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$(Z_2)^k$-ACTIONS WITH FIXED POINT SET OF CONSTANT CODIMENSION $2^k + 8$[1]

Jingyan Li* and Yanying Wang**

**Department of Mathematics and Physics, Shijiazhuang Railway Institute, Shijiazhuang*, 050043, *Peoples' Republic of China*
e-mail : yanjinglee@163.com
***College of Mathematics and Information Science*,
Hebei Normal University, Shijiazhuang, 050016, *Peoples' Republic of China*
e-mail : wyanying2003@yahoo.com.cn

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The ideal $J_{*,k}^{2^k+8}$ of cobordism classes in the unoriented cobordism ring MO_* containing a representative admitting a $(Z_2)^k$-action with fixed point set of constant codimension $2^k + 8$ is determined for $k \geq 4$.

Key words : $(Z_2)^k$-action, indecomposable cobordism class, fixed point set, constant codimension, Dold manifold.

1. Introduction

Let $\phi : (Z_2)^k \times M^n \to M^n$ be a smooth action of the group $(Z_2)^k = \{T_1, T_2, \cdots, T_k | \; T_i^2 = 1, T_iT_j = T_jT_i\}$ on a closed n-dimensional manifold. The fixed point set F of the action is a disjoint union of closed submanifolds

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of M^n. If each component of F is $(n-r)$-dimensional, then F has constant codimension r. Let $J^r_{n,k}$ denote the set of n-dimensional cobordism classes containing a representative M^n admitting a $(Z_2)^k$-action with fixed point set of constant codimension r and $J^r_{*,k} = \sum_{n\geq r} J^r_{n,k}$. From [1] we know that $J^r_{n,k}$ is a subgroup in the unoriented cobordism group MO_n and $J^r_{*,k}$ forms an ideal of the cobordism ring $MO_* = \sum_{n\geq 0} MO_n$. It is not difficult to see that $J^r_{*,k} \subset J^r_{*,k+1}$ and $J^{r_1}_{*,k} J^{r_2}_{*,k} \subset J^{r_1+r_2}_{*,k}$.

In 1965, Conner and Floyd showed that $J^1_{*,1} = (0)$ in [2]. Inspired in this fact, Stong introduced the question of determining the ideal $J^r_{*,1}$ and computed $J^2_{*,1}$ in [3]. Soon after, Capobianco computed $J^3_{*,1}$ in [4] and $J^4_{*,1}$ in [5]. After this, Iwata, Wada, Wu and Kikuchi obtained $J^5_{*,1}, J^6_{*,1}, J^7_{*,1}$ and $J^8_{*,1}$ respectively in [6-9]. In 1992, Pergher introduced the same question of determining $J^r_{*,k}$ with $k \geq 2$ in [10]. Also, he computed $J^1_{*,k}$ and $J^2_{*,k}$ for $k \geq 2$, and made a lot of characteristic numbers calculations to determine certain sub-ideals of $J^3_{*,2}$ defined in terms of certain restrictions on the fixed data of the actions. In 1994, Shaker got the most important advance in the direction of computing $J^r_{*,k}$ in [11]. The crucial point was the discovery of a method for constructing models of $(Z_2)^k-$actions on certain manifolds so that one can control two involved properties: the indecomposability of the manifold and the codimension of the fixed point set of the action, thus providing a method to construct generators for the unoriented cobordism ring with desired properties. Using this method, Shaker computed $J^r_{*,k}$ for $r < 2^k$, and concluded that

$$J^r_{*,k} = \begin{cases} (0), & r = 1, \\ \oplus_{n=r}^{\infty} MO_n, & \text{r } \textit{even}, \\ \oplus_{n=r}^{\infty} MO_n \cap Ker\chi, & \text{r } \textit{odd}, r \geq 3, \end{cases}$$

where $\chi : MO_* \to Z_2$ denotes the mod 2 Euler characteristic. Later, in [12], Shaker solved the case $r = 2^k$. The key point was a way to equip Dold manifolds with certain $(Z_2)^k-$actions and the fact that there is a method to recognize the indecomposability of Dold manifolds. These two articles of Shaker gave a clear scheme to attack the next cases. Using this scheme and the complicated formula of Kosniowski and Stong given in [13], Wang, Wu and Ma solved the case $r = 2^k + 1$ in [14], leaving open some particular cases. In [15], Wang, Wu and Ding gave a crucial contribution concerning the determination of the next cases and made an important advance in the setting of computing $J^r_{*,k}$ for $r > 2^k$. The authors showed that, for a given

