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SCALAR CURVATURE OF QR-SUBMANIFOLDS WITH MAXIMAL QR-DIMENSION IN A QUATERNIONIC PROJECTIVE SPACE

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In this paper we derive an integral formula on an n-dimensional, compact, minimal QR-submanifold M of $(p-1)$ QR-dimension immersed in a quaternionic projective space $QP^{(n+p)/4}$. Using this integral formula, we give a sufficient condition concerning with the scalar curvature of M in order that such a submanifold M is to be a tube over a quaternionic projective space.

Key words : quaternionic projective space, scalar curvature, QR-submanifold, maximal QR-dimension, quaternionic invariant distribution, minimal.

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1. Introduction

Let M be a connected real n-dimensional submanifold of real codimension p immersed in a real $(n+p)$-dimensional quaternionic Kähler manifold $\overline{M}$ with quaternionic Kähler structure $\{F, G, H\}$. If there exists a subbundle ν of the normal bundle $TM^{\perp}$ such that

$$\begin{aligned} &F\nu_x \subset \nu_x, \quad G\nu_x \subset \nu_x, \quad H\nu_x \subset \nu_x, \\ &F\nu_x^{\perp} \subset T_xM, \quad G\nu_x^{\perp} \subset T_xM, \quad H\nu_x^{\perp} \subset T_xM \end{aligned} \tag{1.1}$$

for each $x \in M$, where $\nu^{\perp}$ is the complementary orthogonal subbundle to ν in $TM^{\perp}$, then such a submanifold is called *QR-submanifold* of $\overline{M}$ ([2]) and the dimension of ν, the *QR-dimension* of that ([5, 7, 10, 11]). A real hypersurface is a QR-submanifold of zero QR-dimension. When the ambient manifold $\overline{M}$ is a quaternionic projective space $QP^{(n+p)/4}$, real hypersurfaces have been investigated by many authors ([3, 12, 13]) under conditions concerning with shape operator.

In this paper we shall study n-dimensional QR-submanifolds of $(p-1)$ QR-dimension immersed in $QP^{(n+p)/4}$ and prove the following theorem as a generalization of Lawson's result ([12, Theorem 4]) about real hypersurfaces.

***Theorem* 1** — *Let M be an n-dimensional, compact, minimal QR-submanifold of $(p-1)$ QR-dimension immersed in $QP^{(n+p)/4}$. If the scalar curvature of M is greater or equal to n^2+7n-6, then M is $\pi(S^{4s+3}(r_1) \times S^{4t+3}(r_2))$ $(r_1^2 + r_2^2 = 1,\ s+t = (n-3)/4)$, where π is the Hopf fibration $S^{n+4} \rightarrow QP^{(n+1)/4}$.*

Remark : The above main theorem was provided in [10] under the condition that the distinguished normal vector field ξ is parallel with respect to the normal connection $\nabla^{\perp}$.

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2. Preliminaries

Let $\overline{M}$ be a real $(n+p)$-dimensional quaternionic Kähler manifold. Then, by definition, there is a 3-dimensional vector bundle V consisting of tensor fields of type (1,1) over $\overline{M}$ satisfying the following conditions (a), (b) and (c) :

(a) In any coordinate neighborhood $\overline{\mathcal{U}}$, there is a local basis $\{F, G, H\}$ of V such that

$$\begin{aligned} &F^2 = -I,\ G^2 = -I,\ H^2 = -I, \\ &FG = -GF = H,\ GH = -HG = F,\ HF = -FH = G. \end{aligned} \tag{2.1}$$

(b) There is a Riemannian metric g which is Hermitian with respect to all of F, G and H.

(c) For the Riemannian connection $\overline{\nabla}$ with respect to g

$$\begin{pmatrix} \overline{\nabla}F \\ \overline{\nabla}G \\ \overline{\nabla}H \end{pmatrix} = \begin{pmatrix} 0 & r & -q \\ -r & 0 & p \\ q & -p & 0 \end{pmatrix} \begin{pmatrix} F \\ G \\ H \end{pmatrix} \tag{2.2}$$

where p, q and r are local 1-forms defined in $\overline{\mathcal{U}}$. Such a local basis $\{F, G, H\}$ is called a *canonical local basis* of the bundle V in $\overline{\mathcal{U}}$(cf. [6]).

As a direct consequence of (2.1), we can easily see that every quaternionic Kähler manifold is orientable(cf. [6]).

From now on, we consider a real n-dimensional QR-submanifold M of $(p-1)$ QR-dimension immersed in $\overline{M}$. Then it is clear from (1.1) that there is a naturally distinguished unit normal vector field ξ to M such that $F\xi,\ G\xi,\ H\xi \in TM$. We denote these unit vector fields tangent to M by

$$U = -F\xi, \quad V = -G\xi, \quad W = -H\xi. \tag{2.3}$$

Then it is clear from (2.1) and (2.3) that

$$FU = \xi, \quad GV = \xi. \quad HW = \xi. \tag{2.4}$$

On the other hand, each tangent space T_xM is decomposed as

$$T_xM = \mathcal{D}_x \oplus \mathcal{D}_x^{\perp},$$

where $\mathcal{D}_x$ is the maximal quaternionic invariant subspace of T_xM defined by

$$\mathcal{D}_x = T_xM \cap FT_xM \cap GT_xM \cap HT_xM$$

and $\mathcal{D}_x^{\perp}$ its orthogonal complement in T_xM. In this case, we can see that $\mathcal{D}_x^{\perp} = Span\{U, V, W\}$ because of (2.1), (2.3), (2.4) and the definition of $\mathcal{D}_x$, and so

