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SUPER MAGIC STRENGTH OF A GRAPH

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By $G(p,q)$ we denote a graph having p vertices and q edges, by V and E the vertex set and edge set of G respectively. A graph $G(p,q)$ is said to have an $edge\ magic$ labeling (valuation) with the constant (magic number) $c(f)$ if there exists a one-to-one and onto function $f : V \cup E \longrightarrow \{1, 2, ..., p + q\}$ such that $f(u)+f(v)+f(uv) = c(f)$ for all $uv \in E$. An edge magic labeling f of G is called a $super\ magic\ labeling$ if $f(E)$ =$\{1, 2, ..., q\}$. In this paper the concepts of the $super\ magic$ and $super\ magic\ strength$ of a graph are introduced. The super magic strength (sms) of a graph G is defined as the minimum of all constants $c'(f)$ where the minimum is taken over all super magic labeling of G and is denoted by sms(G). This minimum is defined only if the graph has at least one such super magic labeling. In this paper, the super magic strength of some well known graphs P_{2n}, P_{2n+1}, $K_{1,n}$, $B_{n,n}$, $< K_{1,n} : 2 >$, P_n^2 and $(2n+1)P_2$, C_n and W_n are obtained, where P_n is a path on n vertices, $K_{1,n}$ is a star graph on $n+1$ vertices, n-bistar $B_{n,n}$ is the graph obtained from two copies of $K_{1,n}$ by joining the centres of two copies of $K_{1,n}$ by an edge e, if e is subdivided then $B_{n,n}$ becomes $< K_{1,n} : 2 >$, $(2n + 1)\ P_2$ is $2n + 1$ disjoint copies of P_2, P_n^2 is a square graph of P_n. C_n is a cycle on n vertices and $W_n = C_n + K_1$ is wheel on $n + 1$ vertices.

Key words: Edge magic labeling, magic strength, super magic strength, super edge magic strength.

1. INTRODUCTION

Throughout this paper by the word graph we mean a finite, undirected graph without loops or multiple edges having at least one edge. For notations and terminology we follow Bondy and Murthy [3]. Throughout this paper we denoted the path on n vertices by P_n, the cycle on n vertices by C_n, the star graph $K_{1,n}$ with $n+1$ vertices. n-bistar $B_{n,n}$ is the graph obtained from two copies of star $K_{1,n}$ by joining the centers of two $K_{1,n}$ by an edge e. If e is subdivided then $B_{n,n}$ becomes $< K_{1,n} : 2 >$. $(2n+1)P_2$ is $2n+1$ disjoint copies of P_2, P_n^2 is a square graph of P_n, and $W_n = C_n + K_1$ is a wheel on n + 1 vertices. In [5], Kotzig and Rosa defined a magic labeling (valuation) of a graph $G(V, E)$ is a bijection $f : V \cup E \longrightarrow \{1, 2, \ldots, p+q\}$ such that for all edges $uv, f(u) + f(v) + f(uv)$ are the same (see Fig. 1). A graph G is said to be magic if it has a magic labeling. In [5], the magic labeling of $K_{m,n}$ and C_n are given. Ringel and Llado [6] called this graph edge magic. They have proved that if G is a graph in which both p and q are even such that $p + q \equiv 2$ (mod 4) and in which each vertex has odd degree then G is not magic.

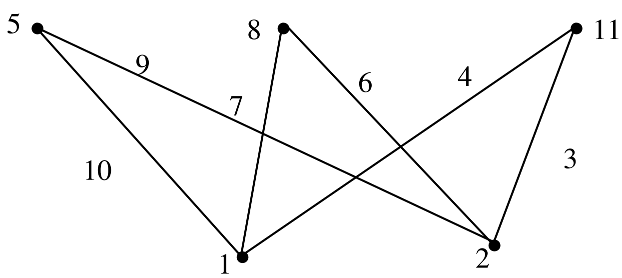

Figure 1:

If there is a one-one and onto function f: $V \cup E \longrightarrow \{1, 2, ..., p+q\}$ such that for every edge $uv, f(u) + f(v) + f(uv)$ are all distinct constants then G is called $antimagic$. (Or edge antimagic) Ringel and Llado [6] proved that any graph is an antimagic.