$r > 2^k$, there exists a number $g(r)$ much bigger than r so that, if $n \geq g(r)$, then $J^r_{n,k}$ is explicitly computed. For $r > 2^k$, this completely solves the question of computing $J^r_{n,k}$ for all sufficiently large n. In other words, for $r > 2^k$, this reduces the computation of $J^r_{n,k}$ to a finite and explicit list of values of n. The authors also solved the case $r = 2^k + 3$ for $k \geq 4$ and $n \geq 2^k + 4$. In [16], Liu and Wu studied the case $r = 2^k + 2$. They used the generalized Dold manifold given in [17] to construct certain indecomposable manifolds equipped with $(Z_2)^k-$actions. This allowed to obtain, besides the case $r = 2^k + 2$, the solution for certain cases left previously open. For $r = 2^k + 4, 2^k + 5, 2^k + 6$ and $2^k + 7$, $J^r_{*,k}$ have been determined in [18-21] respectively.

In this paper, we will construct some special generators of MO_* to determine the ideal $J^{2^k+8}_{*,k}$. The main result is

Theorem — *For $k \geq 4$, the ideal $J^{2^k+8}_{*,k}$ of MO_* consists of all classes in dimensions greater than 2^k+8 and the decomposable classes in dimension $2^k + 8$ which contain only factors with dimension less than 2^k.*

Throughout this paper manifolds will be smooth, compact and without boundary but not necessarily connected. S^m denotes the m-dimensional sphere. The coefficient group is Z_2 (the integers mod 2) and $\equiv$ denotes congruence mod 2. Binomial coefficients are $\binom{m}{n} = m!/n!(m-n)!$.

2. Preliminaries

It is well known that the unoriented cobordism ring MO_* is a Z_2-polynomial algebra with a single generator in each dimension n which is not of the form $2^u - 1$ (see [1]). If the cobordism class $[M^n]$ of the smooth closed manifold M^n can be expressed as a sum of products of lower dimensional cobordism classes, then $[M^n]$ is called decomposable. Otherwise it is indecomposable. The indecomposable classes can be chosen as generators of the Z_2-polynomial algebra MO_*. For our purpose, the indecomposable classes will come from the following sources.

Lemma 2.1 — [3; Lemma 3.4] Let $RP(n_1, n_2, \cdots, n_l)$ be the projective space bundle of $\lambda_1 \oplus \lambda_2 \oplus \cdots \oplus \lambda_l$ over $RP(n_1) \times RP(n_2) \times \cdots \times RP(n_l)$, where λ_i is the pullback of the canonical line bundle over the i-th factor. Then for $l > 1$, $[RP(n_1, n_2, \cdots, n_l)]$ is indecomposable in MO_* if and only if

$$\binom{n+l-2}{n_1} + \binom{n+l-2}{n_2} + \cdots + \binom{n+l-2}{n_l} \equiv 1 \; mod \; 2,$$

where $n = n_1 + n_2 + \cdots + n_l$.

The manifold $RP(n_1, n_2, \cdots, n_l)$ has dimension $n + l - 1$. If $n_{i+1} = n_{i+2} = \cdots = n_l = 0$, then $RP(n_1, n_2, \cdots, n_l)$ will sometimes be written as $RP(n_1, n_2, \cdots, n_i; l)$.

In [17], Brown constructed the generalized Dold manifold. For a space X and a positive integer m, let $P(m, X)$ be formed from $S^m \times X \times X$ by identifying (u, x, y) with $(-u, y, x)$. If X is an n-dimensional manifold, then $P(m, X)$ is an $m + 2n$-dimensional manifold. He also gave

Lemma 2.2 — [17; Proposition 4.1] $[P(m, M^n)]$ is indecomposable in MO_* if and only if $[M^n]$ is indecomposable in MO_* and the binomial coefficient

$$\binom{m+n-1}{m-1} \equiv 1 \; mod \; 2.$$

To calculate binomial coefficients mod 2, we recall Kummer's result: If

$$m = \sum_{i=0}^{l} m_i 2^i \text{ and } n = \sum_{i=0}^{l} n_i 2^i$$

with $0 \leq m_i, n_i \leq 1$, then $\binom{m}{n} \equiv 1 \; mod \; 2$ if and only if $n_i \leq m_i$ for every i.