$\mathcal{D} : x \mapsto \mathcal{D}_x$ defines an $(n-3)$-dimensional distribution on M (for details, see [2, 11]) and $n = 4m+3$ for some integer m. Thus M can be dealt with a quternionic CR-submanifold in the sense of Barros-Chen-Urbano ([1]). Furthermore, we have

$$FT_xM,\ GT_xM,\ HT_xM \subset T_xM \oplus Span\{\xi\}$$

and consequently, for any tangent vector field X and for a local orthonormal basis $\{\xi_\alpha\}_{\alpha=1,\dots,p}$ $(\xi_1 := \xi)$ of normal vectors to M, we have the following decomposition in tangential and normal components :

$$FX = \phi X + u(X)\xi, \quad GX = \psi X + v(X)\xi, \quad HX = \theta X + w(X)\xi, \tag{2.5}$$

$$F\xi_\alpha = -U_\alpha + P_1\xi_\alpha, \quad G\xi_\alpha = -V_\alpha + P_2\xi_\alpha, \quad H\xi_\alpha = -W_\alpha + P_3\xi_\alpha \tag{2.6}$$

for $\alpha = 1, \dots, p$. Then it is easily seen that $\{\phi, \psi, \theta\}$ and $\{P_1, P_2, P_3\}$ are skew-symmetric endomorphisms acting on T_xM and $T_xM^\perp$, respectively. Moreover, the Hermitian property of $\{F, G, H\}$ combined with (2.5) and (2.6) implies

$$g(X, U_\alpha) = u(X)\delta_{1\alpha}, \quad g(X, V_\alpha) = v(X)\delta_{1\alpha}, \quad g(X, W_\alpha) = w(X)\delta_{1\alpha}$$

and hence we obtain

$$\begin{aligned} &g(U_1, X) = u(X),\ \ g(V_1, X) = v(X),\ \ g(W_1, X) = w(X), \\ &U_\alpha = 0,\ \ V_\alpha = 0,\ \ W_\alpha = 0, \quad \alpha = 2, \dots, p. \end{aligned} \tag{2.7}$$

On the other hand, comparing (2.3) with (2.6) in the case of $\alpha = 1$, we have $U_1 = U$, $V_1 = V$, $W_1 = W$, which together with (2.3) and (2.7) implies

$$g(U, X) = u(X), \quad g(V, X) = v(X) \quad g(W, X) = w(X), \tag{2.8}$$

$$P_1\xi = 0, \quad P_2\xi = 0, \quad P_3\xi = 0. \tag{2.9}$$

So (2.6), (2.7) and (2.9) yield

$$F\xi_\alpha = \sum_{\beta=2}^{p} P_{1\alpha\beta}\xi_\beta, \quad G\xi_\alpha = \sum_{\beta=2}^{p} P_{2\alpha\beta}\xi_\beta, \quad H\xi_\alpha = \sum_{\beta=2}^{p} P_{3\alpha\beta}\xi_\beta \tag{2.10}$$

for $\alpha = 2, \dots, p$, where we have put

$$P_1\xi_\alpha = \sum_{\beta=1}^{p} P_{1\alpha\beta}\xi_\beta, \quad P_2\xi_\alpha = \sum_{\beta=1}^{p} P_{2\alpha\beta}\xi_\beta, \quad P_3\xi_\alpha = \sum_{\beta=1}^{p} P_{3\alpha\beta}\xi_\beta.$$

In the sequel we shall use the notations U, V, W instead of U_1, V_1, W_1, respectively.

Applying F to the first equation of (2.5), and using (2.1), (2.3), (2.5) and (2.8), we have

$$\phi^2 X = -X + u(X)U, \quad u(\phi X) = 0.$$

Similarly, the second and the third equations of (2.5) also imply

$$\phi^2 X = -X + u(X)U, \quad \psi^2 X = -X + v(X)V, \quad \theta^2 X = -X + w(X)W, \tag{2.11}$$

$$u(\phi X) = 0, \quad v(\psi X) = 0, \quad w(\theta X) = 0. \tag{2.12}$$

From (2.12) and the skew-symmetry of ϕ, ψ and θ, it follows that

$$\phi U = 0, \quad \psi V = 0, \quad \theta W = 0. \tag{2.13}$$

Applying G and H respectively to the first equation of (2.5), and using (2.1), (2.3) and (2.5), we have

$$\begin{aligned} &\theta X + w(X)\xi = -\psi(\phi X) - v(\phi X)\xi + u(X)V, \\ &\psi X + v(X)\xi = \theta(\phi X) + w(\phi X)\xi - u(X)W, \end{aligned}$$

respectively. Thus we can see that

$$\begin{aligned} &\psi(\phi X) = -\theta X + u(X)V, \quad v(\phi X) = -w(X), \\ &\theta(\phi X) = \psi X + u(X)W, \quad w(\phi X) = v(X). \end{aligned} \tag{2.14}$$

Therefore, according to similar method as the above, the second and the third equations of (2.5) also yield respectively

$$\begin{aligned} &\phi(\psi X) = \theta X + v(X)U, \quad u(\psi X) = w(X), \\ &\theta(\psi X) = -\phi X + v(X)W, \quad w(\psi X) = -u(X), \end{aligned} \tag{2.15}$$

$$\begin{aligned} &\phi(\theta X) = -\psi X + w(X)U, \quad u(\theta X) = -v(X), \\ &\psi(\theta X) = \phi X + w(X)V, \quad v(\theta X) = u(X). \end{aligned} \tag{2.16}$$

From (2.13), (2.14), (2.15) and (2.16), it follows that

$$\begin{aligned} &\psi U = -W, \quad v(U) = 0, \quad \theta U = V, \quad w(U) = 0, \\ &\phi V = W, \quad u(V) = 0, \quad \theta V = -U, \quad w(V) = 0, \\ &\phi W = -V, \quad u(W) = 0, \quad \psi W = U, \quad v(W) = 0. \end{aligned} \tag{2.17}$$

The equations (2.8) and (2.11) - (2.17) tell us that M admits the so-called almost contact 3-structure (for definition, see [8]).

