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A HYBIRD MEAN VALUE INVOLVING GAUSS SUMS AND CHARACTER SUMS

Tian Qing*,**[1] and Zhang Wenpeng*

**Department of Mathematics, Northwest University,*
Xi'an, Shaanxi, 710127, *Peoples' Republic of China*
***School of Science, Xi'an University of Architecture and Technology,*
Xi'an, Shaanxi, 710055, *Peoples' Republic of China*
e-mails : tianqingv@hotmail.com, wpzhang@nwu.edu.cn

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Let χ be a Dirichlet character modulo $q > 2$, and $L(s, \chi)$ denotes the Dirichlet L-function corresponding to χ. The main purpose of this paper is using the estimate for character sums and the analytic method to study the mean value properties of $\frac{L'}{L}(1, \chi)$ with the weight of Gauss sums and character sums, and give an interesting mean value formula for it.

Key words : L-functions; hybrid mean value; Gauss sums; character in short interval; asymptotic formula.

1. Introduction

Let $q > 2$ be an integer and χ denotes a Dirichlet character modulo q. $L(s, \chi)$ denotes the Dirichlet L-function corresponding to χ. It seems that

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one can hardly estimate $\frac{L'}{L}(1,\chi) \equiv \frac{L'(1,\chi)}{L(1,\chi)}$, though it has long history and plays an important role in number theory (see [1]). However, $\frac{L'}{L}(1,\chi)$ enjoys good mean value properties. For example, the second author [2] studied the asymptotic properties of the summations

$$\sum_{q\le Q}\frac{q}{\phi(q)}\sum_{\substack{\chi \bmod q\\ \chi\ne\chi_0}}\left|\frac{L'}{L}(1,\chi)\right|^4 \quad \text{and} \quad \sum_{q\le Q}\frac{1}{\phi(q)}\sum_{\substack{\chi \bmod q\\ \chi\ne\chi_0}}\left|\frac{L'}{L}(1,\chi)\right|^4,$$

where χ_0 is the principal character modulo q and $\phi(q)$ is the Euler function. Liu and Zhang [3] considered the mean value of $\left|\frac{L'}{L}(1,\chi)\right|^{2k}$, the hybrid mean value of $\frac{L'}{L}(1,\chi)$ with general Kloosterman sums, general quadratic Gauss sums respectively, and obtained some asymptotic formulae. The second author [4] studied the mean value property of

$$\sum_{\substack{\chi \bmod q\\ \chi\ne\chi_0}}\left|\sum_{n\le N}\chi(n)\right|^2\cdot|L(1,\chi)|^2,$$

by using the estimate for character sums and analytic method.

Gauss sum plays an important role in number theory. The classical Gauss sum $G(m,\chi)$ is defined as:

$$G(m,\chi)=\sum_{a=1}^{q}\chi(a)e\left(\frac{ma}{q}\right),$$

where $e(y)=e^{2\pi iy}$. In particular for $m=1$, we write $G(1,\chi)=\tau(\chi)$. The various properties and applications of $\tau(\chi)$ have appeared in many analytic number theory books. Perhaps its most famous property is $|\tau(\chi)|=\sqrt{q}$ when χ is a primitive character mod q. Contrarily, when χ is nonprimitive character, the value of $\tau(\chi)$ is more irregular, even more it may be zero. However, many scholars have found it enjoys good value distribution properties in some problems of weighted mean value (see [5-7]).

In this paper, we shall study the asymptotic properties of the hybrid mean value

$$\sum_{\substack{\chi \bmod q\\ \chi\ne\chi_0}}|\tau(\chi)|^m\cdot\left|\sum_{n\le Q}\chi(n)\right|^2\cdot\left|\frac{L'}{L}(1,\chi)\right|^2.$$

This problem is interesting, because it can help us to find some relations between the $\frac{L'}{L}(1,\chi)$, Gauss sum and Character sums.

By using the estimate for Character sums and the analytic method, we shall prove the following main conclusion.

