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GEOMETRY OF LIGHTLIKE HYPERSURFACES OF AN INDEFINITE SASAKIAN MANIFOLD

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In this paper, we study the geometry of lightlike hypersurfaces of an indefinite Sasakian manifold. The main result is to prove three characterization theorems for such a lightlike hypersurface. In addition to these main theorems, we study the geometry of totally geodesic lightlike hypersurfaces of an indefinite Sasakian manifold.

Key words : Totally umbilical, screen conformal, indefinite Sasakian manifold.

1. Introduction

The theory of non-degenerate submanifolds [3, 14] of Riemannian or semi-Riemannian manifolds is one of the most important topics of differential geometry. But the theory of lightlike submanifolds of semi-Riemannian manifolds is relatively new and in a developing stage. The geometry of lightlike submanifolds becomes more difficult and is completely different from that of non-degenerate submanifolds. In 1996, Duggal and Bejancu published their work [4] on lightlike submanifolds of semi-Riemannian manifolds and indefinite Kaehler manifolds. Many authors studied the geometry of lightlike submanifolds of semi-Riemannian or indefinite Kaehler manifolds [1, 6, 9, 10-12]. Recently several authors have studied the geometry of lightlike submanifolds M of indefinite Sasakian manifolds $\bar{M}$ [7, 8, 13]. The authors in above papers principally assumed that M is totally umbilical (or totally geodesic) [4, 7, 13]or screen conformal [1, 6, 9, 11], or the screen distribution $S(TM)$ of M is totally umbilical in M [4, 10, 12].

The purpose of this paper is to prove the following three characterization theorems for lightlike hypersurfaces of an indefinite Sasakian manifold: (1) There exist no screen conformal lightlike hypersurfaces of an indefinite Sasakian manifold (Theorem 3.1). (2) There exist no lightlike hypersurfaces M of an indefinite Sasakian manifold such that the screen distribution $S(TM)$ of M is totally umbilic in M (Theorem 3.1). (3) There exist no totally umbilical lightlike hypersurfaces M of an indefinite Sasakian manifold such that the characteristic vector field ζ is tangent to M (Theorem 3.2). In addition to these main theorems, we study the geometry of totally geodesic lightlike hypersurfaces M of an indefinite Sasakian manifold $\bar{M}$ in which the characteristic vector field ζ of $\bar{M}$ is not tangent to M.

2. Lightlike Hypersurfaces

An odd dimensional smooth manifold $(\bar{M}, \bar{g})$ is called a contact metric manifold [7, 13] if there exists a $(1,1)$-type tensor field J, a vector field ζ, called the characteristic vector field, and its 1-form θ satisfying

$$\begin{aligned} &J^2X = -X + \theta(X)\zeta,\ J\zeta = 0,\ \theta \circ J = 0,\ \theta(\zeta) = 1,\\ &\bar{g}(\zeta,\zeta) = \epsilon, \quad \bar{g}(JX, JY) = \bar{g}(X,Y) - \epsilon\,\theta(X)\theta(Y),\\ &\theta(X) = \epsilon\bar{g}(\zeta, X), \quad d\theta(X,Y) = \bar{g}(JX,Y), \quad \epsilon = \pm 1, \end{aligned} \tag{2.1}$$

for any vector fields X, Y on $\bar{M}$. Then the set $(J,\, \theta,\, \zeta,\, \bar{g})$ is called a contact metric structure on $\bar{M}$. We say that $\bar{M}$ has a normal contact structure if $N_J + d\theta \otimes \zeta = 0$, where N_J is the Nijenhuis tensor field of J [7, 8]. A normal contact metric manifold is called a Sasakian manifold [15] for which we have

$$\bar{\nabla}_X \zeta = JX, \tag{2.2}$$

$$(\bar{\nabla}_X J)Y = \epsilon\,\theta(Y)X - \bar{g}(X,Y)\zeta. \tag{2.3}$$

The next ingredient we consider is a semi-Riemannian metric $\bar{g}$ of index $\mu(>0)$ on the Sasakian manifold $\bar{M} = (\bar{M},\, J,\, \zeta,\, \theta,\, \bar{g})$. Then we say that $\bar{M}$ is an *indefinite Sasakian manifold*. First of all, we have the following result:

***Theorem* 2.1.** — *In an indefinite Sasakian manifold $\bar{M}$, the characteristic vector field ζ is a unit spacelike vector field on $\bar{M}$.*

Proof : Apply the operator $\bar{\nabla}_X$ to $J\zeta = 0$ and use (2.2) and (2.3), we get $J^2X = \epsilon J^2 X$ for any vector field X in $\bar{M}$. If $\epsilon = -1$, then we have $X = \theta(X)\zeta$ for any vector field X on $\bar{M}$. It is a contradiction to $\operatorname{rank} T\bar{M} > 1$ which proves $\epsilon = 1$. Thus ζ is a unit spacelike vector field.