On the other hand, by means of (2.10), we can take a local orthonormal basis $\{\xi, \xi_a, \xi_{a^*}, \xi_{a^{**}}, \xi_{a^{***}}\}_{a=1,\ldots,q:=(p-1)/4}$ of normal vectors to M such that

$$\xi_{a^*} := F\xi_a, \quad \xi_{a^{**}} := G\xi_a, \quad \xi_{a^{***}} := H\xi_a. \tag{2.18}$$

Now, let ∇ be the Levi-Civita connection on M and let $\nabla^\perp$ the normal connection of $TM^\perp$ induced from $\overline{\nabla}$. Then Gauss and Weingarten formulae are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{2.19}$$

$$\text{(i)} \quad \overline{\nabla}_X \xi = -AX + \nabla^\perp_X \xi = -AX + \sum_{a=1}^{q}\{s_a(X)\xi_a + s_{a^*}(X)\xi_{a^*} + s_{a^{**}}(X)\xi_{a^{**}} + s_{a^{***}}(X)\xi_{a^{***}}\},$$

(2.20)

$$\text{(ii)} \quad \overline{\nabla}_X \xi_a = -A_a X - s_a(X)\xi + \sum_{b=1}^{q}\{s_{ab}(X)\xi_b + s_{ab^*}(X)\xi_{b^*} + s_{ab^{**}}(X)\xi_{b^{**}} + s_{ab^{***}}(X)\xi_{b^{***}}\},$$

$$\text{(iii)} \quad \overline{\nabla}_X \xi_{a^*} = -A_{a^*} X - s_{a^*}(X)\xi + \sum_{b=1}^{q}\{s_{a^*b}(X)\xi_b + s_{a^*b^*}(X)\xi_{b^*} + s_{a^*b^{**}}(X)\xi_{b^{**}} + s_{a^*b^{***}}(X)\xi_{b^{***}}\},$$

$$\text{(iv)} \quad \overline{\nabla}_X \xi_{a^{**}} = -A_{a^{**}} X - s_{a^{**}}(X)\xi + \sum_{b=1}^{q}\{s_{a^{**}b}(X)\xi_b + s_{a^{**}b^*}(X)\xi_{b^*} + s_{a^{**}b^{**}}(X)\xi_{b^{**}} + s_{a^{**}b^{***}}(X)\xi_{b^{***}}\},$$

$$\text{(v)} \quad \overline{\nabla}_X \xi_{a^{***}} = -A_{a^{***}} X - s_{a^{***}}(X)\xi + \sum_{b=1}^{q}\{s_{a^{***}b}(X)\xi_b + s_{a^{***}b^*}(X)\xi_{b^*} + s_{a^{***}b^{**}}(X)\xi_{b^{**}} + s_{a^{***}b^{***}}(X)\xi_{b^{***}}\}$$

for any vector fields X, Y tangent to M, where $s's$ are coefficients of the normal connection $\nabla^\perp$. Here and in the sequel h denotes the second fundamental form and A, A_a, A_{a^*}, $A_{a^{**}}$ and $A_{a^{***}}$ the shape operators corresponding to the normals N, N_a, N_{a^*}, $N_{a^{**}}$ and $N_{a^{***}}$, respectively. They are related by

$$h(X,Y) = g(AX,Y)\xi + \sum_{a=1}^{q}\{g(A_aX,Y)\xi_a + g(A_{a^*}X,Y)\xi_{a^*} + g(A_{a^{**}}X,Y)\xi_{a^{**}} + g(A_{a^{***}}X,Y)\xi_{a^{***}}\}. \tag{2.21}$$

By means of (2.1), (2.2), (2.3), (2.5), (2.18) and $(2.20)_{\mathrm{(i)-(v)}}$, it can be easily verified that

$$\begin{aligned}
&\text{(i)} \quad A_aX = -\phi A_{a^*}X + s_{a^*}(X)U = -\psi A_{a^{**}}X + s_{a^{**}}(X)V = -\theta A_{a^{***}}X + s_{a^{***}}(X)W,\\
&\text{(ii)} \quad A_{a^*}X = \phi A_aX - s_a(X)U = \psi A_{a^{***}}X - s_{a^{***}}(X)V = -\theta A_{a^{**}}X + s_{a^{**}}(X)W,\\
&\text{(iii)} \quad A_{a^{**}}X = -\phi A_{a^{***}}X + s_{a^{***}}(X)U = \psi A_aX - s_a(X)V = \theta A_{a^*}X - s_{a^*}(X)W,\\
&\text{(iv)} \quad A_{a^{***}}X = \phi A_{a^{**}}X - s_{a^{**}}(X)U = -\psi A_{a^*}X + s_{a^*}(X)V = \theta A_aX - s_a(X)W,
\end{aligned} \tag{2.22}$$

$$\begin{aligned}
&\text{(i)} \quad s_a(X) = -u(A_{a^*}X) = -v(A_{a^{**}}X) = -w(A_{a^{***}}X),\\
&\text{(ii)} \quad s_{a^*}(X) = u(A_aX) = v(A_{a^{***}}X) = -w(A_{a^{**}}X),\\
&\text{(iii)} \quad s_{a^{**}}(X) = -u(A_{a^{***}}X) = v(A_aX) = w(A_{a^*}X),\\
&\text{(iv)} \quad s_{a^{***}}(X) = u(A_{a^{**}}X) = -v(A_{a^*}X) = w(A_aX).
\end{aligned} \tag{2.23}$$

Moreover, since ϕ, ψ, θ are skew-symmetric and A_a, A_{a^*}, $A_{a^{**}}$, $A_{a^{***}}$ are symmetric, $(2.22)_{\mathrm{(i)-(iv)}}$ together with (2.8) yield

$$g((A_a\phi + \phi A_a)X, Y) = s_a(X)u(Y) - s_a(Y)u(X),$$

(i) $g((A_a\psi+\psi A_a)X,Y)=s_a(X)v(Y)-s_a(Y)v(X)$,

$g((A_a\theta+\theta A_a)X,Y)=s_a(X)w(Y)-s_a(Y)w(X)$,

$g((A_{a^*}\phi+\phi A_{a^*})X,Y)=s_{a^*}(X)u(Y)-s_{a^*}(Y)u(X)$, (ii)

