Indian J. Pure Appl. Math., **41**(4): 595-606, August 2010

ESTIMATES ON CONJECTURES OF MINKOWSKI AND WOODS

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(*Received* 9 *April* 2010; *accepted* 14 *May* 2010)

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space. Let $\wedge$ be a lattice of determinant 1 such that there is a sphere $|X| < R$ which contains no point of $\wedge$ other than the origin O and has n linearly independent points of $\wedge$ on its boundary. A well known conjecture in the geometry of numbers asserts that any closed sphere in $\mathbb{R}^n$ of radius $\sqrt{n/4}$ contains a point of $\wedge$. This is known to be true for $n \leq 8$. Here we give estimates on a more general conjecture of Woods for $n \geq 9$. This leads to an improvement for $9 \leq n \leq 22$ on estimates of Il'in (1991) to the long standing conjecture of Minkowski on product of n non-homogeneous linear forms.

Key words : Lattice, covering, non-homogeneous, product of linear forms, critical determinant, Korkine and Zolotareff reduction, Hermite's constant, centre density.

[1]The author acknowledges the support of Indian National Science Academy as INSA Senior Scientist during the period of preparation of this paper.

1. Introduction

Let $L_i = a_{i1}x_1 + \cdots + a_{in}x_n$, $1 \le i \le n$, be n real linear forms in n variables $x_1, \cdots, x_n$ and determinant $\Delta = \det(a_{ij}) \ne 0$. Let $c_1, \cdots, c_n$ be real numbers. Minkowski is believed to have made the following conjecture.

Conjecture I.(Minkowski) There exist integers $u_1, \cdots, u_n$ such that

$$\prod_{i=1}^{n} |(L_i(u_1, \cdots, u_n) + c_i)| \le \frac{1}{2^n}|\Delta|.$$

Minkowski [24] proved it for $n = 2$. Since then several authors have given a variety of proofs for the 2-dimensional case. For $n = 3$, the conjecture was first proved by Remak [29] . A simpler proof was given by Davenport [11] on the same lines. (Different proofs were given by Birch and Swinnerton-Dyer [5] and Narzullaev [27].) For $n = 4$, the conjecture was proved by Dyson [13], Bambah and Woods [2] and Skubenko [30]. Following this approach which has come to be known as Remak-Davenport approach, Minkowski's Conjecture has been further proved for $5 \le n \le 8$ by the efforts of various mathematicians : Skubenko [31], Bambah and Woods [3], McMullen [23], Woods [35, 36] and authors [18, 19]. For a detailed history of Minkowski's conjecture and related results, see Gruber [16], Gruber and Lekkerkerker [15] and Bambah *et al.* [4].

Define

$$M_n = \mathrm{Sup}_{L_1,\cdots,L_n}\mathrm{Sup}_{(c_1,\cdots,c_n)\in\mathbb{R}^n}\mathrm{Inf}_{(u_1,\cdots,u_n)\in\mathbb{Z}^n}\prod_{1\le i\le n}|(L_i(u_1,\cdots,u_n)+c_i)|.$$

Minkowski's Conjecture is equivalent to saying that

$$M_n \le \frac{1}{2^n}|\Delta|. \tag{1}$$

Chebotarev [8] proved the weaker inequality

$$M_n \le \frac{1}{2^{n/2}}|\Delta|.$$

Since then several authors have tried to improve upon this estimate. The bounds have been obtained in the form