Theorem — *Let $q = MN > 2$ be an integer, $M = \prod_{p\|q} p$, $(M,N) = 1$. Then, for any real number Q and any fixed positive number ϵ with $1 < Q < q^{1-\epsilon}$, we have the asymptotic formula*

$$\sum_{\substack{\chi \bmod q \\ \chi\neq\chi_0}} |\tau(\chi)|^m \cdot \left|\sum_{n\leq Q}\chi(n)\right|^2 \cdot \left|\frac{L'}{L}(1,\chi)\right|^2 = \frac{\phi(q)\phi^2(N)}{q} Q N^{\frac{m}{2}-1} C \prod_{p|M}\left(p^{\frac{m}{2}+1} - 2p^{\frac{m}{2}} + 1\right) + O\left(2^{\omega(q)} q^{\frac{m}{2}+\epsilon}\right) + O\left(Q^2 q^{\frac{m}{2}+\epsilon}\right),$$

where $\sum_{\chi\neq\chi_0}$ denotes the summation over all non-principal characters modulo q, $\phi(q)$ is the Euler function, $\omega(q)$ denotes the number of all different prime divisors of q and C is a constant depending only on q.

2. Several Lemmas

In order to complete the proof of the theorem, we need following several lemmas. First we have

Lemma 1 — Let $q = uv$, $u > 2$, $v > 2$ and $(u,v) = 1$. Then for any $\chi \bmod q$, there exists one and only one character $\chi_u \bmod u$, one and only one character $\chi_v \bmod v$, such that $\chi = \chi_u\chi_v$, and

$$|\tau(\chi)| = |\tau(\chi_u)| \times |\tau(\chi_v)|.$$

Proof : See Theorem 13.3.1 of Ref. [8].

Lemma 2 — Let q and r be integers with $q \geq 2$ and $(r,q) = 1$. Then we have the identities

$$\sideset{}{^*}\sum_{\chi \bmod q} \chi(r) = \sum_{d|(q,r-1)} \mu\left(\frac{q}{d}\right)\phi(d), \qquad J(d) = \sum_{d|q}\mu(d)\phi\left(\frac{q}{d}\right),$$

where $\sum^*$ denotes the summation over all primitive characters, $\mu(n)$ is the *Möbius* function and $J(q)$ denotes the number of primitive characters mod q.

PROOF : See Lemma 4 of Ref. [9].

Lemma 3 — Let p be a prime, α be a positive integer and $\alpha \geq 2$, $n = p^\alpha$. Then for any nonprimitive character χ_1 mod n, we have the identity

$$\sum_{a=1}^{p^\alpha} \chi_1(a) e\left(\frac{a}{p^\alpha}\right) = 0.$$

PROOF : See Ref. [6].

Lemma 4 — Let $q = MN > 2$ be a positive integer, where $N = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$, $\alpha_i \geq 2(1 \leq i \leq s)$, positive integer M has no square factor. $(M, N) = 1$, $T = \exp(q^\epsilon)$ and ϵ is any fixed positive number. Then for any given positive integer $1 \leq n, k \leq q$, $(nk, q) = 1$ and $d|M$, we have

$$\sum_{\chi \bmod Nd}^{*} \chi\chi_M^0(n\overline{k}) \left| \frac{L'}{L}(1, \chi\chi_M^0) \right|^2 = \sum_{\chi \bmod Nd}^{*} \chi\chi_M^0(n\overline{k}) \left| \sum_{1 \leq a \leq T} \frac{\chi\chi_M^0(a)\Lambda(a)}{a} \right|^2 + O(1),$$

where $\Lambda(n)$ is the Mangoldt function and χ_M^0 denotes the principal character modulo M.