$g((A_{a^*}\psi+\psi A_{a^*})X,Y)=s_{a^*}(X)v(Y)-s_{a^*}(Y)v(X)$,

$g((A_{a^*}\theta+\theta A_{a^*})X,Y)=s_{a^*}(X)w(Y)-s_{a^*}(Y)w(X)$,

$g((A_{a^{**}}\phi+\phi A_{a^{**}})X,Y)=s_{a^{**}}(X)u(Y)-s_{a^{**}}(Y)u(X)$,

(iii) $g((A_{a^{**}}\psi+\psi A_{a^{**}})X,Y)=s_{a^{**}}(X)v(Y)-s_{a^{**}}(Y)v(X)$,

$g((A_{a^{**}}\theta+\theta A_{a^{**}})X,Y)=s_{a^{**}}(X)w(Y)-s_{a^{**}}(Y)w(X)$,

$g((A_{a^{***}}\phi+\phi A_{a^{***}})X,Y)=s_{a^{***}}(X)u(Y)-s_{a^{***}}(Y)u(X)$,

(iv) $g((A_{a^{***}}\psi+\psi A_{a^{***}})X,Y)=s_{a^{***}}(X)v(Y)-s_{a^{***}}(Y)v(X)$,

$$g((A_{a^{***}}\theta+\theta A_{a^{***}})X,Y)=s_{a^{***}}(X)w(Y)-s_{a^{***}}(Y)w(X). \tag{2.24}$$

On the other side, we can take an orthonormal basis $\{e_i\}_{i=1,\dots,n=4m+3}$ of tangent vectors to M such that

$$\begin{aligned}&e_{m+1}:=\phi e_1,\dots,e_{2m}:=\phi e_m,\;\; e_{2m+1}:=\psi e_1,\dots,e_{3m}:=\psi e_m,\\ &e_{3m+1}:=\theta e_1,\dots,e_{4m}:=\theta e_m,\;\; e_{4m+1}:=U,\;\; e_{4m+2}:=V,\;\; e_{4m+3}:=W.\end{aligned} \tag{2.25}$$

Replacing X by ϕe_i in $(2.23)_{(\mathrm{i})}$, we have

$$s_a(\phi e_i)=-g(A_{a^*}\phi e_i,U)=-g(A_{a^{**}}\phi e_i,V)=-g(A_{a^{***}}\phi e_i,W),$$

which together with (2.13) and $(2.24)_{(\mathrm{ii})}$ implies

$$s_a(\phi e_i)=-s_{a^*}(e_i)=-g(A_{a^{**}}\phi e_i,V).$$

On the other hand, it follows from (2.8), (2.17), $(2.23)_{(\mathrm{ii})}$ and $(2.24)_{(\mathrm{iii})}$ that

$$g(A_{a^{**}}\phi e_i,V)=g(A_{a^{**}}e_i,W)=-s_{a^*}(e_i)$$

and consequently

$$s_a(\phi e_i) = s_{a^*}(e_i) = 0, \quad i = 1, \dots, m. \tag{2.26}$$

Similarly, replacing X by ψe_i and θe_i in $(2.23)_{(\mathrm{i})}$, respectively, and using (2.8), (2.17) and $(2.23)_{(\mathrm{iii})} - (2.24)_{(\mathrm{iv})}$, we also have

$$\begin{aligned} s_a(\psi e_i) &= -g(A_{a^*}\psi e_i, U) = -s_{a^{**}}(e_i) = -g(A_{a^{***}}\psi e_i, W), \\ s_a(\theta e_i) &= -g(A_{a^*}\theta e_i, U) = -g(A_{a^{**}}\theta e_i, V) = -s_{a^{***}}(e_i). \end{aligned}$$

But it follows from $(2.24)_{(\mathrm{ii})}$ and $(2.24)_{(\mathrm{iv})}$ that

$$\begin{aligned} g(A_{a^*}\psi e_i, U) &= g(A_{a^*}e_i, \psi U) = -g(A_{a^*}e_i, W) = -s_{a^{**}}(e_i), \\ g(A_{a^*}\theta e_i, U) &= g(A_{a^*}e_i, \theta U) = g(A_{a^*}e_i, V) = -s_{a^{***}}(e_i), \end{aligned}$$

and consequently

$$s_a(\psi e_i) = s_a(\theta e_i) = s_{a^{**}}(e_i) = s_{a^{***}}(e_i) = 0, \quad i = 1, \dots, m. \tag{2.27}$$

Next, replacing X by ϕe_i in $(2.23)_{(\mathrm{ii})} - (2.23)_{(\mathrm{iv})}$, respectively, and using (2.8), (2.17), $(2.23)_{(\mathrm{iii})}$, $(2.23)_{(\mathrm{iv})}$ and $(2.24)_{(\mathrm{i})}$, we can obtain

$$\begin{aligned} &s_{a^*}(\phi e_i) = s_a(e_i) = g(A_{a^{***}}e_i, W) = g(A_{a^{**}}e_i, V), \\ &s_{a^{**}}(\phi e_i) = -s_{a^{***}}(e_i), \quad s_{a^{***}}(\phi e_i) = s_{a^{**}}(e_i). \end{aligned}$$

But, it is clear from $(2.23)_{(\mathrm{i})}$ that

$$s_a(e_i) = -g(A_{a^{**}}e_i, V) = -g(A_{a^{***}}e_i, W),$$

which together with (2.27) and the above equations yields

$$s_a(e_i) = s_{a^*}(\phi e_i) = s_{a^{**}}(\phi e_i) = s_{a^{***}}(\phi e_i) = 0, \quad i = 1, \dots, m. \tag{2.28}$$