In 2000, Avadayappan *et al.* [1] introduce the concept of $edge\ magic\ strength$ of a graph. We know that for any edge magic labeling f of G, there is a constant $c(f)$ such that $f(u) + f(v) + f(uv)$ for every edge $uv \in E$. The edge magic strength of $G, m(G)$ is defined as the minimum of all $c(f)$ where the minimum taken over all edge magic labeling of G. This minimum is defined only if the graphs has atleast one such edge magic labeling. That is $m(G)$ = min { $c(f) : f$

is an edge magic labeling of $G\}$ and also obtained the edge magic strength of P_{2n}, $K_{1,n}$, $B_{n,n}$,$< K_{1,n} : 2 >$, C_{2n+1} and $(2n+1)P_2$.

An edge magic labeling of a graph $G(V, E)$ is called a *super edge magic* labeling of graph G, if $f(V)$ = $\{1, 2, ..., p\}$ and $f(E)$ = $\{p+1, p+2, \ldots, p+q\}$ (see Fig. 2).

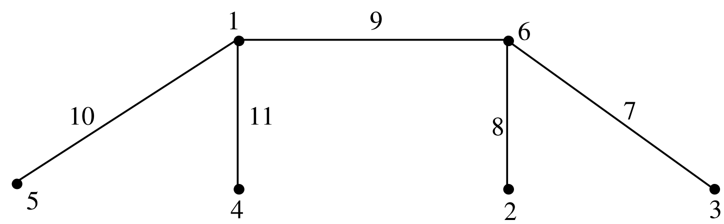

Figure 2:

A graph is said to be *super edge magic* if it has a super edge magic labeling. In [4], the super edge magic of C_n and $K_{m,n}$ are obtained.

In 2001, Avadayappan, Jeyanthi and Vasuki [2], introduce the concept of *super edge magic strength* of a graph. The super edge magic strength of a graph G, sm(G) is defined as the minimum of all $c(f)$ where the minimum is taken over all super edge magic labeling f of G if there exists at least one such super edge magic labeling. That is sm(G) = min$\{c(f) : f$ is a super edge magic labeling of $G\}$. They obtained the super edge magic labelings of P_{2n}, $B_{n,n}$, $< K_{1,n} : 2 >$, C_{2n+1} and $(2n+1)P_2$.

In this paper, we introduced the concept of super magic and super magic strength of a graph. An edge magic labeling of a graph $G(V, E)$ is called *super magic labeling* of G if $f(E)$ = $\{$ 1, 2, ...,q $\}$ and $f(V)$ = $\{$ $q+1, q+2, ..., p+q$ $\}$ (see Fig. 3).

The super magic strength of a graph G, sms(G) is defined as the minimum of all $c'(f)$ where the minimum is taken over all super magic labeling f of G if there exist at least one such super magic labeling. That is sms(G) = min $\{$ $c'(f)$; f is a super magic labeling of $G\}$. In this paper, we present the super magic strength of some well known graphs P_{2n}, P_{2n+1}, $B_{n,n}$, $< K_{1,n} : 2 >$, $K_{1,n}$, P_n^2, $(2n+1)P_2$, C_n and W_n.

One can easily see that, since the labels are from the set $\{$ 1, 2, ..., $p+q\}$, $3q+3 \leq$ sms$(G) \leq 2(p+q)$.