$$M_n \le \frac{1}{\nu_n 2^{n/2}}|\Delta|,$$

where $\nu_n > 1$. Clearly $\nu_n \leq 2^{n/2}$ by considering the linear forms $L_i = x_i$ and $c_i = \frac{1}{2}$ for $1 \leq i \leq n$. Davenport [12], Woods [33], Bombieri [7], Mordell [25], Gruber [14], obtained results with $\nu_n \sim c(2e-1)$ as $n \to \infty$, with $c \leq 3.0001$. Skubenko [31] obtained ν_n, where $\nu_n \sim e^{-2}n^{1/3}(log\ n)^{-2/3}$ as $n \to \infty$. Narazullaev [28] and Andrijasjan, Il'in and Malyshev [1] obtained improvements for large n (for more details see Gruber and Lekkerkerker [15]). Mukhsinov [26] obtained improvements for small values of n. Il'in [20] improved the estimates for $6 \leq n \leq 17$. In 1991, Il'in [21] obtained further improvements for $6 \leq n \leq 31$. Since Minkowski's Conjecture is now proved for $n \leq 8$, we shall consider $n \geq 9$ and get improvements on results of Il'in for $n \leq 22$. (Refining our method, we expect to obtain further improvements for $9 \leq n \leq 31$.) The results of Il'in and our results are given in Table I.

Table 1

	Estimates of Il'in		Our Estimates
n	ν_n		ν_n
9	3.3151283		19.3967939
10	3.4798928		22.6239298
11	3.5229055		25.3402874
12	3.5502417		27.2130945
13	3.5785628		27.9834142
14	3.6020935		27.5240464
15	3.6111553		25.871948
16	3.6190753		23.2241420
17	3.6392444		19.8972896
18	3.6617581		16.2628000
19	3.6673429		12.6763203
20	3.6723611		9.4205568
21	3.6769169		6.6737319
22	3.684080		4.5063277

We shall follow the Remak-Davenport approach. If $L_i = a_{i1}x_1 + \cdots + a_{in}x_n$, $1 \leq i \leq n$, are n real linear forms of determinant $\Delta = \det(a_{ij}) \neq 0$, then the associated lattice

$$\wedge = \{(L_1(u_1, \cdots, u_n), \cdots, L_n(u_1, \cdots, u_n)) \ : \ (u_1, \cdots, u_n) \in \mathbb{Z}^n\}$$

is of determinant $|\Delta|$. Clearly we can state Minkowski's Conjecture in the terminology of lattices as :

Any lattice $\wedge$ of determinant $d(\wedge)$ in $\mathbb{R}^n$ is a covering lattice for the set

$$S : |x_1 x_2 \dots x_n| \leq \frac{d(\wedge)}{2^n}.$$

Define the homogeneous minimum of $\wedge$ as

$$M_H(\wedge) \ = \ \text{Inf}\{|x_1 x_2 \dots x_n| : \ X = (x_1, x_2, \cdots, x_n) \in \wedge, X \neq O\}.$$

Birch and Swinnerton-Dyer [5] proved.

Proposition 1 — Suppose that Minkowski Conjecture has been proved for dimensions $1, 2, \cdots, n-1$. Then it holds for all lattices $\wedge$ in $\mathbb{R}^n$ with $M_H(\wedge) = 0$.

McMullen [23] proved

Proposition 2 — If $\wedge$ is a lattice in $\mathbb{R}^n$ with $M_H(\wedge) \neq 0$ then there exists an ellipsoid having n linearly independent points of $\wedge$ on its boundary and no point of $\wedge$ other than O in its interior.

It is well known that using these results Minkowski's Conjecture would follow from :

Conjecture II — If $\wedge$ is a lattice in $\mathbb{R}^n$ of determinant 1 and there is a sphere $|X| < R$ which contains no point of $\wedge$ other than O in its interior and has n linearly independent points of $\wedge$ on its boundary then $\wedge$ is a covering lattice for the closed sphere of radius $\sqrt{n/4}$. Equivalently, every closed sphere of radius $\sqrt{n/4}$ lying in $\mathbb{R}^n$ contains a point of $\wedge$.