Replacing X by ψe_i and θe_i, respectively, in $(2.23)_{(\mathrm{ii})} - (2.23)_{(\mathrm{iv})}$, we can also have

$$\begin{aligned} &s_{a^*}(\psi e_i) = s_{a^{***}}(e_i), \quad s_{a^{**}}(\psi e_i) = s_a(e_i) \quad s_{a^{***}}(\psi e_i) = -s_{a^*}(e_i), \\ &s_{a^*}(\theta e_i) = -s_{a^{**}}(e_i), \quad s_{a^{**}}(\theta e_i) = s_{a^*}(e_i) \quad s_{a^{***}}(\theta e_i) = s_a(e_i), \end{aligned}$$

which combined with (2.26) - (2.28) implies

$$\begin{aligned} s_{a^*}(\psi e_i) &= s_{a^{**}}(\psi e_i) = s_{a^{***}}(\psi e_i) = s_{a^*}(\theta e_i) = s_{a^{**}}(\theta e_i) \\ &= s_{a^{***}}(\theta e_i) = 0, \quad i = 1, \ldots, m. \end{aligned} \tag{2.29}$$

Since the ambient manifold is a quaternionic Kähler manifold, differentiating the first equation of (2.5) covariantly, we can obtain

$$\begin{aligned} (\nabla_Y \phi)X &= r(Y)\psi X - q(Y)\theta X + u(X)AY - g(AY, X)U, \\ (\nabla_Y u)X &= r(Y)v(X) - q(Y)w(X) + g(\phi AY, X) \end{aligned} \tag{2.30}$$

with the help of (2.2), (2.5), (2.10), (2.19), $(2.20)_{(\mathrm{i})}$ and (2.21).

Similarly, from the second and the third equations of (2.5), we also obtain

$$\begin{aligned} (\nabla_Y \psi)X &= -r(Y)\phi X + p(Y)\theta X + v(X)AY - g(AY, X)V, \\ (\nabla_Y v)X &= -r(Y)u(X) + p(Y)w(X) + g(\psi AY, X), \end{aligned} \tag{2.31}$$

$$\begin{aligned} (\nabla_Y \theta)X &= q(Y)\phi X - p(Y)\psi X + w(X)AY - g(AY, X)W, \\ (\nabla_Y w)X &= q(Y)u(X) - p(Y)v(X) + g(\theta AY, X). \end{aligned} \tag{2.32}$$

Differentiating the first equation of (2.3) covariantly, and using (2.2), (2.3), (2.5) and $(2.20)_{(\mathrm{i})}$, we have

$$\nabla_Y U = r(Y)V - q(Y)W + \phi AY. \tag{2.33}$$

Similarly, from the second and the third equations of (2.3), we also have

$$\nabla_Y V = -r(Y)U + p(Y)W + \psi AY, \tag{2.34}$$

$$\nabla_Y W = q(Y)U - p(Y)V + \theta AY. \tag{2.35}$$

If the ambient manifold $\overline{M}$ is a quaternionic space form $\overline{M}^{(n+p)/4}(c)$, that is, a quaternionic Kähler manifold of constant Q-sectional curvature c, then its curvature tensor $\overline{R}$ satisfies

$$\begin{aligned} \overline{R}(X,Y)Z =& \frac{c}{4}\{g(Y,Z)X - g(X,Z)Y + g(FY,Z)FX - g(FX,Z)FY \\ & - 2g(FX,Y)FZ + g(GY,Z)GX - g(GX,Z)GY - 2g(GX,Y)GZ \\ & + g(HY,Z)HX - g(HX,Z)HY - 2g(HX,Y)HZ\} \end{aligned}$$

for X, Y, Z tangent to M(cf. [6]). Hence by means of the equation of Gauss, we can easily verify that the Ricci tensor $\mathrm{R}ic(Y,Z)$ turns out to be

$$\begin{aligned}\mathrm{R}ic(Y,Z) = \frac{c}{4}\{(n+8)g(Y,Z) - 3(u(Y)u(Z) + v(Y)v(Z) + w(Y)w(Z))\}\\ + (\mathrm{tr}A)g(AY,Z) - g(AY,AZ) + \sum_{a=1}^{q}\{(\mathrm{tr}A_a)g(A_aY,Z) + (\mathrm{tr}A_{a^*})g(A_{a^*}Y,Z)\\ + (\mathrm{tr}A_{a^{**}})g(A_{a^{**}}Y,Z) + (\mathrm{tr}A_{a^{***}})g(A_{a^{***}}Y,Z)\} - \sum_{a=1}^{q}\{g(A_aY,A_aZ)\\ + g(A_{a^*}Y,A_{a^*}Z) + g(A_{a^{**}}Y,A_{a^{**}}Z) + g(A_{a^{***}}Y,A_{a^{***}}Z)\}\end{aligned}$$

and consequently the scalar curvature ρ is given by

$$\begin{aligned}\rho =& \frac{c}{4}(n+9)(n-1) + (\mathrm{tr}A)^2 - \mathrm{tr}A^2 + \sum_{a=1}^{q}\{(\mathrm{tr}A_a)^2 + (\mathrm{tr}A_{a^*})^2\\ &+ (\mathrm{tr}A_{a^{**}})^2 + (\mathrm{tr}A_{a^{***}})^2\} - \sum_{a=1}^{q}\{\mathrm{tr}A_a^2 + \mathrm{tr}A_{a^*}^2 + \mathrm{tr}A_{a^{**}}^2 + \mathrm{tr}A_{a^{***}}^2\}.\end{aligned} \tag{2.36}$$

Moreover, by means of the equation of Codazzi, we also have

$$\begin{aligned}g((\nabla_X A)Y - (\nabla_Y A)X, Z) = \frac{c}{4}\{g(\phi Y,Z)u(X) - g(\phi X,Z)u(Y)\\ - 2g(\phi X,Y)u(Z) + g(\psi Y,Z)v(X) - g(\psi X,Z)v(Y) - 2g(\psi X,Y)v(Z)\\ + g(\theta Y,Z)w(X) - g(\theta X,Z)w(Y) - 2g(\theta X,Y)w(Z)\}\\ - \sum_{a=1}^{q}\{g(A_aX,Z)s_a(Y) - g(A_aY,Z)s_a(X) + g(A_{a^*}X,Z)s_{a^*}(Y)\\ - g(A_{a^*}Y,Z)s_{a^*}(X) + g(A_{a^{**}}X,Z)s_{a^{**}}(Y) - g(A_{a^{**}}Y,Z)s_{a^{**}}(X)\\ + g(A_{a^{***}}X,Z)s_{a^{***}}(Y) - g(A_{a^{***}}Y,Z)s_{a^{***}}(X)\}\end{aligned} \tag{2.37}$$

for any X, Y, Z tangent to M.