PROOF : For convenience, firstly we let

$$A(\chi\chi_M^0, y) = \sum_{T < a \leq y} \chi\chi_M^0(a)\Lambda(a).$$

Then applying Abel's identity, we have

$$\frac{L'}{L}(1, \chi\chi_M^0) = \sum_{a=1}^{\infty} \frac{\chi\chi_M^0(a)\Lambda(a)}{a} = \sum_{1 \leq a \leq T} \frac{\chi\chi_M^0(a)\Lambda(a)}{a} + \int_T^\infty \frac{A(\chi\chi_M^0, y)}{y^2} dy. \quad (1)$$

From (1) we may get

$$\begin{aligned}
\sideset{}{^*}\sum_{\chi \bmod Nd} \chi\chi_M^0(n\overline{k}) \left|\frac{L'}{L}(1,\chi\chi_M^0)\right|^2 &= \sideset{}{^*}\sum_{\chi \bmod Nd} \chi\chi_M^0(n\overline{k}) \\
&\left|\sum_{1\le a\le T} \frac{\chi\chi_M^0(a)\Lambda(a)}{a} + \int_T^\infty \frac{A(\chi\chi_M^0,y)}{y^2}dy\right|^2 \\
&= \sideset{}{^*}\sum_{\chi \bmod Nd} \chi\chi_M^0(n\overline{k}) \left|\sum_{1\le a\le T} \frac{\chi\chi_M^0(a)\Lambda(a)}{a}\right|^2 \\
&+ \sideset{}{^*}\sum_{\chi \bmod Nd} \chi\chi_M^0(n\overline{k}) \left|\int_T^\infty \frac{A(\chi\chi_M^0,y)}{y^2}dy\right|^2 \\
&+ \sideset{}{^*}\sum_{\chi \bmod Nd} \chi\chi_M^0(n\overline{k}) \left(\sum_{1\le a\le T} \frac{\chi\chi_M^0(a)\Lambda(a)}{a}\right) \\
&\left(\int_T^\infty \frac{A(\overline{\chi\chi_M^0},y)}{y^2}dy\right) \\
&+ \sideset{}{^*}\sum_{\chi \bmod Nd} \chi\chi_M^0(n\overline{k}) \left(\sum_{1\le a\le T} \frac{\overline{\chi\chi_M^0}(a)\Lambda(a)}{a}\right) \\
&\left(\int_T^\infty \frac{A(\chi\chi_M^0,y)}{y^2}dy\right). \qquad (2)
\end{aligned}$$

From the property of L-function (See [1 p.121-132].) we know, in all of the Dirichlet characters modulo q, there at most exists a character χ_1 with the exceptional zero β, $\beta > 1 - \frac{C}{\ln q}$. For the other characters, we have

$$|A(\chi,y)| \ll y \exp(-c_1\sqrt{\ln y}), \qquad (3)$$

when $y > T$. When coming to the possible exceptional character χ_1, from the famous Siegel's theorem, we know

$$|A(\chi_1,y)| \ll y \exp(-c_2\sqrt{\ln y}), \qquad (4)$$

when $y > \exp(q^\epsilon)$ with c_1 and c_2 are both positive numbers.

Using (3) and (4), we get

$$\left|\int_T^\infty \frac{A(\chi\chi_M^0,y)}{y^2}dy\right| \ll \int_T^{+\infty} \frac{e^{-c_1\sqrt{\ln y}}}{y}dy + \int_T^{\exp(q^\epsilon)} \frac{1}{y}dy + \int_{\exp(q^\epsilon)}^{+\infty} \frac{e^{-c_2\sqrt{\ln y}}}{y}dy.$$

Because of

$$\int_T^{+\infty} \frac{e^{-c_1\sqrt{\ln y}}}{y} dy \ll e^{-\frac{c_1}{2}\sqrt{\ln T}} \int_T^{+\infty} \frac{e^{-\frac{c_1}{2}\sqrt{\ln y}}}{y} dy \ll e^{-\frac{c_1}{2}\sqrt{\ln T}} = e^{-cq^{\epsilon}},$$

one has

$$\left| \int_T^{\infty} \frac{A(\chi\chi_M^0, y)}{y^2} dy \right| \ll e^{-cq^{\epsilon}}.$$