Lemma 2.3 — [11; Lemma 3.1] Let $\lambda_i \longrightarrow X_i$ be line bundles, and let $(Z_2)^{k_i} (k_i \geq 0)$ act on λ_i as bundle maps with fixed point set F_i on X_i for all $1 \leq i \leq l$ with $\sum_{i=1}^{l} 2^{k_i} \leq 2^k$. Then $RP(n_1, n_2, \cdots, n_l)$ over $X_1 \times X_2 \times \cdots \times X_l$ admits a $(Z_2)^k$-action with fixed point set $F_1 \times F_2 \times \cdots \times F_l \times E$ with E being a set of l points.

Lemma 2.4 — [13] If (M^n, ϕ) is a $(Z_2)^k$-action on an indecomposable n-manifold, then some component of the fixed point set of M is of dimension at least $[n/2^k]$.

Lemma 2.5 — [13] If (M^n, ϕ) is a $(Z_2)^k$-action on a manifold with

$$s_{(\lambda_1,\cdots,\lambda_i)}[M] \neq 0,$$

then some component of the fixed point set of M is of dimension at least

$$[\lambda_1/2^k] + \cdots + [\lambda_i/2^k].$$

3. Existence of Indecomposables

The main task of this section is to exhibit indecomposable classes in $J_{*,k}^{2^k+8}$.

Lemma 3.1 — There exist indecomposable classes $x_n \in J_{*,k}^{2^k+8}$ for $k \geq 4$, $n \geq 2^k + 9$ odd and not of the form $2^u - 1$.

Proof : Since n is not of the form $2^u - 1$, $\frac{n-1}{2}$ is not of the form $2^u - 1$. We have $\frac{n-1}{2} \geq \frac{2^k+9-1}{2} = 2^{k-1} + 4$. From [11; 5.1], there exists an indecomposable class $X \in J_{\frac{n-1}{2},k}^{2^{k-1}+4}$ (for $2^{k-1} + 4 < 2^k$), so that X has a representative M admitting a $(Z_2)^k$-action with fixed point set F', where $\dim(F') = \dfrac{n-1}{2} - (2^{k-1}+4) = \dfrac{n-2^k-9}{2}$. Let $T_i'(i = 1, 2, \cdots, k)$ denote the $(Z_2)^k$-action on M. Then we can define a $(Z_2)^k$-action on $S^1 \times M \times M$ as follows:

$$T_1(u,x,y) = (u, T_1'(x), T_1'(y)),$$

$$T_2(u,x,y) = (u, T_2'(x), T_2'(y)),$$

$$\cdots\cdots,$$

$$T_k(u,x,y) = (u, T_k'(x), T_k'(y)).$$

These involutions commute with the map $T(u,x,y) = (-u,y,x)$, so that induce a $(Z_2)^k$-action on $P(1,M)$ with fixed point set $F = S^1 \times F' \times F'/T = P(1,F')$, where $\dim F = n - 2^k - 8$. Taking $x_n = [P(1,M)]$, we have that $x_n \in J_{*,k}^{2^k+8}$. By Lemma 2.2, x_n is indecomposable. □

Lemma 3.2 — There exist indecomposable classes $x_n \in J_{*,k}^{2^k+8}$ for $k \geq 4$, $2^k + 10 \leq n \leq 2^{k+1}$ and n even.

Proof : We will consider the following cases.

Case 1 — $n = 2^k + 2^{r_1} + 2^{r_2} + \cdots + 2^{r_m} + 2, k > r_1 > r_2 > \cdots > r_m \geq 2$ and $m \geq 1$.

From [11; 5.1], there exists an indecomposable class $X \in J^{2^{k-1}+4}_{\frac{n-2}{2},k}$ (for $2^{k-1} + 4 < 2^k$), so that X has a representative M admitting a $(Z_2)^k$-action with fixed point set F', where $\dim(F') = \frac{n-2}{2} - (2^{k-1} + 4) = \frac{n-2^k-10}{2}$. Take $x_n = [P(2, M)]$, with the $(Z_2)^k$-action given in the proof of Lemma 3.1. The fixed point set of this action is $F = S^2 \times F' \times F'/T = P(2, F')$ with $\dim F = n - 2^k - 8$, then $x_n \in J^{2^k+8}_{*,k}$. By Lemma 2.2, x_n is indecomposable.