3. An Integral Formula on a Compact QR-Submanifold of Maximal QR-Dimension

Let M be a real n-dimensional QR-submanifold of $(p-1)$ QR-dimension immersed in a quaternionic space form $\overline{M}^{(n+p)/4}(c)$.

We now put

$$T := \nabla_U U + \nabla_V V + \nabla_W W + (\mathrm{d}ivU)U + (\mathrm{d}ivV)V + (\mathrm{d}ivW)W$$

and take the same orthonormal basis $\{e_i\}_{i=1,\ldots,n=4m+3}$ of tangent vectors to M as given in (2.25). Since $\mathrm{d}ivU = \sum_{i=1}^{n} g(e_i, \nabla_{e_i} U)$, making use of (2.17) and (2.32) - (2.34), we obtain

$$T = \phi AU + \psi AV + \theta AW, \tag{3.1}$$

$$g(T,U) = g(T,V) = g(T,W) = 0. \tag{3.2}$$

We note that T is a global vector field defined on M.

Now, for later use we shall compute $\mathrm{d}ivT = \sum_{i=1}^{n} g(e_i, \nabla_{e_i} T)$.

First of all, differentiating (3.1) covariantly, and using (2.29) - (2.34), we have

$$\begin{aligned}\nabla_X T =& \{u(AU) + v(AV) + w(AW)\}AX \\ & - g(A^2U, X)U - g(A^2V, X)V - g(A^2W, X)W \\ & + \phi A\phi AX + \psi A\psi AX + \theta A\theta AX \\ & + \phi(\nabla_X A)U + \psi(\nabla_X A)V + \theta(\nabla_X A)W,\end{aligned}$$

from which, taking account of (2.8), (2.11), (2.13), (2.14) - (2.17) and (2.25),

$$\begin{aligned}divT =& trA\{u(AU) + v(AV) + w(AW)\} - u(A^2U) - v(A^2V) - w(A^2W) \\ & + \sum_{i=1}^{n}\{g(\phi A\phi Ae_i, e_i) + g(\psi A\psi Ae_i, e_i) + g(\theta A\theta Ae_i, e_i)\} \\ & - \sum_{i=1}^{m}\{g((\nabla_{e_i}A)\phi e_i - (\nabla_{\phi e_i}A)e_i + (\nabla_{\psi e_i}A)\theta e_i - (\nabla_{\theta e_i}A)\psi e_i, U) \\ & + g((\nabla_{e_i}A)\psi e_i - (\nabla_{\psi e_i}A)e_i + (\nabla_{\theta e_i}A)\phi e_i - (\nabla_{\phi e_i}A)\theta e_i, V) \\ & + g((\nabla_{e_i}A)\theta e_i - (\nabla_{\theta e_i}A)e_i + (\nabla_{\phi e_i}A)\psi e_i - (\nabla_{\psi e_i}A)\phi e_i, W)\} \\ & - g((\nabla_V A)W - (\nabla_W A)V, U) - g((\nabla_W A)U - (\nabla_U A)W, V) \\ & - g((\nabla_U A)V - (\nabla_V A)U, W).\end{aligned} \tag{3.3}$$

On the other hand, using (2.11), (2.13), $(2.23)_{\text{(i)}} - (2.23)_{\text{(iv)}}$, (2.26)-(2.28) and (2.37), we can easily obtain

$$\text{(i)} \qquad \sum_{i=1}^{m} g((\nabla_{e_i}A)\phi e_i - (\nabla_{\phi e_i}A)e_i, U) = -\frac{(n-3)c}{8}. \tag{3.4}$$

Similarly, taking account of (2.11), (2.13), $(2.23)_{\text{(i)}} - (2.23)_{\text{(iv)}}$, (2.26) - (2.28) and (2.37), we also have

$$\begin{aligned} &\text{(ii)} \qquad \sum_{i=1}^{m} g((\nabla_{e_i}A)\psi e_i - (\nabla_{\psi e_i}A)e_i, V) = -\frac{(n-3)c}{8}, \\ &\text{(iii)} \qquad \sum_{i=1}^{m} g((\nabla_{e_i}A)\theta e_i - (\nabla_{\theta e_i}A)e_i, W) = -\frac{(n-3)c}{8}. \end{aligned} \tag{3.4}$$

Moreover, making use of (2.11), (2.14) - (2.16), $(2.23)_{\text{(i)}} - (2.23)_{\text{(iv)}}$, (2.26) - (2.28) and (2.37), it can be similarly verified that

$$\begin{aligned} &\text{(i)} \qquad \sum_{i=1}^{m} g((\nabla_{\psi e_i}A)\theta e_i - (\nabla_{\theta e_i}A)\psi e_i, U) = -\frac{(n-3)c}{8}, \\ &\text{(ii)} \qquad \sum_{i=1}^{m} g((\nabla_{\theta e_i}A)\phi e_i - (\nabla_{\phi e_i}A)\theta e_i, V) = -\frac{(n-3)c}{8}, \\ &\text{(iii)} \qquad \sum_{i=1}^{m} g((\nabla_{\phi e_i}A)\psi e_i - (\nabla_{\psi e_i}A)\phi e_i, W) = -\frac{(n-3)c}{8}. \end{aligned} \tag{3.5}$$

