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k-IDEMPOTENCY OF LINEAR COMBINATIONS OF AN IDEMPOTENT MATRIX AND A TRIPOTENT MATRIX THAT COMMUTE

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A complete solution is established to the problem of characterizing all situations in which a linear combination $C = c_1A + c_2B$ of an idempotent matrix A and a tripotent matrix B is *k*-idempotent. As a special case of this, a set of necessary and sufficient conditions for a linear combination $C = c_1A + c_2B$ to be *k*-idempotent when A and B are idempotent matrices, is also studied in this paper.

Key words : *k*-idempotent matrix, idempotent, tripotent, linear combination.

1. Introduction

Baksalary and Baksalary [1] studied and listed a set of situations for a linear combination of two idempotent matrices to be idempotent. This result was generalized in [2], where the idempotency of linear combinations of an idempotent matrix and a tripotent matrix was studied. In this paper, we shall study the problem of characterizing situations for a linear combination $C = c_1A + c_2B$ of an idempotent matrix A and a tripotent matrix B to be *k*-idempotent.

The concept of *k*-idempotent matrices was introduced in [4] , and the spectral and *k*-spectral properties of such matrices were obtained in [5]. Let '*k*' be a fixed product of disjoint transpositions in S_n-the set of all permutations on $\{1, 2, ...n\}$. Then let K be the associated permutation matrix of *k*. Hence, K is involutory (i.e.,$K^2 = I$, where I is the identity matrix of order n). A matrix $A = \langle a_{ij} \rangle$ in $C_{n\times n}$(the set of all complex matrices of order n) is k-idempotent if $\sum_{t=1}^{n} a_{k(i)t}a_{tk(j)}$ for all i and j in $\{1, 2.....n\}$. This is equivalent to $KA^2K = A$.

$$\text{For example, } A = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & -2 & -\sqrt{3} & 0 \\ 0 & \sqrt{3} & 1 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

Here A is a *k*-idempotent matrix, where *k* = (1,4)(2,3). The associated permutation matrix K is a matrix with ones on its southwest - northeast diagonal and zeros everywhere else.

2. Preliminaries and Notations

It is well-known that a tripotent matrix B can uniquely be represented as a difference of two idempotent matrices, say B_1 and B_2 (i.e., $B = B_1 - B_2$), which are disjoint, in the sense that $B_1B_2 = B_2B_1 = 0$ (cf. Lemma 5.6.6 of [6]). If B_1 and B_2 are non-zero, then B is known as an essentially tripotent matrix. Otherwise, B reduces to a scalar multiple of an idempotent matrix. The *k*-idempotency of linear combinations of two commuting idempotent matrices is thoroughly studied in Theorem 1 of Section 3. A set of necessary conditions are listed in Theorem 2 for linear combinations of a commuting idempotent and an essentially tripotent matrix to be k-idempotent.

Let $A \in C_{n\times n}$. It can be proved that the set $\{X \in C_{n\times n} : AXA = A, XAX = X$ and $AX = XA\}$ is empty or a singleton. When it is a singleton, it is customary to say that X is the group inverse of A, and it is then denoted by $A^{\#}$. A matrix $A \in C_{n\times n}$ satisfying $A^{\#} = A^{t-1}$ for $t = 2, 3...$ is called a $\{t\}$-group periodic matrix [3]. Let Ω_3 denotes the set of all cube roots of unity [i.e., Ω_3 is the set containing $1, \omega$ and ω^2 ,where $\omega = exp(2\pi i/3)$]. It was proved in [4] that if A is *k*-idempotent, then it is quadripotent (i.e., $A^4 = KAKKAK = KA^2K = A$).

3. Main Results

Lemma 1 — All *k*-idempotent matrices are $\{3\}$-group periodic.