An indefinite Sasakian manifold $\bar{M}$ is called an *indefinite Sasakian space form*, denoted by $\bar{M}(c)$, if it has the constant J-sectional curvature c [16]. The curvature tensor $\bar{R}$ of this space form $\bar{M}(c)$ is given by

$$\begin{aligned} 4\bar{R}(X,Y)Z &= (c+3)\{\bar{g}(Y,Z)X - \bar{g}(X,Z)Y\} \\ &+(c-1)\{\theta(X)\theta(Z)Y - \theta(Y)\theta(Z)X + \bar{g}(X,Z)\theta(Y)\zeta \\ &-\bar{g}(Y,Z)\theta(X)\zeta + \bar{g}(JY,Z)JX + \bar{g}(JZ,X)JY - 2\bar{g}(JX,Y)JZ\}, \end{aligned} \tag{2.4}$$

for any vector fields X, Y and Z in $\bar{M}$.

A hypersurface M of $\bar{M}$ is called a *lightlike hypersurface* if the normal bundle $TM^{\perp}$ of M is a vector subbundle of the tangent bundle TM of M, of rank 1. Then there exists a non-degenerate complementary vector bundle $S(TM)$ of $TM^{\perp}$ in TM, called a *screen distribution* on M, such that

$$TM = TM^{\perp} \oplus_{orth} S(TM), \tag{2.5}$$

where $\oplus_{orth}$ denotes the orthogonal direct sum. We denote such a lightlike hypersurface by $(M, g, S(TM))$. Denote by $F(\bar{M})$ the algebra of smooth functions on $\bar{M}$ and by $\Gamma(E)$ the $F(\bar{M})$ module of smooth sections of a vector bundle E over $\bar{M}$. We known [4] that, for any null section ξ of $TM^{\perp}$ on a coordinate neighborhood $\mathcal{U} \subset M$, there exists a unique null section N of a unique vector bundle $tr(TM)$ of rank 1 in $S(TM)^{\perp}$ satisfying

$$\bar{g}\,(\xi, N) = 1, \quad \bar{g}(N,N) = \bar{g}(N,X) = 0, \;\; \forall\, X \in \Gamma(S(TM)). \tag{2.6}$$

In this case, the tangent bundle $T\bar{M}$ of $\bar{M}$ is decomposed as follow:

$$T\bar{M} = TM \oplus tr(TM) = \{TM^{\perp} \oplus tr(TM)\} \oplus_{orth} S(TM). \tag{2.7}$$

We call $tr(TM)$ and N the *transversal vector bundle* and the *null transversal vector field* of M with respect to the screen $S(TM)$ respectively.

Let $\bar{\nabla}$ be the Levi-Civita connection of $\bar{M}$ and P the projection morphism of $\Gamma(TM)$ on $\Gamma(S(TM))$ with respect to the decomposition (2.5). Then the local Gauss and Weingartan formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + B(X,Y)N, \tag{2.8}$$

$$\bar{\nabla}_X N = -A_N X + \tau(X)N, \tag{2.9}$$

$$\nabla_X PY = \nabla^*_X PY + C(X,PY)\xi, \tag{2.10}$$

$$\nabla_X \xi = -A^*_\xi X - \tau(X)\xi, \tag{2.11}$$

for all X, $Y \in \Gamma(TM)$, where ∇ and ∇^* are the liner connections on TM and $S(TM)$ respectively, B and C are the local second fundamental forms on TM and $S(TM)$ respectively, A_N and A^*_ξ are the shape operators on TM and $S(TM)$ respectively and τ is a 1-form on TM defined by $\tau(X) = \nabla^{\perp}_X N = \bar{g}(\bar{\nabla}_X N, \xi)$. Since the connection $\bar{\nabla}$ of $\bar{M}$ is torsion-free, the induced connection ∇ of M is also torsion-free and the second fundamental form B is symmetric on TM. From the fact that $B(X,Y) = \bar{g}(\bar{\nabla}_X Y, \xi)$ for all X, $Y \in \Gamma(TM)$, we show that the local second fundamental form B is independent of the choice of a screen distribution and satisfies