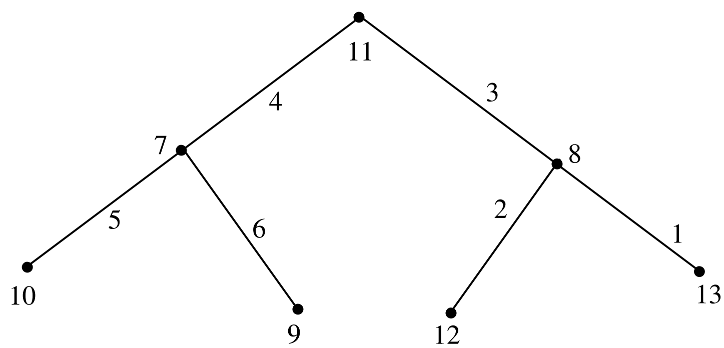

Figure 3:

Before proceeding further, we make the following note.

Note 1 : Let f be a super magic labeling of G with the constant $c'(f)$. Then adding all the constants obtained at each edge we get $q\ c'(f) = \sum_{v \in V} d(v)f(v) + \sum_{e \in E} f(e)$

Lemma 4 — If a non trivial graph G is super magic then $q \leq 2p - 3$.

2. Main Results

1. *Super magic strength of some trees:*
In this section, we establish super magic strength of P_{2n}, P_{2n+1}, $K_{1,n}$, $B_{n,n}$, $< K_{1,n} : 2 >$, P_n^2, and $(2n + 1)P_2$, C_n and W_n.

Lemma 1 — sms (P_{2n}) $\leq$ 7n-1 and sms (P_{2n+1}) $\leq 7n$+2 for all n.

Proof : We show this lemma by assigning super magic labeling to P_{2n} and P_{2n+1}. Let $V(P_{2n})$ = $\{v_1,v_2,...,v_{2n}\}$ and $E(P_{2n})$ = $\{e_1,e_2,....,e_{2n-1}\}$ such that e_i = $v_i\ v_{i+1}$ for $1 \leq i \leq 2n - 1$. Then the following labeling f is a super magic labeling of P_{2n}, $f(v_{2i})$ = $4n - i$, $f(v_{2i-1})$ = $3n - i$, for $1 \leq i \leq n$ and $f\ (e_i)$ = i, $1 \leq i \leq 2n - 1$ (see Fig. 4).

Similarly we define a super magic labeling g of P_{2n+1} as follows: $g(v_{2i}) = 3n+1-i$, $g(v_{2i-1}) = 4n+2-i$ for $1 \leq i \leq n$ and $g(e_i) = i$,$1 \leq i \leq 2n$ (see Fig. 5).

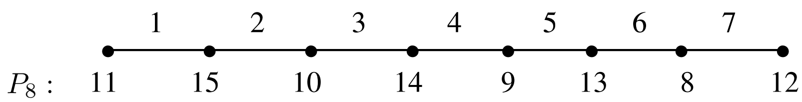

Figure 4:

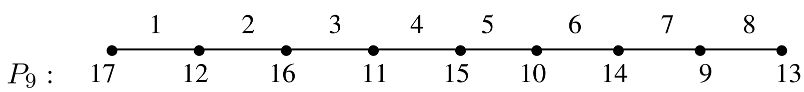

Figure 5:

Thus sms $(P_{2n}) \leq 7n - 1$ and sms $(P_{2n+1}) \leq 7n + 2$.

Lemma 2 — sms $(P_{2n}) \geq 7n - 1$ and sms $(P_{2n+1}) \geq 7n + 2$, for all n.

PROOF : In this case $q = 2n - 1$ and $p + q = 4n - 1$. Suppose f is super magic labeling of P_{2n} with magic constant $c'(f)$. Then by Note 1

$$\begin{aligned} qc'(f) &= \sum_{v \in V} d(v)f(v) + \sum_{e \in E} f(e) \\ &\textit{or} \\ (2n-1)c'(f) &= \sum_{i=2}^{2n-1} 2f(v_i) + f(v_1) + f(v_{2n}) + \sum_{i=1}^{2n-1} f(e_i) \\ &= \sum_{i=1}^{2n} f(v_i) + \sum_{i=1}^{2n-1} f(e_i) + \sum_{i=2}^{2n-1} f(v_i) \\ &= 1 + 2 + 3 + \ldots + (4n-1) + \sum_{i=2}^{2n-1} f(v_i). \end{aligned}$$

Therefore $c'(f) = (2n(4n-1))/(2n-1) + \sum_{i=2}^{2n-1} f(v_i)/(2n-1)$
$\geq 4n - 1 + 2 + \{[2n + (2n+1) + ... + 2n + (2n-1)] - (6n-1)\}/2n - 1$
$= (4n-1) + 2 + ... + (3n - 2 - 1) = 7n - 2$.