Woods [34, 35, 36] proved this conjecture for $n = 4, 5$ and 6 using Korkine and Zolotareff reduction. Korkine and Zolotareff [22] proved that a cartesian co-ordinate system can be chosen in $\mathbb{R}^n$ in such a way that $\wedge$ has a basis of the form

$$(A_1, 0, \cdots, 0), (a_{21}, A_2, 0, \cdots, 0), \cdots, (a_{n-1,1}, \cdots, A_{n-1}, 0),$$

$$(a_{n1}, \cdots, a_{n,n-1}, A_n),$$

where $A_1, A_2, \cdots, A_n$ are all positive. Further for each i, $1 \leq i \leq n$, the lattice $\wedge_i$ in $\mathbb{R}^{n-i+1}$ generated by

$$(A_i, 0, \cdots, 0), (a_{i+1,i}, A_{i+1}, 0, \cdots, 0), \cdots (a_{ni}, a_{n,i+1}, \cdots, A_n)$$

has minimum A_i i.e.

$$A_i = \text{Inf}\{|P| : \; P \in \wedge_i, \; P \neq O\}.$$

We shall call such a basis a reduced basis of $\wedge$. Woods [35] made the following Conjecture:

Conjecture III (Woods) — If $d(\wedge) = A_1 \ldots A_n = 1$ and $A_i \leq A_1$ for $i = 2, \cdots, n$, then any closed sphere in $\mathbb{R}^n$ of radius $\sqrt{n}/2$ contains a point of $\wedge$.

Woods [36] showed that Conjecture III implies Conjecture II. He [34, 35, 36] proved Conjecture III for $n = 4, 5$ and 6. Hans-Gill *et al.* [17] gave a unified simpler proof for $n \leq 6$. Authors [18, 19] proved it for $n = 7$ and 8. In this paper we shall obtain estimates for Woods Conjecture for $n \geq 9$ (see Theorem 1). Estimates on Minkowski's conjecture follow from this (see Theorem 4).

2. Statements of Results and their Proofs

Let $\Delta(S_n)$ denote the critical determinant of the unit sphere S_n with centre O in $\mathbb{R}^n$ i.e.

$$\Delta(S_n) = \text{Inf}\{d(\wedge) : \quad \wedge \text{ has no non-zero point of } S_n\}.$$

Let γ_n be the Hermite's constant i.e. γ_n is the smallest real number such that for any positive definite quadratic form Q in n variables of determinant D, there exist integers $u_1, u_2, \cdots, u_n$ satisfying

$$Q(u_1, u_2, \cdots, u_n) \leq \gamma_n D^{1/n}.$$

It is well known that $\Delta^2(S_n) = \gamma_n^{-n}$. Let $\mathbb{L}$ be a lattice in $\mathbb{R}^n$ reduced in the sense of Korkine and Zolotareff. Let $A_1, A_2, \cdots, A_n$ be as defined in Section 1. We state below some preliminary lemmas. Write $B_i = A_i^2$. Lemma 1 is due to Korkine and Zolotareff [22] and Lemma 2 is due to Woods [34]. In Lemma 3, the cases $n = 2$ and 3 are classical results of Lagrange and Gauss; $n = 4$ and 5 are due to Korkine and Zolotareff [22] while $n = 6, 7$ and 8 are due to Blichfeldt [6].

Lemma 1 — For all relevant i, $\quad B_{i+1} \geq \frac{3}{4} B_i$ and $B_{i+2} \geq \frac{2}{3} B_i$.

Lemma 2 — For a fixed integer i with $1 \leq i \leq n-1$, denote by $\mathbb{L}_1$ the lattice in $\mathbb{R}^i$ with the reduced basis

$$(A_1, 0, \cdots, 0), (a_{2,1}, A_2, 0, \cdots, 0), \cdots, (a_{i,1}, a_{i,2}, \cdots, a_{i,i-1}, A_i)$$

and denote by $\mathbb{L}_2$ the lattice in $\mathbb{R}^{n-i}$ with the reduced basis

$$(A_{i+1}, 0, \cdots, 0), (a_{i+2,i+1}, A_{i+2}, 0, \cdots, 0), \cdots, (a_{n,i+1}, a_{n,i+2}, \cdots, a_{n,n-1}, A_n).$$

If any sphere in $\mathbb{R}^i$ of radius r_1 contains a point of $\mathbb{L}_1$ and if any sphere in $\mathbb{R}^{n-i}$ of radius r_2 contains a point of $\mathbb{L}_2$ then any sphere in $\mathbb{R}^n$ of radius $(r_1^2 + r_2^2)^{1/2}$ contains a point of $\mathbb{L}$.