Case 2 — $n = 2^k + 2^{r_1} + 2^{r_2} + \cdots + 2^{r_m}, k > r_1 > r_2 > \cdots > r_m \geq 2$ and $m \geq 2$.

(1) If $m \geq 3$ or $m = 2$ and $r_1 \geq 5$, then $2^{k-1} - 2^{r_1-1} - \cdots - 2^{r_{m-1}-1} + 8 < 2^{k-1}$. From [11; 5.1], there exists an indecomposable class $X \in J^{2^{k-1}-2^{r_1-1}-\cdots-2^{r_{m-1}-1}+8}_{\frac{n-2^{r_m}}{2},k-1}$, so that X has a representative M admitting a $(Z_2)^{k-1}$-action with fixed point set F', where $\dim(F') = \frac{n-2^{r_m}}{2} - (2^{k-1} - 2^{r_1-1} - \cdots - 2^{r_{m-1}-1} + 8) = 2^{r_1} + 2^{r_2} + \cdots + 2^{r_{m-1}} - 8$. Let $T'_i (i = 1, 2, \cdots, k-1)$ denote the $(Z_2)^{k-1}$-action on M. Then we can define a $(Z_2)^k$-action on $S^{2^{r_m}} \times M \times M$ as follows:

$$T_1(u, x, y) = (u, y, x),$$

$$T_2(u, x, y) = (u, T'_1(x), T'_1(y)),$$

$$\cdots\cdots,$$

$$T_k(u, x, y) = (u, T'_{k-1}(x), T'_{k-1}(y)).$$

These involutions commute with the map $T(u, x, y) = (-u, y, x)$, so that induce a $(Z_2)^k$-action on $P(2^{r_m}, M)$ with fixed point set $F = S^{2^{r_m}} \times \Delta(F' \times F')/T$, where $\dim F = \dim(F') + 2^{r_m} = n - 2^k - 8$. Taking $x_n = [P(2^{r_m}, M)]$, we have $x_n \in J^{2^k+8}_{*,k}$. By Lemma 2.2, x_n is indecomposable.

(2) If $m = 2$ and $r_1 = 4$, we take $x_n = [RP(19, 2^{r_2} - 1; 2^k - 1)]$. From [15; Lemma 2.4], $x_n \in J^{2^k+8}_{*,k}$.

$$\binom{2^k + 2^4 + 2^{r_2} - 1}{2^4 + 2 + 1} + \binom{2^k + 2^4 + 2^{r_2} - 1}{2^{r_2} - 1} + 2^k - 3 \equiv 1 \ mod\ 2.$$

By Lemma 2.1, x_n is indecomposable.

(3) If $m = 2$ and $r_1 = 3$, we take $x_n = [RP(11, 11; 2^k - 9)]$. Let

$$T_1[x_1, x_2, \cdots, x_{12}] = [x_1, x_2, \cdots, x_6, -x_7, -x_8, \cdots, -x_{12}],$$

$$T_2[x_1, x_2, \cdots, x_{12}] = [x_1, x_2, x_3, -x_4, -x_5, -x_6, x_7, x_8, x_9, -x_{10}, -x_{11}, -x_{12}].$$

Then (T_1, T_2) defines a $(Z_2)^2$-action on $RP(11)$ with fixed point set being four copies of $RP(2)$. Let $(Z_2)^0$ act as the identity on the rest of the base. Since $2^2 + 2^2 + 2^k - 9 - 2 = 2^k - 3 < 2^k$, by Lemma 2.3, $x_n \in J_{n,k}^{2^k+8}$. By Lemma 2.1, x_n is indecomposable.

Case 3 — $n = 2^k + 2^{r_1}$ and $k > r_1 \geq 4$.

If $r_1 > 4$, we take $x_n = [RP(2n - 2^{k+1} - 17, 1, 1; 2^{k+1} - n + 16)]$. From [22; Lemma 2.3], $x_n \in J_{n,k}^{2^k+8}$.

$$\begin{aligned} & \binom{n-1}{2n - 2^{k+1} - 17} + 2\binom{n-1}{1} + 2^{k+1} - n + 16 - 3 \\ \equiv & \binom{2^k + 2^{r_1 - 1} + \cdots + 2 + 1}{2^{r_1} + \cdots + 2^5 + 2^3 + 2^2 + 2} + 1 \\ \equiv & \ 1 \ mod\ 2. \end{aligned}$$

By Lemma 2.1, x_n is indecomposable.