$$B(X,\xi) = 0, \quad \forall\, X \in \Gamma(TM). \tag{2.12}$$

The induced connection ∇ of M is not metric and satisfies

$$(\nabla_X g)(Y,Z) = B(X,Y)\,\eta(Z) + B(X,Z)\,\eta(Y), \tag{2.13}$$

for any X, Y, $Z \in \Gamma(TM)$, where η is a 1-form such that

$$\eta(X) = \bar{g}(X,N), \quad \forall\, X \in \Gamma(TM). \tag{2.14}$$

But the connection ∇^* on $S(TM)$ is metric. Two local second fundamental forms B and C are related to their shape operators by

$$B(X,Y) = g(A^*_\xi X, Y), \qquad \bar{g}(A^*_\xi X, N) = 0, \tag{2.15}$$

$$C(X,PY) = g(A_N X, PY), \quad \bar{g}(A_N X, N) = 0. \tag{2.16}$$

From (2.15), the operator A^*_ξ is $S(TM)$-valued self-adjoint on TM such that

$$A^*_\xi \xi = 0. \tag{2.17}$$

3. Characterization Theorems

Let $(M, g, S(TM))$ be a lightlike hypersurface of an indefinite Sasakian manifold $\bar{M}$. In general, the characteristic vector field ζ belongs to $T\bar{M}$. Thus, from the decomposition (2.7) of $T\bar{M}$, ζ is decomposed by

$$\zeta = W + a\xi + bN, \tag{3.1}$$

where W is a smooth vector field on $S(TM)$, and a and b are smooth functions on $\bar{M}$ defined by $a = \theta(N)$ and $b = \theta(\xi)$. In this case we have

Lemma 3.1 — Let $(M, g, S(TM))$ be a lightlike hypersurface of an indefinite Sasakian manifold $\bar{M}$. Then the distributions $J(TM^{\perp})$ and $J(tr(TM))$ are vector subbundles of $S(TM)$ of rank 1.

Proof : If $J\xi = 0$, then we have $0 = \bar{g}(J\xi, J\xi) = -b^2$ and $0 = \bar{g}(J\xi, JN) = 1 - ab$ due to the first equation of (2.1). This two equations deduce a contradiction $1 = 0$. Thus we have $J\xi \neq 0$. Also if $JN = 0$, then we have $0 = \bar{g}(JN, JN) = -a^2$ and $0 = \bar{g}(J\xi, JN) = 1 - ab$. It is also a contradiction. Thus we also have $JN \neq 0$. From the fact that $\bar{g}(J\xi, \xi) = 0$, we see that $J\xi$ is tangent to M and $J(TM^{\perp})$ is a distribution on M of rank 1 such that $TM^{\perp} \cap J(TM^{\perp}) = \{0\}$. In fact, if $TM^{\perp} \cap J(TM^{\perp}) \neq \{0\}$, then there exists a non-vanishing smooth real valued function f such that $J\xi = f\xi$. Apply J to this equation and use (2.1), we have $(f^2 + 1)\xi = b\zeta$. Taking the scalar product with ξ and N in this equation by turns, we get $b = 0$ and $f^2 + 1 = 0$ respectively. It is an impossible case for the real M. Therefore we have $TM^{\perp} \cap J(TM^{\perp}) = \{0\}$. This enables one to choose a screen distribution $S(TM)$ such that it contains $J(TM^{\perp})$ as a vector subbundle. From the fact that $\bar{g}(JN, \xi) = -\bar{g}(N, J\xi) = 0$, JN is also tangent to M. As $\bar{g}(JN, N) = 0$, $J(tr(TM))$ is also a vector subbundle of $S(TM)$ of rank 1.

Lemma 3.2 — Let $(M, g, S(TM))$ be a lightlike hypersurface of an indefinite Sasakian manifold $\bar{M}$. The function b, defined by $b = \theta(\xi)$, satisfies

$$b = C(\xi, J\xi). \tag{3.2}$$

Proof : Apply the operator $\bar{\nabla}_X$ to $\bar{g}(J\xi, N) = 0$ and use the equations (2.3), (2.8), (2.9), (2.11), (2.15) and (2.16), we have

$$b\eta(X) + B(X, JN) = C(X, J\xi), \quad \forall\, X \in \Gamma(TM). \tag{3.3}$$

Replace X by ξ in this equation and use (2.12), we have $b = C(\xi, J\xi)$.