Using the Cauchy inequality and the estimate $\sum_{a\le T} \frac{\Lambda(a)}{a} \ll \ln T$, we have the estimates

$$\begin{aligned} & \sideset{}{^*}\sum_{\chi \bmod Nd} \chi\chi_M^0(n\bar{k}) \left(\sum_{1\le a\le T} \frac{\chi\chi_M^0(a)\Lambda(a)}{a} \right) \left(\int_T^{\infty} \frac{A(\overline{\chi\chi_M^0}, y)}{y^2} dy \right) \\ \ll\ & \left[\sideset{}{^*}\sum_{\chi \bmod Nd} \left| \sum_{1\le a\le T} \frac{\overline{\chi\chi_M^0}(a)\Lambda(a)}{a} \right|^2 \right]^{\frac{1}{2}} \left[\sideset{}{^*}\sum_{\chi \bmod Nd} \left| \int_T^{\infty} \frac{A(\chi\chi_M^0, y)}{y^2} dy \right|^2 \right]^{\frac{1}{2}} \\ \ll\ & q^{\epsilon} e^{-cq^{\epsilon}} J(Nd) \ll 1, \end{aligned} \tag{5}$$

$$\left| \sideset{}{^*}\sum_{\chi \bmod Nd} \chi\chi_M^0(a) \left(\sum_{1\le a\le T} \frac{\chi\chi_M^0(a)\Lambda(a)}{a} \right) \left(\int_T^{\infty} \frac{A(\overline{\chi\chi_M^0}, y)}{y^2} dy \right) \right| \ll 1 \tag{6}$$

and

$$\left| \sideset{}{^*}\sum_{\chi \bmod Nd} \chi\chi_M^0(a) \left| \int_T^{\infty} \frac{A(\chi\chi_M^0, y)}{y^2} dy \right|^2 \right| \ll 1. \tag{7}$$

Combining (5), (6) and (7) we may immediately get the asymptotic formula

$$\sideset{}{^*}\sum_{\chi \bmod Nd} \chi\chi_M^0(n\bar{k}) \left| \frac{L'}{L}(1, \chi\chi_M^0) \right|^2 = \sideset{}{^*}\sum_{\chi \bmod Nd} \chi\chi_M^0(n\bar{k}) \left| \sum_{1\le a\le T} \frac{\chi\chi_M^0(a)\Lambda(a)}{a} \right|^2 + O(1).$$

This proves Lemma 4.

Lemma 5 — Let Q and T be any real number with $1 < Q < q^{1-\epsilon}$, ϵ is any fixed positive number, χ_M^0 is the principle character mod M, then we have

$$\begin{aligned}
&\sum_{n\le Q}{}'\sum_{k\le Q}{}'\sum_{\chi \bmod Nd}{}^{*}\chi\chi_M^0(n\overline{k})\left|\sum_{1\le a\le T}\frac{\chi\chi_M^0(a)\Lambda(a)}{a}\right|^2\\
=\ &\frac{\phi(q)}{q}QJ(Nd)C+O\left(2^{\omega(q)}q^{\epsilon}J(Nd)\right)+O\left(Q^2(qNd)^{\epsilon}\right),
\end{aligned}$$

where $\sum_{n}{}'$ denotes that the sum is over all n with $(n,q)=1$.