If $r_1 = 4$, we take $x_n = [RP(19, 4; 2^k - 6)]$. Let

$$T_1[x_1, x_2, \cdots, x_{20}] = [x_1, x_2, \cdots, x_{10}, -x_{11}, -x_{12}, \cdots, -x_{20}],$$

$$T_2[x_1, x_2, \cdots, x_{20}] = [x_1, \cdots, x_5, -x_6, \cdots, -x_{10}, x_{11}, \cdots, x_{15}, -x_{16}, \cdots, -x_{20}].$$

Then (T_1, T_2) defines a $(Z_2)^2$-action on $RP(19)$ with fixed point set being four copies of $RP(4)$. Let $(Z_2)^0$ act as the identity on the rest of the base. By Lemma 2.3, $x_n \in J_{n,k}^{2^k+8}$.

$$\binom{2^k + 2^4 - 1}{19} + \binom{2^k + 2^4 - 1}{4} + 2^k - 8 \equiv 1 \ mod\ 2.$$

By Lemma 2.1, x_n is indecomposable. □

Lemma 3.3 — There exist indecomposable classes $x_n \in J_{*,k}^{2^k+8}$ for $k \geq 4$, $2^{k+1} \leq n < 2^{k+2}$ and n even.

PROOF : We will consider the following cases.

Case 1 — $2^{k+1} \leq n \leq 2^{k+1} + 2^k$ and $k \geq 5$.

Let $x_n = [RP(2^k + 1, 1, n - 2^k - 2^{k-1} - 8; 2^{k-1} + 7)]$. From [22; Lemma 2.3], $x_n \in J_{n,k}^{2^k+4}$.

$$\binom{n-1}{2^k+1} + \binom{n-1}{1} + \binom{n-1}{n-2^k-2^{k-1}-8} + 2^{k-1} + 7 - 3$$
$$\equiv \binom{n-1}{2^k+1} + \binom{n-1}{2^k+2^{k-1}+7} + 1.$$

If $n = 2^{k+1}$,

$$\binom{2^{k+1}-1}{2^k+1} + \binom{2^{k+1}-1}{2^k+2^{k-1}+7} + 1 \equiv 1 + 1 + 1 \equiv 1 \; mod \; 2.$$

If $n = 2^{k+1} + 2^k$,

$$\binom{2^{k+1}+2^k-1}{2^k+1} + \binom{2^{k+1}+2^k-1}{2^k+2^{k-1}+7} + 1 \equiv 0 + 0 + 1 \equiv 1 \; mod \; 2.$$

If $n = 2^{k+1} + 2^{r_1} + \cdots + 2^{r_m}$ and $r_1 < 2^k$,

$$\binom{2^{k+1}+2^{r_1}+\cdots+2^{r_m}-1}{2^k+1} + \binom{2^{k+1}+2^{r_1}+\cdots+2^{r_m}-1}{2^k+2^{k-1}+7} + 1$$
$$\equiv 0 + 0 + 1 \equiv 1 \; mod \; 2.$$

By Lemma 2.1, x_n is indecomposable.

Case 2 — $2^{k+1} \leq n \leq 2^{k+1} + 2^k$ and $k = 4$.

If $n = 2^5$, we take $x_n = [RP(26; 7)]$. From [15; Lemma 2.5], $x_n \in J_{n,k}^{2^k+8}$. By Lemma 2.1, x_n is indecomposable.

If $2^5 < n \leq 2^5 + 2^4$, we take $x_n = [RP(11, 11, n - 34; 13)]$. From [22; Lemma 2.3], $x_n \in J_{n,k}^{2^k+8}$.

$$2\binom{n-1}{11} + \binom{n-1}{n-34} + 13 - 3 \equiv \binom{n-1}{2^5+1} \equiv 1 \; mod \; 2,$$

By Lemma 2.1, x_n is indecomposable.

Case 3 — $2^{k+1}+2^k < n < 2^{k+2}$ and $k \geq 4$.

Suppose $n = 2^{k+1}+2^{r_1}+2^{r_2}+\cdots+2^{r_s}, k = r_1 > r_2 > \cdots > r_s \geq 1, s \geq 2$. Write $S = \{i|$there exists some i in the 2-adic expansion of n such that $r_i > r_{i+1}+1\}$.