Definition 1 — (1) A lightlike hypersurface $(M, g, S(TM))$ of $\bar{M}$ is called *screen conformal* [1] if there exist a non-vanishing smooth function φ on a neighborhood $\mathcal{U}$ in M such that $A_N = \varphi A^*_\xi$, or equivalently,

$$C(X, PY) = \varphi B(X, Y), \ \forall X, Y \in \Gamma(TM).$$

(2) A screen distribution $S(TM)$ is called *totally umbilical* [4] in M if there exist a smooth function γ on a neighborhood $\mathcal{U}$ in M such that

$$C(X, PY) = \gamma\, g(X, Y), \quad \forall X, Y \in \Gamma(TM).$$

***Theorem* 3.1.** (1) — *There exist no screen conformal lightlike hypersurfaces M of an indefinite Sasakian manifold $\bar{M}$.*

(2) *There exist no lightlike hypersurfaces M of an indefinite Sasakian manifold $\bar{M}$ such that $S(TM)$ is totally umbilical in M.*

PROOF : (1) If M is screen conformal, then, from (2.12), we have

$$b = C(\xi, J\xi) = \varphi B(\xi, J\xi) = 0.$$

Thus, from (3.1) and (3.3), we show that ζ is tangent to M and

$$B(X, JN) = C(X, J\xi), \quad \forall X \in \Gamma(TM). \tag{3.4}$$

Since ζ is tangent to M, substituting (2.8) in (2.2), we obtain

$$JX = \nabla_X \zeta + B(X, \zeta)N, \quad \forall X \in \Gamma(TM).$$

Taking the scalar product with ξ in this equation, we have

$$B(X, \zeta) = -g(X, J\xi), \quad \forall X \in \Gamma(TM). \tag{3.5}$$

Replace X by $J\xi$ and JN in (3.5) by turns and use $b = 0$, we have

$$B(J\xi, \zeta) = 0, \quad B(JN, \zeta) = -1,$$

respectively.

Since M is screen conformal, using (3.4), we have

$$-1 = B(JN, \zeta) = C(J\xi, \zeta) = \varphi B(J\xi, \zeta) = \varphi\, 0 = 0.$$

It is a contradiction. Thus there exist no screen conformal lightlike hypersurfaces M of an indefinite Sasakian manifold $\bar{M}$.

(2) If $S(TM)$ is totally umbilical in M, then we have

$$b = C(\xi, J\xi) = \gamma g(\xi, J\xi) = 0.$$

Thus ζ is tangent to M and we have (3.4) and (3.5). From (3.4), we get

$$B(X, JN) = \gamma\, g(X, J\xi), \quad \forall\, X \in \Gamma(TM).$$

Replace X by JN in (3.5) and use the last equation, we have

$$-1 = -g(JN, J\xi) = B(JN,\, \zeta) = \gamma\, g(J\xi,\, \zeta) = \gamma\, \theta(J\xi) = 0.$$

It is also a contradiction. Thus there exist no lightlike hypersurfaces M of an indefinite Sasakian manifold $\bar{M}$ such that $S(TM)$ is totally umbilical.

Definition 2 — We say that M is *totally umbilical* [4] if, on any coordinate neighborhood $\mathcal{U}$, there is a smooth function β such that

$$B(X, Y) = \beta\, g(X, Y),$$

for all X, $Y \in \Gamma(TM)$.

In case $\beta = 0$ on $\mathcal{U}$, we say that M is *totally geodesic*.

***Theorem* 3.2** — *There exist no totally umbilical lightlike hypersurfaces M of an indefinite Sasakian manifold $\bar{M}$ such that ζ is tangent to M.*

Proof : Assume that ζ is tangent to M. Then we have the equation (3.5). If M is totally umbilical, then, from (3.5), we have

$$\beta\, g(X,\, \zeta) = -g(X,\, J\xi), \quad \forall\, X \in \Gamma(TM).$$

Replace X by JN in this equation and use (2.1), we have

$$0 = \beta\, \theta(JN) = \beta\, g(JN,\, \zeta) = -g(JN, J\xi) = -1.$$

It is a contradiction. Thus there exist no totally umbilical lightlike hypersurfaces M of an indefinite Sasakian manifold $\bar{M}$ such that ζ is tangent to M.