Proof : From Lemma 2 we have

$$\begin{aligned}
&\sum_{n\le Q}{}'\sum_{k\le Q}{}'\sum_{\chi \bmod Nd}{}^{*}\chi\chi_M^0(n\overline{k})\left|\sum_{1\le a\le T}\frac{\chi\chi_M^0(a)\Lambda(a)}{a}\right|^2\\
=\ &\sum_{n\le Q}{}'\sum_{k\le Q}{}'\sum_{1\le a\le T}{}'\sum_{1\le b\le T}{}'\frac{\chi_M^0(na\overline{kb})\Lambda(a)\Lambda(b)}{ab}\sum_{\chi \bmod Nd}{}^{*}\chi(na\overline{kb})\\
=\ &\sum_{n\le Q}{}'\sum_{k\le Q}{}'\sum_{1\le a\le T}{}'\sum_{1\le b\le T}{}'\frac{\chi_M^0(na\overline{kb})\Lambda(a)\Lambda(b)}{ab}\sum_{l\mid(Nd,na\overline{kb}-1)}\mu\left(\frac{Nd}{l}\right)\phi(l)\\
=\ &\sum_{l\mid Nd}\mu\left(\frac{Nd}{l}\right)\phi(l)\sum_{n\le Q}{}'\sum_{k\le Q}{}'\sum_{\substack{1\le a\le T\\ na=kb}}{}'\sum_{1\le b\le T}{}'\frac{\Lambda(a)\Lambda(b)}{ab}\\
+\ &\sum_{l\mid Nd}\mu\left(\frac{Nd}{l}\right)\phi(l)\sum_{n\le Q}{}'\sum_{k\le Q}{}'\sum_{\substack{1\le a\le T\\ na\equiv kb(\bmod l)\\ na\ne kb}}{}'\sum_{1\le b\le T}{}'\frac{\Lambda(a)\Lambda(b)}{ab}\\
=\ &E_1+E_2. \qquad (8)
\end{aligned}$$

Now we estimate E_1 and E_2 respectively.

Let $K = [q^\epsilon]$, then we have

$$E_2 \leq \sum_{l|Nd} \mu\left(\frac{Nd}{l}\right)\phi(l) \sum_{i,j=0}^{K} \sum_{e^i\leq a\leq e^{i+1}} \frac{\Lambda(a)}{a} \sum_{e^j\leq b\leq e^{j+1}} \frac{\Lambda(b)}{b} \sum_{\substack{n,k=1\\ na\equiv kb(\bmod l)\\ na\neq kb}}^{Q} 1$$

$$\leq 2\sum_{l|Nd} \mu\left(\frac{Nd}{l}\right)\phi(l) \sum_{0\leq i\leq j\leq K} \sum_{e^i\leq a\leq e^{i+1}} \frac{\Lambda(a)}{a} \sum_{e^j\leq b\leq e^{j+1}} \frac{\Lambda(b)}{b} \sum_{\substack{n,k=1\\ na\equiv kb(\bmod l)\\ na\neq kb}}^{Q} 1$$

$$\leq 2\sum_{l|Nd} \mu\left(\frac{Nd}{l}\right)\phi(l) \sum_{0\leq i\leq j\leq K} (i+1)(j+1)e^{-i-j}T_{ij},$$

where

$$T_{ij} = \sum_{\substack{n,k=1\\ na\equiv kb(\bmod l)\\ na\neq kb}}^{Q} 1$$

is just the number of solutions (n,k,a,b) to the congruence

$$na \equiv kb(\mathrm{mod} l), \quad 1\leq n,k\leq Q, \quad e^i\leq a\leq e^{i+1}, \quad e^j\leq b\leq e^{j+1},$$

with $nk \neq ab$.

If for a solution (n,k,a,b) we write $na = kb + tl$ with an integer t, then we see that

$$1\leq |t| \leq l^{-1}\max\{na, kb\} \leq l^{-1}Q\max\{e^{i+1}, e^{j+1}\} = e^{j+1}Q/l.$$

Thus, there are $O(e^jQ/l)$ possible values for t. Clearly there are at most e^{i+1} possible values for a and Q possible values n. Thus the product $kb = na - tl$ can take at most $e^{i+j+2}Q^2/l$ possible values. Therefore, we see from the bound on the divisor function $2^{\omega(l)} \leq \tau(l) = l^\epsilon$ ($\tau(l)$ is the number of positive integer divisors of l) that when n, a and t are fixe then k and b can take at most l^ϵ possible values. Hence

$$T_{ij} \ll e^{i+j}Q^2l^{-1+\epsilon}.$$

That is,

$$E_2 \ll \sum_{l|Nd} \mu\left(\frac{Nd}{l}\right)\phi(l)Q^2 q^\epsilon l^{-1+\epsilon} \ll Q^2(qNd)^\epsilon.$$