Thus $c'(f) \geq 7n - 2$ and hence $c'(f) \geq 7n - 1$ which implies that sms (P_{2n}) $\geq 7n - 1$.

Similarly we can prove that sms$(P_{2n+1}) \geq 7n + 2$. Combining Lemmas 1 and 2 we can state the following Theorem.

***Theorem* 1** — *sms* $(P_{2n}) = 7n - 1$ *and sms*$(P_{2n+1}) = 7n + 2$*, for all* n.

***Theorem* 2** — *sms*$(B_{n,n})$ *= 7n + 6, for all n.*

PROOF : Let $V(B_{n,n})$ = $\{uv; u_1, u_2, \ldots, u_n; v_1, v_2, \ldots, v_n\}$ and $E(B_n, n)$ = $\{uv, uu_i, vv_i, 1 \le i \le n\}$

First we show that sms$(B_{n,n}) \le 7n + 6$ by assigning super magic labeling of $B_{n,n}$. Consider the following labeling f of $B_{n,n}$:

$$f(u) = 2n + 2, f(v) = 4n + 3, f(u_i) = 3n + 2 + i, f(v_i) = 2n + 2 + i, \text{ for } 1 \le i \le n$$
$$f(uu_i) = 2n + 2 - i, f(vv_i) = n + 1 - i, \text{ for } 1 \le i \le n, f(uv) = n + 1.$$

It is easy to see that f is a super magic labeling of $B_{n,n}$ with $c'(f) \le (7n + 6)$ and hence sms$(B_{n,n}) \ge 7n + 6$.

For example, a super magic labeling of $B_{6,6}$ is shown in Fig. 6.

Now it is remains to prove that sms$(B_{n,n}) \ge 7n + 6$. Let f be a super magic labeling of $B_{n,n}$ with constant $c'(f)$. Then by Note 1,

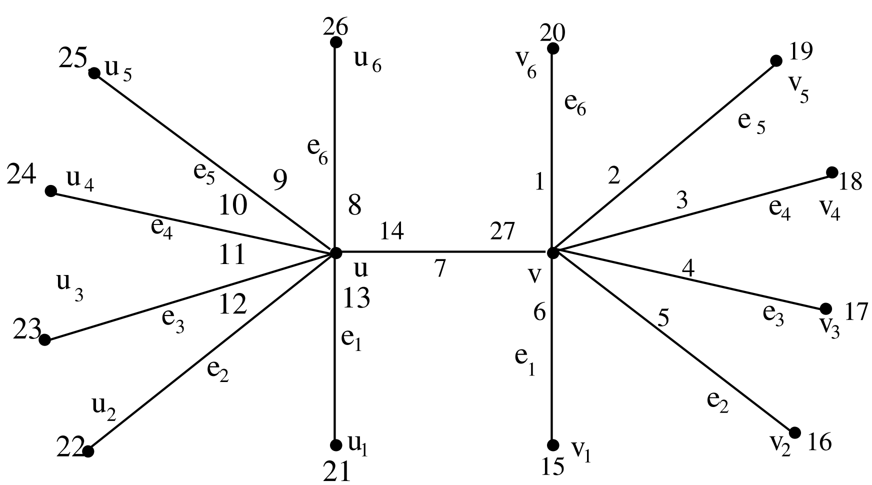

Figure 6:

$$qc'(f) = \sum_{v \in V} d(v) f(v) + \sum_{e \in E} f(e)$$

That is

$$\begin{aligned}(2n+1)c'(f) &= \sum_{i=1}^{n} f(u_i) + \sum_{i=1}^{n} f(v_i) + (n+1)f(u) + (n+1)f(v) + \sum_{e\in E} f(e)\\ &= \sum_{v\in V} f(v) + \sum_{e\in E} f(e) + n[f(u)+f(v)]\\ &= [1+2+.....+(4n+3)] + n[f(u)+f(v)]\end{aligned}$$