Lemma 3.3 — Let $(M, g, S(TM))$ be a lightlike hypersurface of an indefinite Sasakian manifold $\bar{M}$. Then ζ does not belong to $TM^{\perp}$ and $tr(TM)$.

Proof : Assume that ζ belongs to $TM^\perp$ or $tr(TM)$. Then we have $\zeta = a\xi$ or $\zeta = bN$ respectively, where $a \neq 0$ and $b \neq 0$. From this facts, we have

$$1 = \bar{g}(\zeta, \zeta) = a^2\bar{g}(\xi, \xi) = 0 \quad \text{or} \quad 1 = \bar{g}(\zeta, \zeta) = b^2\bar{g}(N, N) = 0.$$

Which are contradictions. From this result we deduce our assertion.

Note 1 : (i) If ζ is tangent to M, then, by Lemma 3.3, ζ does not belong to $TM^\perp$. This enables one to choose a screen distribution $S(TM)$ which contains ζ. This implies that *if ζ is tangent to M, then it belongs to $S(TM)$*. Călin also proved this result in his book [2] which Kang *et al.* [13] and Duggal-Sahin [7, 8] assumed in their papers.

(ii) Kang et al assumed that ζ belongs to $S(TM)$ and M is totally umbilical or totally geodesic in their paper [13] which is not correct. Because, by Theorem 3.2, we show that if ζ is tangent to M, then M is neither totally umbilical nor totally geodesic. Moreover, by Theorem 3.1, M is not screen conformal and its screen $S(TM)$ is not also totally umbilical in M.

Denote by $\bar{R}$ and R the curvature tensors of the Levi-Civita connection $\bar{\nabla}$ of $\bar{M}$ and the induced connection ∇ of M respectively. Using the local Gauss-Weingarten formulas for M, we obtain the Gauss equation for M:

$$\begin{aligned}\bar{R}(X,Y)Z = R(X,Y)Z &+ B(X,Z)A_NY - B(Y,Z)A_NX \\ &+ \{(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) \\ &+ \tau(X)B(Y,Z) - \tau(Y)B(X,Z)\}N,\end{aligned} \tag{3.6}$$

for all $X, Y, Z \in \Gamma(TM)$.

Replace Z by ξ in this equation and use (2.12) and the fact $B(Y, A^*_\xi X) = B(X, A^*_\xi Y)$ for all $X, Y \in \Gamma(TM)$, we have

$$\bar{R}(X,Y)\xi = R(X,Y)\xi, \quad \forall\, X, Y \in \Gamma(TM). \tag{3.7}$$

Using (3.7) and the fact $R(X,Y)Z \in \Gamma(TM)$ for $X, Y, Z \in \Gamma(TM)$, we get

$$\begin{aligned}\bar{g}(\bar{R}(X,Y)Z, \xi) &= -\bar{g}(\bar{R}(X,Y)\xi, Z) = -g(R(X,Y)\xi, Z) \\ &= g(R(X,Y)Z, \xi) = 0, \quad \forall\, X, Y, Z \in \Gamma(TM).\end{aligned} \tag{3.8}$$

***Theorem* 3.3** — *Let $(M, g, S(TM))$ be a lightlike hypersurface of an indefinite Sasakian space form $\bar{M}(c)$. Then we have $c = 1$.*

Proof : Assume that $b \neq 0$. Then, taking the scalar product with ξ in (2.4) and using the equations (2.1), (3.1) and (3.8), we obtain

$$(c-1)\{bg(X,Z)\theta(Y) - bg(Y,Z)\theta(X) - \bar{g}(JY,Z)g(X,J\xi) \qquad (3.9)$$
$$-\bar{g}(JZ,X)g(Y,J\xi) + 2\bar{g}(JX,Y)g(Z,J\xi)\} = 0,\ \forall\, X,Y,Z \in \Gamma(TM).$$

Replace Z by $J\xi$ and Y by ξ in this equation and use (2.1), we have

$$4b^2(c-1)g(X,J\xi) = 0, \quad \forall\, X \in \Gamma(TM).$$

Replace X by $J\xi$ in this equation, we obtain $b^4(c-1) = 0$. Since $b \neq 0$ and $(c-1)$ is a constant, we have $c = 1$.