Furthermore, by means of (2.37), a direct computation yields

$$\begin{aligned} g((\nabla_V A)W - (\nabla_W A)V, U) + g((\nabla_W A)U - (\nabla_U A)W, V) \\ + g((\nabla_U A)V - (\nabla_V A)U, W) = 0. \end{aligned} \tag{3.6}$$

Inserting $(3.4)_{\text{(i)}} - (3.4)_{\text{(iii)}}$, $(3.5)_{\text{(i)}}$ - $(3.5)_{\text{(iii)}}$ and (3.6) back into (3.3), the equation (3.3) turns out to be

$$\begin{aligned} divT = \frac{3(n-3)c}{4} + \text{tr}A\{u(AU) + v(AV) + w(AW)\} - u(A^2U) - v(A^2V) \\ - w(A^2W) + \sum_{i=1}^{n}\{g(\phi A\phi Ae_i, e_i) + g(\psi A\psi Ae_i, e_i) + g(\theta A\theta Ae_i, e_i)\}. \end{aligned} \tag{3.7}$$

On the other hand, by using (2.8), (2.11) - (2.12) and (2.14) - (2.17), we can easily verify that

$$\begin{aligned}&\sum_{i=1}^{n}\{g(\phi A\phi Ae_i,e_i)+g(\psi A\psi Ae_i,e_i)+g(\theta A\theta Ae_i,e_i)\}\\ =&\frac{1}{2}\{\|\phi A-A\phi\|^2+\|\psi A-A\psi\|^2+\|\theta A-A\theta\|^2\}-3\mathrm{tr}A^2\\ &+u(A^2U)+v(A^2V)+w(A^2W).\end{aligned}$$

Then, the last equation together with (3.7) implies

$$\begin{aligned}divT=&\frac{3(n-3)c}{4}+\mathrm{tr}A\{u(AU)+v(AV)+w(AW)\}-3\mathrm{tr}A^2\\ &+\frac{1}{2}\{\|\phi A-A\phi\|^2+\|\psi A-A\psi\|^2+\|\theta A-A\theta\|^2\}.\end{aligned} \tag{3.8}$$

Moreover, from (2.36) and (3.8), we have

$$\begin{aligned}divT=&\frac{1}{2}\{\|\phi A-A\phi\|^2+\|\psi A-A\psi\|^2+\|\theta A-A\theta\|^2\}-3\sum_{\alpha=1}^{p}(\mathrm{tr}A_\alpha)^2\\ &+\mathrm{tr}A\{u(AU)+v(AV)+w(AW)\}+3\{\rho-\frac{c}{4}(n^2+7n-6)\}\\ &+3\sum_{a=1}^{q}\{\mathrm{tr}A_a^2+\mathrm{tr}A_{a^*}^2+\mathrm{tr}A_{a^{**}}^2+\mathrm{tr}A_{a^{***}}^2\}.\end{aligned}$$

Thus we have

Lemma 3.1 — Let M be an n-dimensional compact QR-submanifold of $(p-1)$ QR-dimension immersed in $\overline{M}^{(n+p)/4}(c)$. Then the following equality is valid :

$$\begin{aligned}&\int_M[\frac{1}{2}\{\|\phi A-A\phi\|^2+\|\psi A-A\psi\|^2+\|\theta A-A\theta\|^2\}-3\sum_{\alpha=1}^{p}(\mathrm{tr}A_\alpha)^2\\ &+\mathrm{tr}A\{u(AU)+v(AV)+w(AW)\}+3\{\rho-\frac{c}{4}(n^2+7n-6)\}\\ &+3\sum_{a=1}^{q}\{\mathrm{tr}A_a^2+\mathrm{tr}A_{a^*}^2+\mathrm{tr}A_{a^{**}}^2+\mathrm{tr}A_{a^{***}}^2\}]*1=0.\end{aligned}$$

4. The Proof of Main Theorem

Let M be a real n-dimensional compact, minimal QR-submanifold of $(p-1)$ QR-dimension immersed in a quaternionic space form $\overline{M}^{(n+p)/4}(c)$.

Owing to Lemma 3.1, we have

Lemma 4.1 — Let M be an $n(\geq 7)$-dimensional compact, minimal QR-submanifold of $(p-1)$ QR-dimension immersed in $\overline{M}^{(n+p)/4}(c)$. If the scalar curvature ρ of M is greater or equal to $\frac{c}{4}(n^2+7n-6)$, then $A_\alpha = 0$, $\alpha = 2, \ldots, p$ and $s_a = s_{a^*} = s_{a^{**}} = s_{a^{***}} = 0$, namely, the distinguished normal vector field ξ is parallel with respect to the normal connection $\nabla^\perp$. Moreover the equalities

$$A\phi = \phi A, \quad A\psi = \psi A, \quad A\theta = \theta A \tag{4.1}$$

are valid on M and $c > 0$.

Proof : It is clear from Lemma 3.1 that our assumption implies both (4.1) and the following

$$A_a = 0, \quad A_{a^*} = 0, \quad A_{a^{**}} = 0, \quad A_{a^{***}} = 0, \quad a = 1, \cdots, q,$$

which together with $(2.23)_{\text{(i)}} - (2.23)_{\text{(iv)}}$ yields

$$s_a = s_{a^*} = s_{a^{**}} = s_{a^{***}} = 0, \quad a = 1, \cdots, q,$$

and consequently the distinguished normal vector field ξ is parallel with respect to the normal connection because of $(2.20)_{\text{(i)}}$. On the other hand, (4.1) combined with (2.11) yields

$$AU = \lambda U, \quad AV = \lambda V, \quad AW = \lambda W \quad (\lambda := u(AU) = v(AV) = w(AW))$$

and hence (3.1) implies $T = 0$ identically on M. Thus from (3.8) and the above facts, it follows that $\mathrm{t}rA^2 - (n-3)c/4 = 0$ and hence $c \geq 0$. When $c = 0$, that is, the ambient manifold $\overline{M}^{(n+p)/4}(c)$ is a real $(n+p)$-dimensional Euclidean space $\mathcal{R}^{\backslash+}\surd$ identified with the quaternionic number space $\mathcal{Q}^{(\backslash+\surd)/\triangle}$, the shape operator A is also identically zero on M. Hence M is totally geodesic in $\mathcal{R}^{\backslash+}\surd$, which is a contradiction since M is compact. Thus c can not be zero which implies our final assertion. □