Lemma 3 — $\Delta(S_n) = 1/\sqrt{2},\ 1/2,\ 1/2\sqrt{2},\ \sqrt{3}/8,\ 1/8$ and $1/16$ for $n = 3,$ 4, 5, 6, 7 and 8 respectively.

***Theorem* 1** — *Let $\mathbb{L}$ be a lattice in $\mathbb{R}^n$ with determinant $A_1 A_2 \ldots A_n = 1$ and $A_i \leq A_1$ for $i = 2, \cdots, n$. Let $0 < \ell_n \leq A_n^2 = B_n \leq m_n$, where ℓ_n and m_n are real numbers. Then $\mathbb{L}$ is a covering lattice for the sphere $\mid X \mid \leq \sqrt{\omega_n}/2$, where*

$$\omega_n = n \quad \text{for} \quad 1 \leq n \leq 8$$

and for $n \geq 9$ it is defined inductively by

$$\omega_n = \max\,\{\omega_{n-1}\ell_n^{-1/(n-1)} + \ell_n,\ \ \omega_{n-1}m_n^{-1/(n-1)} + m_n\}. \tag{2}$$

PROOF : As stated earlier the result is already known for $n \leq 8$. For $n \geq 9$ we apply induction on n and use Lemma 2 with $i = n-1$. Let $\mathbb{L}_1$ be the lattice in $\mathbb{R}^{n-1}$ with the reduced basis $(A_1, 0, \cdots, 0), (a_{2,1}, A_2, 0, \cdots, 0), \cdots,$ $(a_{n-1,1}, a_{n-1,2}, \cdots, a_{n-1,n-2}, A_{n-1})$ of determinant $A_1 A_2 \ldots A_{n-1} = \frac{1}{A_n}$. So $A_n^{1/n-1}\mathbb{L}_1$ is a lattice of determinant 1. The corresponding basis is reduced and $A_n^{1/n-1}A_i \leq A_n^{1/n-1}A_1$ for $i = 1, \cdots, n-1$. Therefore by induction hypothesis any closed sphere in $\mathbb{R}^{n-1}$ of radius $\sqrt{\omega_{n-1}}/2$ contains a point of $A_n^{1/n-1}\mathbb{L}_1$. Hence any closed sphere in $\mathbb{R}^{n-1}$ of radius $r_1 = A_n^{-1/n-1}\sqrt{\omega_{n-1}}/2$ contains a point of $\mathbb{L}_1$. Also any closed sphere in $\mathbb{R}^1$ of radius $r_2 = A_n/2$ contains a point of the 1-dimensional lattice $\mathbb{L}_2$ with basis (A_n). Therefore by Lemma 2, any closed sphere in $\mathbb{R}^n$ of radius $R = (r_1^2 + r_2^2)^{1/2}$ contains a point of $\mathbb{L}$ where

$$\begin{aligned} 4R^2 &= \omega_{n-1}A_n^{-2/(n-1)} + A_n^2 \\ &= \omega_{n-1}B_n^{-1/(n-1)} + B_n \\ &= f(B_n) \quad \text{say.} \end{aligned}$$

We notice that the second derivative $f''(B_n) > 0$ and so the maximum of $f(B_n)$ on the interval $[\ell_n,\ m_n]$ is attained at one of the end points. Therefore (2) follows.

***Theorem* 2** —

$$\{9^{\frac{1}{5}}\gamma_n^{\frac{1}{n-1}}\gamma_{n-1}^{\frac{1}{n-2}}\dots\gamma_6^{\frac{1}{5}}\}^{-1} \le B_n \le \gamma_{n-1}^{\frac{n-1}{n}}. \tag{3}$$

We first prove

Lemma 4 — For any integer $s,\ 1 \le s \le n-1$

$$(B_1B_2\dots B_s)^{\frac{1}{n-s}} \le \gamma_n^{\frac{1}{n-1}}\gamma_{n-1}^{\frac{1}{n-2}}\dots\gamma_{n-s+1}^{\frac{1}{n-s}}. \tag{4}$$

Proof : Proof is by induction on s. For $s=1$, inequality (4) reduces to $B_1 \le \gamma_n$ which is true by definition of γ_n.