Assume that $b = 0$, i.e., ζ is tangent to M. Then, by Note 1, ζ belongs to $S(TM)$. This implies that if $b = 0$, then $a = 0$. In this case, $g(J\xi, J\xi) = 0$; $g(J\xi, JN) = 1$ and the equation (3.9) deduce to the following form:

$$(c-1)\{\bar{g}(JY,Z)g(X,J\xi) + \bar{g}(JZ,X)g(Y,J\xi) - 2\bar{g}(JX,Y)g(Z,J\xi)\} = 0,$$

for all X, Y, $Z \in \Gamma(TM)$. Replace Z by JN and Y by ξ in this equation and use the fact $g(J\xi, JN) = 1$, we have

$$3(c-1)g(X,J\xi) = 0, \quad \forall\, X \in \Gamma(TM).$$

Taking $X = JN$ in this equation and using the fact $g(J\xi, JN) = 1$, we have $3(c-1) = 0$, i.e., $c = 1$. Thus we have our assertion.

Corollary 1 — There exist no lightlike hypersurfaces M of indefinite Sasakian space form $\bar{M}(c)$ with $c \neq 1$.

Definition 3 — We say that M is *locally symmetric* [14] if its curvature tensor R be parallel, i.e., have vanishing covariant differential, $\nabla R = 0$.

***Theorem* 3.4** — *Let $(M, g, S(TM))$ be a totally geodesic lightlike hypersurface of an indefinite Sasakian manifold $\bar{M}$. If M is locally symmetric, then M is a space of constant curvature* 1.

Proof : As M is totally geodesic (totally umbilical), by Theorem 3.2, we show that ζ is not tangent to M, i.e., $b \neq 0$ due to (3.1) and, by Definition 2 and the equation (2.15), we get $A^*_\xi = 0$ and $B = 0$. Consider the local timelike vector field V on M and its 1-form v defined by

$$V = -b^{-1}J\xi, \quad v(X) = -g(X,V), \quad \forall\, X \in \Gamma(TM). \qquad (3.10)$$

Substituting (3.1) in (2.2) and using (2.8), (2.9) and (2.11), we obtain

$$JX = \nabla_X W - bA_N X + \{Xa - a\tau(X)\}\xi + \{Xb + b\tau(X)\}N,$$

for all $X \in \Gamma(TM)$. Taking the scalar product with ξ in this equation and using (2.1), (2.16) and (3.10), we have

$$Xb + b\tau(X) = -bv(X), \quad \forall\, X \in \Gamma(TM). \tag{3.11}$$

Differentiating $bV = -J\xi$ with $X \in \Gamma(TM)$ with respect to the connection $\bar{\nabla}$ of $\bar{M}$ and using (2.3), (2.8), (2.11), (3.11) and the fact $b \neq 0$, we get

$$\nabla_X V = -X + v(X)V, \quad \forall\, X \in \Gamma(TM). \tag{3.12}$$

Using (3.12) and the fact that ∇ is torsion free connection, we show that

$$R(X,Y)V = 2dv(X,Y)V + v(X)Y - v(Y)X, \ \forall\, X, Y \in \Gamma(TM).$$

Taking the scalar product with V in this equation and using (3.10), we have $dv = 0$, i.e., v is closed. From this facts we deduce the following equation:

$$R(X,Y)V = v(X)Y - v(Y)X, \ \forall\, X, Y \in \Gamma(TM). \tag{3.13}$$

Differentiate (3.10) with $Y \in \Gamma(TM)$ and use (3.10) and (3.12), we have

$$(\nabla_X v)(Y) = g(X,Y) + v(X)v(Y), \quad \forall\, X, Y \in \Gamma(TM). \tag{3.14}$$

Differentiating (3.13) with $Z \in \Gamma(TM)$ and using (3.13) and the fact that M is locally symmetric, i.e., $\nabla_X R = 0$ for any $X \in \Gamma(TM)$, we have

$$R(X,Y)\nabla_Z V = (\nabla_Z v)(X)Y - (\nabla_Z v)(Y)X, \quad \forall\, X, Y \in \Gamma(TM).$$

Substituting (3.12) and (3.14) in this equation and using (3.13), we obtain

$$R(X,Y)Z = g(Y,Z)X - g(X,Z)Y, \quad \forall\, X, Y \in \Gamma(TM). \tag{3.15}$$

Thus M is a space of constant curvature 1, from which we have our theorem.