Therefore

$$\begin{aligned}c'(f) &= [(4n+3)(2n+2)]/(2n+1) + n[f(u)+f(v)]/(2n+1)\\ &= (4n+5) + [1+n(f(u)+f(v))]/(2n+1).\end{aligned}$$

Since $c'(f)$ is an integer, $[1+n(f(u)+f(v))]/(2n+1)$ is also integer. That means $n(f(u)+f(v))+1 \equiv 0 \bmod (2n+1)$, which implies that $nf(u)+nf(v) \equiv 2n \bmod (2n+1)$ and hence $f(u)+f(v) \equiv 2 \bmod (2n+1)$. For a magic labeling, both $f(u)$ and $f(v)$ are $\geq q+1 = 2n+2$. Thus the minimum sum of $f(u)+f(v)$ is $4n+5$. Hence $f(u)+f(v) = 4n+4$ (which is $2 \bmod 2n+1$) is not possible. Hence, the smallest number for $f(u)+f(v)$ [subject to $f(u)+f(v) \equiv 2 \bmod (2n+1)$] is $6n+5$ and thus $f(u)+f(v) \geq 6n+5$. Therefore $c'(f) \geq 4n+5+[n(6n+5)+1]/(2n+1) = 7n+6$. This shows that sms($B_{n,n}$) $\geq 7n+6$. Hence sms ($B_{n,n}$) $= 7n+6$.

***Theorem* 3** — *sms* ($< K_{1,n} : 2 >$) $= 6n+10$.

PROOF : We show first that $< K_{1,n} : 2 >$ is a super magic labeling. We define a labeling f on $< K_{1,n} : 2 >$ as follows:

$$\begin{aligned}f(u) &= 2n+3, f(v) = 2n+4, f(w) = 3n+5,\\ &\quad f(uw) = n+2, f(vw) = n+1, f(u_i) = 2n+4+i\\ f(v_i) &= 3n+5+i, \text{ for } 1 \leq i \leq n,\\ &\quad f(uu_i) = n+2+i, f(vv_i) = n+1-i, \text{ for } 1 \leq i \leq n.\end{aligned}$$

For example, a super magic labeling of $< K_{1,4} : 2 >$ is shown in Fig. 7.

One can verify the above labeling f is a super magic labeling of ($< K_{1,n} : 2 >$) with magic constant $c'(f) = 6n+10$. Thus sms ($< K_{1,n} : 2 >$) $\leq 6n+10$.

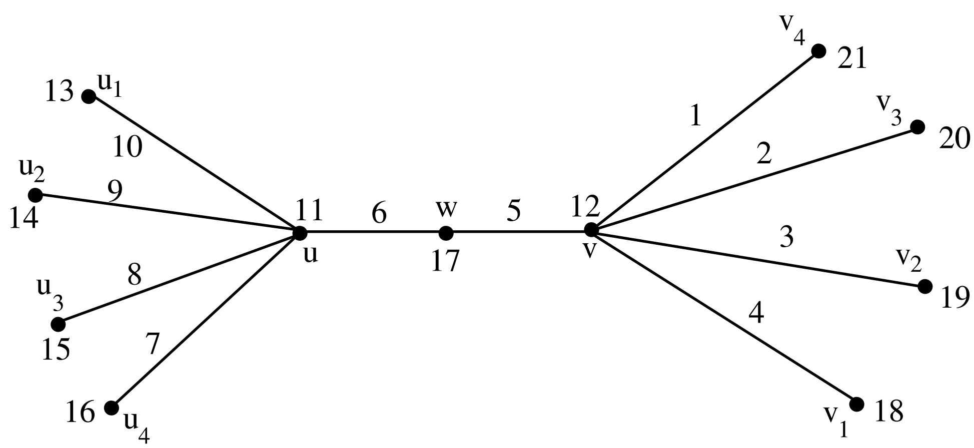

Figure 7:

Now we prove that sms($< K_{1,n} : 2 >$) $\geq 6n + 10$. Let f be a super magic labeling of $< K_{1,n} : 2 >$ with $c'(f)$. Then by Note 1

$$\begin{aligned}
qc'(f) &= \sum_{v\in V} d(v)f(v) + \sum_{e\in E} f(e)\\
(2n+2)c'(f) &= \sum_{i=1}^{n} f(u_i) + \sum_{i=1}^{n} f(v_i) + (n+1)[f(u)+f(v)] + 2f(w)\\
&\quad + \sum_{e\in E} f(e) \sum_{v\in V} f(v) + \sum_{e\in E} f(e) + n[f(u)+f(v)] + f(w)\\
&= 1 + 2 + ... + (4n+5) + n[f(u)+f(v)] + f(w)\\
\text{that is} &= [(4n+5)(2n+3)]/(2n+2)\\
&\quad + \{n[f(u)+f(v)] + f(w)\}/(2n+2)\\
&= 4n + 7 + \{n[f(u)+f(v)] + f(w) + 1\}/(2n+2)\\
&\quad \geq 6n+10\\
&\quad (\text{since } f(u)+f(v) \geq 2 \text{ and since } c'(f) \text{ is an integer})
\end{aligned}$$

Therefore sms ($< K_{1,n} : 2 >$) $\geq 6n + 10$. This implies that sms ($< K_{1,n} : 2 >) = 6n + 10$.

***Theorem* 4** — *sms*($K_{1,n}$) $= 3n + 3$.

PROOF : Let $V(K_{1,n})$ = { $v, v_1, v_2, ..., v_n$ } and E ($K_{1,n}$) = { vv_i }, for $1 \leq$ i$\leq n, f(v) = n + 1, f(v_i) = 2n + 2 - i$, $f(vv_i) = i$ for $1 \leq i \leq n$, thus sms

$(K_{1,n}) \leq 3n+3$ (See Fig. 8).

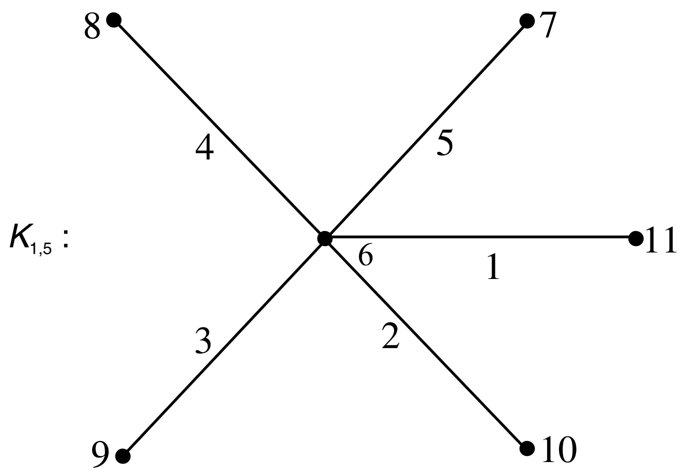

Figure 8:

By Note

$$\begin{aligned} 1, qc'(f) &= \sum_{i=1}^{n} f(v_i) + nf(v) + \sum_{e\in E} f(e) \\ & \quad nc'(f) = 1+2+....+(2n+1)+(n-1)f(v) \\ &= [(2n+1)(2n+2)]/2 + [(n-1)(n+1)] \\ c'(f) &= 3n+3 \text{ that is } c'(f) \geq 3n+3. \text{ Thus } sms(K_{1,n}) = 3n+3. \end{aligned}$$