Now, we estimate E_1. Let $M = [\frac{\log T}{\log p}]$, $M_i = [\frac{\log T}{\log p_i}]$, $i = 1, 2$.

$$E_1 = 2J(Nd)\sum_{s=1}^{M}\sum_{t=1}^{s}\sideset{}{'}\sum_{d\le\frac{Q}{p^{s-t}}}\sideset{}{'}\sum_{1<p\le T}\frac{\log^2 p}{p^{s+t}} - J(Nd)\sum_{s=1}^{M}\sideset{}{'}\sum_{d\le Q}\sideset{}{'}\sum_{1<p\le T}\frac{\log^2 p}{p^{2s}}$$

$$+ \quad J(Nd)\sum_{s=1}^{M_1}\sum_{t=1}^{M_2}\sideset{}{'}\sum_{d\le\min\{\frac{Q}{p_1^s},\frac{Q}{p_2^t}\}}\sideset{}{'}\sum_{1<p_1\le T}\sideset{}{'}\sum_{1<p_2\le T}\frac{\log p_1\log p_2}{p_1^s p_2^t}. \qquad (9)$$

Note that $\sideset{}{'}\sum_{1\le u\le N} 1 = \frac{\phi(q)}{q}N + O\left(2^{\omega(q)}\right)$, then from Lemma 2 we have the first part in (9)

$$\begin{aligned}
&= 2J(Nd)\sum_{s=1}^{M}\sum_{t=1}^{s}\sideset{}{'}\sum_{1<p\le T}\frac{\log^2 p}{p^{s+t}}\left(\frac{\phi(q)}{q}\cdot\frac{Q}{p^{s-t}} + O(2^{\omega(q)})\right)\\
&= 2\frac{\phi(q)J(Nd)}{q}Q\sum_{s=1}^{M}\sum_{t=1}^{s}\sideset{}{'}\sum_{1<p\le T}\frac{\log^2 p}{p^{2s}} + O(2^{\omega(q)}J(Nd)q^\epsilon))\\
&= 2\frac{\phi(q)J(Nd)}{q}Q\sideset{}{'}\sum_{1<p\le T}\log^2 p\left(\frac{1}{4}\sum_{s=1}^{M}\frac{2s}{p^{2s}} + \frac{1}{8}\sum_{s=1}^{M}\frac{(2s)^2}{p^{2s}}\right) + O(2^{\omega(q)}J(Nd)q^\epsilon)\\
&= \frac{\phi(q)J(Nd)}{q}QC_1(q) + O(2^{\omega(q)}J(Nd)q^\epsilon).
\end{aligned}$$

Similarly, we can estimate the other two parts in (9)

$$J(Nd)\sum_{s=1}^{M}\sideset{}{'}\sum_{d\le Q}\sideset{}{'}\sum_{1<p\le T}\frac{\log^2 p}{p^{2s}} = \frac{\phi(q)J(Nd)}{q}QC_2(q) + O(2^{\omega(q)}J(Nd)q^\epsilon)$$

and

$$\begin{aligned}
&J(Nd)\sum_{s=1}^{M_1}\sum_{t=1}^{M_2}\sideset{}{'}\sum_{d\le\min\{\frac{Q}{p_1^s},\frac{Q}{p_2^t}\}}\sideset{}{'}\sum_{1<p_1\le T}\sideset{}{'}\sum_{1<p_2\le T}\frac{\log p_1\log p_2}{p_1^s p_2^t}\\
&= 4\frac{\phi(q)J(Nd)}{q}QC_3(q) + O(2^{\omega(q)}J(Nd)q^\epsilon),
\end{aligned}$$

where

$$C_1(q) = \sum_p{}' \log^2 p \frac{6p^{2M+1} - 8p^{2M} + 3p^{2M-1} - 2Mp^2 + (2M-2)p + 2}{4p^{2M}(p-1)^3},$$

$$C_2(q) = \sum_p{}' \log^2 p \frac{p^{2M} - 1}{p^{2M}(p^2-1)}$$

and

$$C_3(q) = \sum_{p_1<p_2}{}' \log p_1 \log p_2 \frac{(p_1^{2M_1} - 1)(p_2^{2M_2} - 1)}{p_1^{2M_1} p_2^{2M_2} (p_1^2 - 1)(p_2^2 - 1)}.$$

So

$$E_1 = \frac{\phi(q) J(Nd)}{q} QC + O(2^{\omega(q)} J(Nd) q^{\epsilon}).$$

Substituting the estimates of E_1 and E_2 in (8), we get the conclusion.