Let $2 \le s \le n-1$. Suppose the result (4) holds for $s-1$. Since the lattice generated by $(A_s, 0, 0, \cdots, 0)$, $(a_{s+1,s}, A_{s+1}, 0, \cdots, 0), \cdots, (a_{n,s}, \cdots, a_{n,n-1}, A_n)$ in $\mathbb{R}^{n-s+1}$ has no point other than the origin in the interior of the sphere $|X| \le A_s$, it follows that $\Delta(A_sS_{n-s+1}) \le A_sA_{s+1}\dots A_n = \frac{1}{A_1A_2\dots A_{s-1}}$ which gives

$$B_1B_2\dots B_{s-1}B_s^{n-s+1} \le \frac{1}{\Delta^2(S_{n-s+1})} = \gamma_{n-s+1}^{n-s+1}.$$

i.e.

$$B_s \le \frac{\gamma_{n-s+1}}{(B_1B_2\dots B_{s-1})^{\frac{1}{n-s+1}}}.$$

Multiply both sides by $B_1B_2\dots B_{s-1}$ we have

$$\begin{aligned} B_1B_2\dots B_{s-1}B_s &\le \gamma_{n-s+1}(B_1B_2\dots B_{s-1})^{\frac{n-s}{n-s+1}} \\ &\le \gamma_{n-s+1}(\gamma_n^{\frac{1}{n-1}}\gamma_{n-1}^{\frac{1}{n-2}}\dots\gamma_{n-s}^{\frac{1}{n-s-1}})^{n-s} \end{aligned}$$

by induction hypothesis. This proves (4) for s.

Proof of Theorem 2 : To obtain lower bounds on B_n, we notice that, by Lemma 1

$$B_n^5 \ge B_n \times \frac{3}{4}B_{n-1} \times \frac{2}{3}B_{n-2} \times \frac{1}{2}B_{n-3} \times \frac{4}{9}B_{n-4} = \frac{1}{9B_1B_2\dots B_{n-5}}.$$

Therefore using Lemma 4 with $s = n-5$, we get

$$B_n \geq \frac{1}{9^{\frac{1}{5}}(B_1 B_2 \ldots B_{n-5})^{\frac{1}{5}}} \geq \frac{1}{9^{\frac{1}{5}}\gamma_n^{\frac{1}{n-1}}\gamma_{n-1}^{\frac{1}{n-2}}\cdots\gamma_6^{\frac{1}{5}}}.$$

For the upper bound on B_n, we notice that the lattice generated by $(A_1, 0, 0, \cdots, 0)$, $(a_{2,1}, A_2, 0, \cdots, 0), \cdots, (a_{n-1,1}, a_{n-1,2}, \cdots, a_{n-1,n-2}, A_{n-1})$ in $\mathbb{R}^{n-1}$ has no point in the interior of the sphere with radius A_1 centered at the origin, so $\Delta(A_1 S_{n-1}) \leq A_1 A_2 \ldots A_{n-1} = \frac{1}{A_n}$ which gives $B_n B_1^{n-1} \leq \frac{1}{\Delta^2(S_{n-1})} = \gamma_{n-1}^{n-1}$. Since $B_n \leq B_1$, we get $B_n^n \leq \gamma_{n-1}^{n-1}$.

This proves Theorem 2.

Remark 1 : Let

δ_n = the best centre density of packings of unit spheres in $\mathbb{R}^n$,
$\delta_n^\star$ = the best centre density of lattice packings of unit spheres in $\mathbb{R}^n$.