***Theorem* 5** — *sms* (P_n^2) $= 6n-6$.

PROOF : Let $v_1, v_2, ..., v_n$ be the vertices of P_n^2. Then $E(P_n^2)$ = { v_i v_{i+1}: $1 \leq i \leq n-1$;v_i v_{i+2} : $1 \leq i \leq n-2\}$. A super magic labeling f of P_n^2 is as follows:

$$\begin{aligned} f(v_i) &= 3n-(2+i) \text{ for } 1 \leq i \leq n, \\ & \quad f(v_i v_{i+1}) = 2i-1 \text{ for } 1 \leq i \leq n-1 \text{ and} \\ & \quad f(v_i v_{i+2}) = 2i \text{ for } 1 \leq i \leq n-2 \end{aligned}$$

Therefore sms (P_n^2) $\leq 6n-6$. But since $q = 2n-3$

$$\begin{aligned} qc'(f) &= \sum_{v\in V} d(v)f(v) + \sum_{e\in E} f(e). \\ (2n-3)c'(f) &= 2f(v_1) + 2f(v_n) + 3f(v_2) + 3f(v_{n-1}) + \end{aligned}$$

$$\begin{aligned}
& \sum_{i=3}^{n-2} 4f(v_i) + \sum_{e \in E} f(e_e) \\
= \; & 2(3n-3) + 3(3n-4) + 3(2n-1) + 2(2n-2) + 4 \\
& [2n + ... + (3n-5)] + [1 + 2 + ... + (2n-3)] \\
= \; & (6n-6)(2n-3).
\end{aligned}$$

Thus $c'(f) \geq 6n-6$ we have sms $(P_n^2\,) \geq 6n-6$. Hence sms $(P_n^2) = 6n-6$.

For example, a super magic of P_7^2 is shown in Fig. 9.

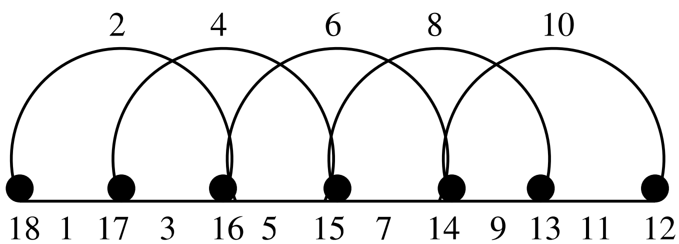

Figure 9:

***Theorem* 6** — *sms ((2n+1)P_2) = 9n+6 (that is nP_2 is super magic if and only if n is odd).*

PROOF : Let the vertices of (2n+1)P_2 be u_1,u_2,...,u_{2n+1}:v_1,v_2,...,v_{2n+1} and let the edges be $\{u_i\ v_i : 1 \leq$ i $\leq 2n+1\}$. Define $f : V \cup E \longrightarrow \{1, 2, ..., 6n+3\}$ as follows:

$$\begin{aligned}
f(u_i) &= 3n+1+i \text{ for } 1 \leq i \leq n+1, f(v_i) = 6n+5-2i, \text{ for } 1 \leq i \leq n+1 \\
f(u_i) &= 3n-2+i \text{ for } 1 \leq i \leq n, f(v_i) = 7n+1-2i, \text{ for } 1 \leq i \leq n \\
f(u_i v_i) &= i, 1 \leq i \leq 2n+1.
\end{aligned}$$

It is easily seen that f is a super magic labeling with the magic number $c'(f) = 9n+6$. Therefore $(2n+1)P_2$ is a super magic.