3. Proof of the Theorem

In this section, we shall complete the proof of the theorem. Let $q = p_1 p_2 \cdots p_k p_{k+1}^{\alpha_{k+1}} \cdots p_r^{\alpha_r}$, $M = p_1 p_2 \cdots p_k$, $N = p_{k+1}^{\alpha_{k+1}} \cdots p_r^{\alpha_r}$, $(M, N) = 1$

With the above lemmas and the orthogonality relation for Character sums modulo q we have

$$\begin{aligned}
&\sum_{\substack{\chi \bmod q \\ \chi \neq \chi_0}} |\tau(\chi)|^m \cdot \left| \sum_{n \le Q} \chi(n) \right|^2 \cdot \left| \frac{L'}{L}(1, \chi) \right|^2 \\
= &\sum_{\chi_1 \bmod M} \sum_{\substack{\chi_2 \bmod N \\ \chi_1\chi_2 \neq \chi_q^0}} |\tau(\chi_1\chi_2)|^m \cdot \left| \sum_{n \le Q} \chi_1\chi_2(n) \right|^2 \cdot \left| \frac{L'}{L}(1, \chi_1\chi_2) \right|^2 \\
= &\sum_{\chi_1 \bmod M} \sideset{}{^*}\sum_{\chi_2 \bmod N} |\tau(\chi_1)|^m N^{\frac{m}{2}} \cdot \left| \sum_{n \le Q} \chi_1\chi_2(n) \right|^2 \cdot \left| \frac{L'}{L}(1, \chi_1\chi_2) \right|^2
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{d|M} d^{\frac{m}{2}} N^{\frac{m}{2}} \sideset{}{^*}\sum_{\chi_1 \bmod d} \sideset{}{^*}\sum_{\chi_2 \bmod N} \left|\sum_{n\le Q} \chi_1\chi_2\chi_M^0(n)\right|^2 \cdot \left|\frac{L'}{L}(1,\chi_1\chi_2\chi_M^0)\right|^2 \\
&= \sum_{d|M} d^{\frac{m}{2}} N^{\frac{m}{2}} \sideset{}{^*}\sum_{\chi \bmod Nd} \left|\sum_{n\le Q} \chi\chi_M^0(n)\right|^2 \cdot \left|\frac{L'}{L}(1,\chi\chi_M^0)\right|^2 \\
&= \sum_{d|M} d^{\frac{m}{2}} N^{\frac{m}{2}} \sum_{n\le Q}\sum_{k\le Q} \sideset{}{^*}\sum_{\chi \bmod Nd} \chi\chi_M^0(n\bar{k}) \cdot \left|\frac{L'}{L}(1,\chi\chi_M^0)\right|^2 \\
&= \frac{\phi(q)\phi^2(N)}{q} Q N^{\frac{m}{2}-1} C \prod_{p|M}\left(p^{\frac{m}{2}+1} - 2p^{\frac{m}{2}+1} + 1\right) \\
&+ O\left(2^{\omega(q)} q^{\frac{m}{2}+\epsilon}\right) + O\left(Q^2 q^{\frac{m}{2}+\epsilon}\right),
\end{aligned}
$$

here we used

$$
\sum_{d|M} d^{\frac{m}{2}} J(d) = \prod_{p|M}\sum_{j=0}^{1} (p^j)^{\frac{m}{2}} J(p^j) = \prod_{p|M}\left(p^{\frac{m}{2}+1} - 2p^{\frac{m}{2}} + 1\right).
$$

This completes the proof of Theorem.

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