_M(I) = 1$ and $r(I, M) \leq 2$ we are in the situation of Theorem 3.6. Thus it follows from Theorem 3.6 that $G_I(M)$ is CM.

(iii) When $\mathrm{ad}_M(I) = 1$ and $r(I, M) \leq 2$ then it follows from Theorem 4.1 that $G_I(M)$ is CM.

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