In the next part we show that sms($(2n+1)P_2$) $= 9n+6$. Assume there exists a super magic labeling f of $(2n+1)P_2$ with $c'(f) = 9n+6$ then by Note 1 we get

$$\begin{aligned}
qc'(f) &= \sum_{v \in V} d(v) f(v) + \sum_{e \in E} f(e) \\
(2n+1)c'(f) &= 1 + 2 + \ldots + (2n+1) + \ldots + (6n+3) \\
&= [(6n+3)(6n+4)]/2
\end{aligned}$$

Therefore $c'(f) = 9n + 6$.

This is true for any super magic labeling f of $(2n + 1)P_2 = 9n + 6$. (see Fig. 10).

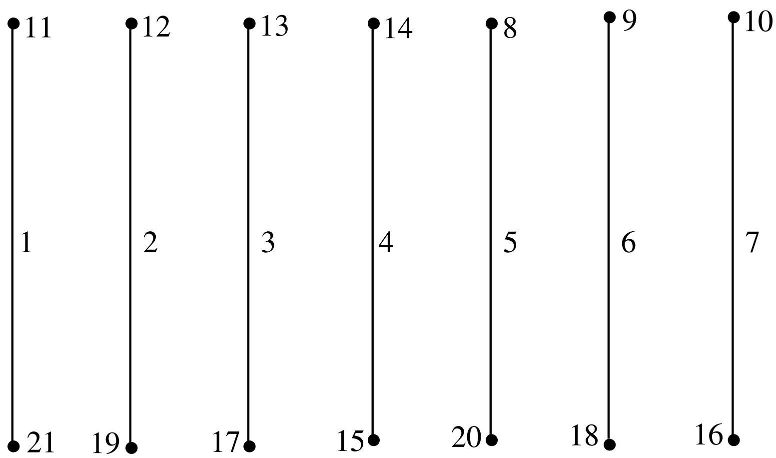

Figure 10:

***Theorem* 7** — *sms*(C_n) $= 1/2\ (7n + 3)$ *where n* $\geq$ *3 odd and* $n = 2m + 1, m \geq$1.

PROOF : Let the vertex sequence of C_n be $v_0, v_1, v_2, ..., v_{n-1}, v_0$ and let edge sequences be $v_i\ v_{i+1}; i = 0, 1...., n-1$. Consider a function $f : V(G) \cup E(G) \longrightarrow \{1, 2, ..., 2n\}$ defined by

$$f(v_i) = \begin{cases} 4m - (i-1)/2 + 2, & 1 \leq i \leq 2m+1; i \text{ is odd} \\ 3m - i/2 + 2, & 0 \leq i \leq 2m; i \text{ is even} \end{cases}$$

$$f(v_i v_{i+1}) = i, \text{ for } 0 \leq i \leq 2m + 1$$

It is not hard to see that f is a super magic labeling with magic constant $c'(f) = 7m + 5 = 1/2\ (7n + 3)$. Thus C_n is super magic.

In the next part we prove that sms(C_n) $\geq 1/2\ (7n + 3)$. Suppose there exists a super magic labeling f of C_n with $c'(f) = 1/2\ (7n + 3)$. Note that

$$nc'(f) \quad = \quad \sum_{v \in V} d(v)f(v) + \sum_{e \in E} f(e)$$

$$
\begin{aligned}
&= 2\sum_{i=n+1}^{2n} f(v_i) + \sum_{i=1}^{n} f(e_i) \\
&= 2\sum_{i=1}^{2n} f(v_i) - \sum_{i=1}^{n} f(e_i) \\
&= 2(2n(2n+1))/2 - n(n+1)/2 \\
&= [(8n+4) - (n+1)]/2 = (7n+3)/2
\end{aligned}
$$

Thus sms $(C_n) = (7n+3)/2$.

***Theorem* 8** — *A wheel W_n = C_n + K_1 is not a super magic.*

PROOF : Lemma [4] is significant in the sense that it eliminates huge number of graphs from being super magic graphs. It is interesting to find families of super magic graph that satisfy q $\leq 2p-3$. Since W^n has $p = n+1, q = 2n$, does not satisfy the above equality. Thus wheel is not a super magic.

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