Proof : Let A be a *k*-idempotent matrix. Since $AA^2A = A$, $A^2AA^2 = A^2$ and $AA^2 = A^2A$, we have $A^{\#} = A^2$. This implies that A is $\{3\}$-group periodic.□

Remark 1 : Let P and Q be two non-zero projectors. If $P = \alpha Q$ for some $\alpha \in C$ (the set of all complex numbers), then $\alpha = 1$.

***Theorem* 1** — *Suppose that A and B are two non-zero, commuting idempotent matrices. Then a linear combination $C = c_1A + c_2B$, with non-zero complex scalars c_1 and c_2, is* k-*idempotent if and only if one of the following conditions* (1) *and* (2) *holds.*

(1) $A - B = 0$ *holds along with either one of the following sets of conditions :*

- *(i)* $c_1 + c_2 = 0$,
- *(ii)* $c_1 + c_2 = 1$ *and* $AK = KA$.

(2) $A - B \neq 0$ *holds along with either one of the following sets of conditions :*

- *(i)* $c_1 = 1$, $c_2 = -1, AB = B$ *and* $K(A - B)K = A - B$,
- *(ii)* $c_1 = \omega, c_2 = \omega^2 - \omega, AB = B$ *and* $K(A - B)K = B$,
- *(iii)* $c_1 = \omega^2, c_2 = \omega - \omega^2, AB = B$ *and* $K(A - B)K = B$,
- *(iv)* $c_1 = \omega, c_2 = \omega^2, AB = 0$ *and* $KBK = A$,
- *(v)* $c_1 = 1$, $c_2 = 1, AB = 0$ *and* $K(A + B)K = A + B$.

The remaining conditions (vi) to (ix) are obtained from (i) to (iv) by interchanging c_1 *by* c_2 *and* A *by* B.

Proof : (1) If $A - B = 0$, then $C = (c_1 + c_2)A$.

The required condition is $K[(c_1 + c_2)A]^2 K = (c_1 + c_2)A$

i.e., $(c_1 + c_2)^2 KAK = (c_1 + c_2)A$ [A is idempotent]

If $c_1 + c_2 = 0$, then $C = 0$, which is situation (i).

If $c_1 + c_2 \neq 0$, then $(c_1 + c_2)KAK = A$.

By Remark 1, we have $c_1 + c_2 = 1$ and hence $KA = AK$, which is situation (ii).

The sufficiency follows by simple computations.

(2) If $A - B \neq 0$, the following relation must be satisfied for C to be *k*-idempotent.

$$K(c_1^2 A + c_2^2 B + 2c_1c_2AB)K = c_1A + c_2B \tag{1}$$

By Lemma 1, $C = c_1A + c_2B$ is $\{3\}$-group periodic. Hence the choice of c_1 and c_2 must be necessarily one among the following cases by Theorem 3.1 of [3].

Case 1 : $c_1 \in \Omega_3$ and $c_1 + c_2 = 0$

By the corresponding sub case $1(b)$ of Theorem 3.1 of [3], we have $AB = B$. It follows from (1) that $c_1K(A - B)K = A - B$. By Remark 1, we have $c_1 = 1$. Therefore $K(A - B)K = A - B$, which is situation(i).

Case 2 : $c_1 \in \Omega_3$ and $c_1 + c_2 \in \Omega_3$

The possibility of $c_2 \in \Omega_3$ is neglected by Note 2 of Section 3 of [3]. Hence we have $AB = B$ by sub case $2(b)$ of Theorem 3.1 of [3].

Pre and Post multiplying (1) by BK and KB respectively, we have

$$(c_1 + c_2)^2 BK = c_1BKA + c_2BKB \tag{2}$$

$$(c_1 + c_2)^2 KB = c_1AKB + c_2BKB \tag{3}$$

Post multiplying (2) by A and B leads respectively to

$$(c_1 + c_2)^2 BKA = c_1BKA + c_2BKB \tag{4}$$

$$(c_1 + c_2)BKB = BKB$$

It follows that $c_1 + c_2 = 1$ or $BKB = 0$.

If $c_1 + c_2 = 1$, then it can be proved from $(2), (3)$ and (4) that $KB = BK$.