From now on, in order to get our main theorem, it is enough to consider that the ambient manifold is a quaternionic projective space $QP^{(n+p)/4}$ of constant Q-sectional curvature 4 because of Lemma 4.1. Let $N_0(x) = \{\xi \in T_xM^\perp : A_\xi = 0\}$ and $H_0(x)$ the maximal quaternionic invariant subspace of $N_0(x)$, that is,

$$H_0(x) = N_0(x) \cap FN_0(x) \cap GN_0(x) \cap HN_0(x).$$

Then Kwon and the second author of this paper have proved the following theorem in their paper [9] :

***Theorem* K-P** — *Let M be an n-dimensional real submanifold of real $(n+p)$-dimensional quaternionic projective space $QP^{(n+p)/4}$. If the orthogonal complement $H_1(x)$ of $H_0(x)$ in $TM^\perp$ is invariant under the parallel translation with respect to the normal connection and q is the constant dimension of $H_1(x)$, then there exists a real $(n+q)$-dimensional totally geodesic quaternionic projective space $QP^{(n+q)/4}$ such that $M \subset QP^{(n+q)/4}$.*

In our case, we see that $N_0(x) = Span\{\xi_2(x), \ldots, \xi_p(x)\}$. In fact, as a consequence of Lemma 4.1, $A_\alpha = 0$ for $\alpha = 2, \ldots, p$. Hence

$$Span\{\xi_2(x), \ldots, \xi_p(x)\} \subset N_0(x).$$

On the other hand, for any ξ' in $N_0(x)$, we can put $\xi' = \sum_{\alpha=1}^{p} \lambda^\alpha \xi_\alpha$. But $A'_\xi = \sum_{\alpha=1}^{p} \lambda^\alpha A_\alpha = \lambda^1 A = 0$ since $A_\alpha = 0$ for $\alpha = 2, \ldots, p$. Hence $\lambda^1 = 0$ and consequently, $\xi' \in Span\{\xi_2(x), \ldots, \xi_p(x)\}$. Moreover, (2.10) implies

$$N_0(x) = H_0(x) = Span\{\xi_2(x), \ldots, \xi_p(x)\}.$$

Thus $H_1(x) = Span\{\xi_1(x)\}$ and so our assumption yields that $H_1(x)$ is invariant under parallel translation with respect to the normal connection. Therefore we can apply Theorem K-P and obtain the following theorem.

***Theorem* 4.2** — *Let M be an n-dimensional compact, minimal QR-submanifold of $(p-1)$ QR-dimension immersed in $QP^{(n+p)/4}$. If the scalar curvature of M is greater or equal to n^2+7n-6, then there exists a real $(n+1)$-dimensional totally geodesic quaternionic projective space $QP^{(n+1)/4}$ such that $M \subset QP^{(n+1)/4}$.*

Finally we shall give the proof of the main theorem stated in §1.

Proof of main theorem : By means of Theorem 4.2, the submanifold M can be regarded as a real hypersurface of $QP^{(n+1)/4}$ which is totally geodesic in $QP^{(n+p)/4}$.

Tentatively we denote $QP^{(n+1)/4}$ by M' and by i_1 the immersion of M into M' and by i_2 the totally geodesic immersion of M' into $QP^{(n+p)/4}$. Then it is clear from (2.19) that

$$\nabla'_{i_1X} i_1Y = i_1\nabla_X Y + h'(X,Y) = i_1\nabla_X Y + g(A'X,Y)\xi',$$

where ∇' is the induced connection on M' from that of $QP^{(n+p)/4}$, h' the second fundamental form of M in M' and A' the corresponding shape operator to a unit normal vector field ξ' to M in M'.

Since $i = i_2 \circ i_1$ and M' is totally geodesic in $QP^{(n+p)/4}$, we can easily see that (2.21) and (4.3) imply

$$\xi = i_2\xi', \quad A_1 = A'.$$

Since M' is a quaternionic invariant submanifold of $QP^{(n+p)/4}$, for any X in TM we get

$$Fi_2X = i_2F'X,\ Gi_2X = i_2G'X,\ Hi_2X = i_2H'X,$$

where $\{F', G', H'\}$ is the induced quaternionic Kähler structure on M'. Thus it follows from the first equation of (2.5) that

$$\begin{aligned} FiX &= Fi_2 \circ i_1X = i_2F'i_1X = i_2(i_1\phi'X + u'(X)\xi') \\ &= i\phi'X + u'(X)i_2\xi' = i\phi'X + u'(X)\xi \end{aligned}$$

for any tangent vector field X to M. Comparing this equation with the first equation of (2.5), we have $\phi = \phi'$ and $u = u'$. Similarly, we also have

$$\phi = \phi',\ \psi = \psi',\ \theta = \theta',\ u = u',\ v = v',\ w = w',$$

which combined with Lemma 4.1 implies

$$A'\phi' = \phi'A', \quad A'\psi' = \psi'A', \quad A'\theta' = \theta'A'.$$

Now applying the theorem ([13], Theorem 10) due to the second author, we can conclude that $M = \pi(S^{4s+3}(r_1) \times S^{4t+3}(r_2))$ $(r_1^2 + r_2^2 = 1,\ s + t = (n-3)/4)$, where π is the Hopf fibration $S^{n+4} \to QP^{(n+1)/4}$. It completes the proof of main theorem. $\square$

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