The induced Ricci type tensor $R^{(0,2)}$ of M is defined by

$$R^{(0,2)}(X,Y) = trace\{Z \to R(Z,X)Y\}, \quad \forall\, X, Y \in \Gamma(TM).$$

In general, the induced Ricci type tensor $R^{(0,2)}$, defined by the method of the geometry of the non-degenerate submanifolds [14], is not symmetric [4, 5, 6]. Hence we need the following definition: A tensor field $R^{(0,2)}$ of lightlike hypersurfaces M is called its *induced Ricci tensor* [5] of M if it is symmetric. A symmetric $R^{(0,2)}$ tensor will be denoted by Ric.

***Theorem* 3.5** [4] — *Let $(M, g, S(TM))$ be a lightlike hypersurface of a semi-Riemannian manifold $\bar{M}$. The Ricci type tensor $R^{(0,2)}$ is symmetric, if and only if, each 1-form τ is closed, i.e., $d\tau = 0$, on any $\mathcal{U} \subset M$.*

We define the curvature tensor $R^{\perp}$ of the transversal bundle of M by

$$R^{\perp}(X,Y)N = \nabla^{\perp}_X \nabla^{\perp}_Y N - \nabla^{\perp}_Y \nabla^{\perp}_X N - \nabla^{\perp}_{[X,Y]} N,$$

for all $X, Y \in \Gamma(TM)$. If $R^{\perp}$ vanishes identically, then the transversal connection $\nabla^{\perp}$ of M is said to be *flat* (or *trivial*) [3].

***Theorem* 3.6** — *Let $(M, g, S(TM))$ be a lightlike hypersurface of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. The following assertions are equivalent:*
(1) *The transversal connection of M is flat, i.e., $R^{\perp} = 0$.*
(2) *Each 1-form τ is closed, i.e., $d\tau = 0$, on any $\mathcal{U} \subset M$.*
(3) *The Ricci type tensor $R^{(0,2)}$ is an induced Ricci tensor of M.*

PROOF : For all $X, Y \in \Gamma(TM)$, using $\bar{\nabla}_X N = -A_N X + \nabla^{\perp}_X N$, we have

$$\begin{aligned} \bar{R}(X,Y)N = &-\nabla_X(A_N Y) + \nabla_Y(A_N X) + A_N[X,Y] + A_{\nabla^{\perp}_X N} Y \\ &- A_{\nabla^{\perp}_Y N} X + B(Y, A_N X)N - B(X, A_N Y)N + R^{\perp}(X,Y)N. \end{aligned}$$

On the other hand, using (2.9): $\bar{\nabla}_X N = -A_N X + \tau(X)N$, we have

$$\begin{aligned} \bar{R}(X,Y)N = &-\nabla_X(A_N Y) + \nabla_Y(A_N X) + A_N[X,Y] + \tau(X)A_N Y \\ &- \tau(Y)A_N X + \{B(Y, A_N X) - B(X, A_N Y) - 2d\tau(X,Y)\}N. \end{aligned} \tag{3.16}$$

From this two equations we have

$$R^{\perp}(X,Y)N = -2d\tau(X,Y)N, \quad \forall\, X,\, Y \in \Gamma(TM).$$

From this result and Theorem 3.5, we have our assertions.

***Theorem* 3.7** — *Let $(M, g, S(TM))$ be a totally geodesic lightlike hypersurface of an indefinite Sasakian manifold $\bar{M}$. If M is locally symmetric,* *then the transversal connection of M is flat and the Ricci type tensor $R^{(0,\,2)}$ is an induced Ricci tensor Ric of M.*

PROOF : Taking the scalar product with ξ in both sides of (3.16), we have

$$\bar{g}(\bar{R}(X,Y)N,\,\xi) = -2d\tau(X,Y), \quad \forall\, X,\, Y \in \Gamma(TM).$$

Using this, (3.7), (3.15) and the fact M is locally symmetric, we have

$$\begin{aligned} 2d\tau(X,Y) &= -\bar{g}(\bar{R}(X,Y)N,\,\xi) = \bar{g}(\bar{R}(X,Y)\xi,\,N) \\ &= \bar{g}(R(X,Y)\xi,\,N) = g(Y,\xi)\eta(X) - g(X,\xi)\eta(Y) = 0, \end{aligned}$$

for all $X,\, Y \in \Gamma(TM)$. Thus, from Theorem 3.6, we have our theorem.

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