Substituting this in (1), we have $c_1 K(A - B)K = A - B$. By Remark 1, we have $c_1 = 1$. This implies that $c_2 = 0$, a contradiction. Therefore, we must have $BKB = 0$.

It follows from (4) that $(c_1 + c_2)^2 = c_1$ or $BKA = 0$.

If $BKA = 0$, then it follows from(2) that $BK = 0$ and then $B = 0$, a contradiction. Therefore, we must have $(c_1 + c_2)^2 = c_1$. Substituting the value of c_2 in (1) implies that

$$c_1 K[A + (c_1^2 - 1)B]K = A + (c_1 - 1)B \tag{5}$$

Cubing (5), we have $KAK = A$. $\quad [AB = B$ and $c_1 \in \Omega_3($ i.e., $c_1^3 = 1)]$

It follows from (5) that $K(A - B)K = B$. $\quad [c_1 \neq 1$ otherwise $c_2 = 0]$

Since $c_1 \in \Omega_3$, we have $c_1 = \omega$ or ω^2. Hence the situation (ii) or (iii) follows.

Case 3 : $c_1 \in \Omega_3$ and $c_2 \in \Omega_3$

It is clear that $c_1 + c_2 \neq 0$. As before, the possibility of $c_1 + c_2 \in \Omega_3$ is neglected by Note 2 of Section 3 of [3]. Hence we have $c_1 + c_2 \notin \{0\} \bigcup \Omega_3$ and $AB = 0$ by sub case $3(b)$ of Theorem 3.1 of [3]. It follows from (1) that

$$K[c_1^2 A + c_2^2 B]K = c_1 A + c_2 B \tag{6}$$

Cubing (6), we have

$$K(A + B)K = A + B \tag{7}$$

Pre and post multiplying (6) by AK and A respectively leads to $c_1^2 AKA = c_1 AKA$, which implies $c_1 = 1$ or $AKA = 0$. Similarly, we get $c_2 = 1$ or $BKB = 0$. Hence we have at least one of the following situations holds.

(a) $AKA = 0$ and $BKB = 0$ (b) $c_1 = 1$ and $c_2 = 1$
(c) $c_1 = 1$ and $BKB = 0$ (d) $c_2 = 1$ and $AKA = 0$

(a) Post multiplying (7) by B and A leads respectively to

$$KAKB = B \text{ and } KBKA = A \tag{8}$$

Pre multiplying (7) by BK and AK leads respectively to

$$AK = AKB \text{ and } BK = BKA \tag{9}$$

It follows from (8) and (9) that $KAK = B$ and $KBK = A$.

Post multiplying (6) by A and B leads respectively to $c_1^2 = c_2$ and $c_2^2 = c_1$.

Hence we have the following three possibilities.

$$c_1 = c_2 = 1 \;\; : \;\; c_1 = \omega, c_2 = \omega^2 \;\; : \;\; c_1 = \omega^2, c_2 = \omega$$

Hence the situations (iv) and (v) are obtained.

(b) The condition (7) follows immediately from (6), which is situation (v).

(c) Post multiplying (6) and (7) by B, we have $KAKB = c_2B = B$.

Since $B \neq 0$, we have $c_2 = 1$. It leads to the situation (v) again along with $BKB = 0$.

(d) This is similar to the sub case (c) and it turns again to the situation (iv).

Interchanging c_1 and c_2 in cases 1 to 3, we see that the role of A and B are interchanged and the conditions (vi) to (ix) are obtained.