Then it is known that, see [10, page 20]

$$\gamma_n = 4(\delta_n^\star)^{\frac{2}{n}} \leq 4(\delta_n)^{\frac{2}{n}}. \tag{5}$$

$\delta_n^\star$ and hence γ_n is known for $2 \leq n \leq 8$. For general n one knows that

$$\gamma_n^n \leq 2^{n-2}\left(\frac{n+2}{V(S_n)}\right)^2,$$

where $V(S_n)$ denotes the volume of the unit sphere in $\mathbb{R}^n$ (see [15]).

For $9 \leq n \leq 22$, we use the bounds on δ_n given by Cohn and Elkies [9] which we list in Table 2. There we also list, for $9 \leq n \leq 22$, bounds on γ_n using inequality (5), lower bound ℓ_n and upper bound m_n of B_n using Theorem 2 and finally the estimates ω_n on Woods Conjecture using Theorem 1. (For these values of n, ℓ_n and m_n we find that $\omega_n = \omega_{n-1}\ell_n^{-1/(n-1)} + \ell_n$ i.e. maximum of $f(B_n)$ occurs at the lower end point ℓ_n.)

To deduce the results on the estimates of Minkowski's conjecture we need the following modification of Proposition 1.

Theorem 3 — *Suppose that we know*

$$M_j \leq \frac{1}{\nu_j 2^{j/2}}|\triangle| \qquad \text{for } \ 1 \leq j \leq n-1.$$

Let $\nu = \min \nu_{k_1}\nu_{k_2}\dots\nu_{k_s}$*, where the minimum is taken over all* $(k_1, k_2, \cdots, k_s)$ *such that* $n = k_1 + k_2 + \cdots + k_s$, k_i *positive integers for all* i *and* $s \geq 2$*. Then for all lattices* $\wedge$ *in* $\mathbb{R}^n$ *with* $M_H(\wedge) = 0$*, the estimate* ν *holds for Minkowski's Conjecture.*

To prove this result one can follow the line of proof of Proposition 1, which is Lemma 7 in the paper of Birch and Swinnerton-Dyer [5] . We shall omit the details which are straight forward.

Table 2

n	δ_n	γ_n	ℓ_n	m_n	ω_n
9	0.05900	2.1326324	0.4342366	1.8517495	9.3134437
10	0.05804	2.2636302	0.3965551	1.9770808	10.7180650
11	0.05932	2.3933470	0.3634151	2.1016019	12.2231803
12	0.06279	2.5217871	0.3341058	2.2254707	13.8382632
13	0.06870	2.6492947	0.3080514	2.3485931	15.5730343
14	0.07750	2.7758041	0.2847845	2.4711931	17.4375252
15	0.08999	2.9014777	0.2639199	2.5931615	19.4421387
16	0.10738	3.0263937	0.2451379	2.7145981	21.5977007
17	0.13150	3.1506793	0.2281708	2.8355395	23.9155092
18	0.16503	3.2743307	0.2127937	2.9560725	26.4073756
19	0.21202	3.3974439	0.1988155	3.0761736	29.0856684
20	0.27855	3.5200620	0.1860734	3.1959111	31.9633532
21	0.37389	3.6422432	0.1744278	3.3153098	35.0540321
22	0.51231	3.7640371	0.1637584	3.4344103	38.3719829

Since by arithmetic-geometric inequality the sphere $\{X \in \mathbb{R}^n : \mid X \mid \leq \frac{\sqrt{\omega_n}}{2}\}$ is a subset of $\{X : |x_1 x_2 ... x_n| \leq \frac{1}{2^{n/2}}(\frac{\omega_n}{2n})^{n/2}\}$, Theorems 1, 2, 3 and Proposition 2 immediately imply

***Theorem* 4** — *The values of* ν_n *for the estimates of Minkowski's Conjecture can be taken as* $(\frac{2n}{\omega_n})^{n/2}$*, where* ω_n *is as listed in Table 2.*

These values are listed in Table 1.

Acknowledgement

Authors are grateful to Professor R. P. Bambah for various discussions throughout the preparation of this paper.

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