By substituting the corresponding sets of conditions (i) to (ix) in (1), the sufficiency can be settled. □

***Theorem* 2** — *Let $B \in C_{n\times n}$ be an essentially tripotent matrix uniquely decomposed as $B = B_1 - B_2$, where B_1 and B_2 are non-zero idempotent matrices such that $B_1B_2 = 0 = B_2B_1$. Let $A \in C_{n\times n}$ be a non-zero idempotent matrix such that $AB = BA$. If a linear combination $C = c_1A + c_2B$ with non-zero $c_1, c_2 \in C$, is a k-idempotent matrix then at least one of the following sets of conditions necessarily hold.*

(i) $c_1 = \frac{1}{2},\ c_2 = -\frac{1}{2},\ AB_1KB_1 = B_1KB_1A$ *and* $AB_2KB_2 = B_2KB_2A$,

(ii) $c_1 = \frac{1}{4},\ c_2 = -\frac{1}{4},\ AB_1KB_1 = B_1KB_1A$ *and* $B_2KB_2A = 0$,

(iii) $c_1 = 2,\ c_2 = -2,\ AB_1KB_1 = B_1KB_1A$ *and* $AB_2KB_2 = 0$,

(iv) $c_1 + c_2 = 0,\ AB_1KB_1 = B_1KB_1A$ *and* $AB_2KB_2 = 0 = B_2KB_2A$,

(v) $c_1 = 2,\ c_2 = 1,\ AB_1KB_1 = 0 = B_1KB_1A$ *and* $B_2KB_2 = B_2KB_2A$,

(vi) $c_1 = \dfrac{5 \pm i\sqrt{7}}{8},\ c_2 = \dfrac{-3 \pm i\sqrt{7}}{8},\ B_1KB_1A = 0$ *and* $AB_2KB_2 = B_2KB_2A$,

(vii) $c_1 = 2 \pm \sqrt{2},\ c_2 = 1 \pm \sqrt{2},\ AB_1KB_1 = 0$ *and* $AB_2KB_2 = B_2KB_2A$,

(viii) $c_1 - c_2 = 1,\ AB_1KB_1A = 0$ *and* $AB_2KB_2 = B_2KB_2A$.

The remaining conditions are obtained from (i) to (viii) by interchanging c_2 *with* $-c_2$ *and* B_1 *with* B_2*. If the choice of* c_1 *and* c_2 *differ from all the above conditions then* $AB_1KB_1A = 0 = AB_2KB_2A$.

PROOF : Since $B = B_1 - B_2$, we have $B^2 = B_1 + B_2$. Solving for B_1 and B_2, we find $B_1 = \dfrac{B + B^2}{2}$ and $B_2 = \dfrac{B^2 - B}{2}$.

Then, it can be easily proved that

'$AB = BA$ holds if and only if $AB_1 = B_1A$ and $AB_2 = B_2A$'

A matrix $C = c_1A + c_2B_1 - c_2B_2$ must satisfy the following to be *k*-idempotent.

$$K[c_1^2A + c_2^2B_1 + c_2^2B_2 + 2c_1c_2AB_1 - 2c_1c_2AB_2]K = c_1A + c_2B_1 - c_2B_2 \quad (10)$$

Pre and post multiplying (10) by B_1K and B_1 respectively, we have

$$[c_1^2AB_1 + c_2^2B_1 + 2c_1c_2AB_1]KB_1 = B_1K[c_1AB_1 + c_2B_1] \quad (11)$$

Pre and post multiplying (11) by A, we have

$$c_1 + c_2 = 0 \text{ otherwise } c_1 + c_2 = 1 \text{ or } AB_1KB_1A = 0 \quad (12)$$

Pre and post multiplying (10) by B_2K and B_2 respectively, we have

$$[c_1^2AB_2 + c_2^2B_2 - 2c_1c_2AB_2]KB_2 = B_2K[c_1AB_2 - c_2B_2] \quad (13)$$

Pre and post multiplying (13) by A, we have

$$c_1 - c_2 = 0 \text{ otherwise } c_1 - c_2 = 1 \text{ or } AB_2KB_2A = 0 \quad (14)$$

While combining the above two sets of situations (12) and (14), we have the following different cases.

Case 1 : $c_1 + c_2 = 0$ and $c_1 - c_2 = 0$

This case is not possible as it leads to $c_2 = 0$, a contradiction.

Case 2 : $c_1 + c_2 = 0$ and $c_1 - c_2 = 1$

It follows immediately that $c_1 = \frac{1}{2}$ and $c_2 = -\frac{1}{2}$. The equation (11) implies that

$$\frac{3}{4}B_1KB_1 = \frac{1}{2}B_1KB_1A + \frac{1}{4}AB_1KB_1 \tag{15}$$

Pre and post multiplying (15) by A, we have $AB_1KB_1 = B_1KB_1A$.

Similarly, we get $AB_2KB_2 = B_2KB_2A$, which is situation (i).

Case 3 : $c_1 + c_2 = 0$ and $AB_2KB_2A = 0$

It follows from (11) that

$$(c_1 + 1)B_1KB_1 = B_1KB_1A + c_1AB_1KB_1 \tag{16}$$

Pre and post multiplying (16) by A, we have $AB_1KB_1 = B_1KB_1A$.

It follows from (13) that

$$3c_1AB_2KB_2 + (c_1 - 1)B_2KB_2 = B_2KB_2A \tag{17}$$

Pre multiplying (17) by A, we have $(4c_1 - 1)AB_2KB_2 = 0$.

Post multiplying (17) by A, we have $(c_1 - 2)B_2KB_2A = 0$.

If $c_1 = \frac{1}{4}$, then $c_2 = -\frac{1}{4}$ and $B_2KB_2A = 0$, which is (ii).

If $c_1 = 2$, then $c_2 = -2$ and $AB_2KB_2A = 0$, which is (iii).

Otherwise $AB_2KB_2 = 0 = B_2KB_2A$, which is (iv).

Case 4 : $c_1 + c_2 = 1$ and $c_1 - c_2 = 1$

This leads to $c_2 = 0$, a contradiction.

Case 5 : $c_1 - c_2 = 1$ and $AB_1KB_1A = 0$

It follows from (13) that

$$(2 - c_1)AB_2KB_2 + (c_1 - 1)B_2KB_2 = B_2KB_2A \tag{18}$$

Pre and post multiplying (18) by A, we have

$$(c_1 - 2)B_2KB_2A = (c_1 - 2)AB_2KB_2 \tag{19}$$

It follows from (11) that

$$(c_1^2 - 3c_1 + 2)B_1KB_1 + (3c_1^2 - 2c_1)AB_1KB_1 = c_1B_1KB_1A \tag{20}$$

Pre and post multiplying (20) by A respectively, we have

$$(4c_1^2 - 5c_1 + 2)AB_1KB_1 = 0 \tag{21}$$

$$(c_1^2 - 4c_1 + 2)B_1KB_1A = 0 \tag{22}$$

(a) Considering (19), if $c_1 = 2$, then $c_2 = 1$. From (18), we have $B_2KB_2 = B_2KB_2A$. It follows from (21) and (22) that $AB_1KB_1 = 0 = B_1KB_1A$, which is (v).

(b) Considering (21), if $c_1 = \dfrac{5 \pm i\sqrt{7}}{8}$, then $c_2 = \dfrac{-3 \pm i\sqrt{7}}{8}$. It follows from (19) and (22) that $AB_2KB_2 = B_2KB_2A$ and $B_1KB_1A = 0$, which is (vi).

(c) Considering (22), if $c_1 = 2 \pm \sqrt{2}$, then $c_2 = 1 \pm \sqrt{2}$. It follows from (19) and (21) that $AB_2KB_2 = B_2KB_2A$ and $AB_1KB_1 = 0$, which is (vii).

(d) If c_1 differ from all the above three subcases (a) to (c), then by (19), we have $AB_2KB_2 = B_2KB_2A$, which is (viii).

The remaining sets of conditions are obtained from the above cases 1 to 5 by interchanging c_2 with $-c_2$ and B_1 with B_2. If c_1 and c_2 do not obey any of the above cases, then the only possibility is $AB_1KB_1A = 0 = AB_2KB_2A$. □

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