(1) If $S \neq \emptyset$, let $l = \min\{i \mid i \in S\}$ and $p = 2^{r_{l+1}} + 2^{r_{l+2}} + \cdots + 2^{r_s}$.

(i) If $p > 9$, then $k \geq 4$. Take $x_n = [RP(2p+1, 1, n-2^k-p-8; 2^k - p+7)]$. From [22; Lemma 2.3], $x_n \in J^{2^k+8}_{n,k}$.

$$
\begin{aligned}
&\binom{n-1}{2p+1} + \binom{n-1}{1} + \binom{n-1}{n-2^k-2p-8} + 2^k - p + 7 - 3 \\
\equiv\ & \binom{2^{k+1}+2^{r_1}+\cdots+2^{r_l}+2^{r_{l+1}}\cdots+2^{r_s}-1}{2^{r_{l+1}+1}+2^{r_{l+2}+1}+\cdots+2^{r_s+1}+1} + \\
& \binom{2^{k+1}+2^{r_1}+\cdots+2^{r_l}+2^{r_{l+1}}\cdots+2^{r_s}-1}{2^k+2^{r_1}+\cdots+2^{r_l}-1-7} + 1 \\
\equiv\ & 1 \ mod\ 2.
\end{aligned}
$$

By Lemma 2.1, x_n is indecomposable.

(ii) If $2 \leq p \leq 8$, the argument is divided into the following cases:

(a) $p = 8$ or $p = 4$.

Take $x_n = [RP(19, n-2^k-17; 2^k-1)]$. From [15; Lemma 2.4], $x_n \in J^{2^k+8}_{n,k}$. If $p = 8$, then $r_l \geq 5$.

$$
\begin{aligned}
&\binom{n-1}{19} + \binom{n-1}{n-2^k-17} + 2^k - 1 - 2 \\
\equiv\ & \binom{2^{k+1}+2^{r_1}+\cdots+2^{r_l}+8-1}{2^4+2+1} \\
& + \binom{2^{k+1}+2^{r_1}+\cdots+2^{r_l}+8-1}{2^k+2^{r_1}+\cdots+2^{r_l}-8-1} + 1 \\
\equiv\ & 0 + \binom{2^{k+1}+2^{r_1}+\cdots+2^{r_l}+8-1}{2^k+2^{r_1}+\cdots+2^{r_l-1}+\cdots+2^4+2^2+1} + 1 \\
\equiv\ & 0+0+1 \equiv 1 \ mod\ 2.
\end{aligned}
$$

If $p = 4$, then $r_l \geq 4$.

$$\begin{aligned}
&\binom{n-1}{19} + \binom{n-1}{n-2^k-17} + 2^k - 1 - 2 \\
\equiv &\binom{2^{k+1} + 2^{r_1} + \cdots + 2^{r_l} + 2^2 - 1}{2^4 + 2 + 1} \\
&+ \binom{2^{k+1} + 2^{r_1} + \cdots + 2^{r_l} + 2^2 - 1}{2^k + 2^{r_1} + \cdots + 2^{r_l} - 1 - 8 - 4} + 1. \quad (*)
\end{aligned}$$

For $r_l > 4$

$$\begin{aligned}
(*) \equiv\ & 0 + \binom{2^{k+1} + 2^{r_1} + \cdots + 2^{r_l} + 2 + 1}{2^k + 2^{r_1} + \cdots + 2^{r_{l+1}} + 2^{r_l - 1} + \cdots + 2 + 1 - 8 - 4} + 1 \\
\equiv\ & 0 + 0 + 1 \equiv 1 \ mod\ 2.
\end{aligned}$$

For $r_l = 4$

$$\begin{aligned}
(*) \equiv\ & 1 + \binom{2^{k+1} + 2^{r_1} + \cdots + 2^4 + 2 + 1}{2^k + 2^{r_1} + \cdots + 2^{r_{l+1}} + 2 + 1} + 1 \\
\equiv\ & 1 + 1 + 1 \equiv 1 \ mod\ 2.
\end{aligned}$$

From above, we know that x_n is indecomposable.

(b) $p = 6$.

If $p = 6$, then $r_l \geq 4$. For $r_l > 4$, we take $x_n = [RP(19, n-2^k-17; 2^k-1)]$. From [15; Lemma 2.4], $x_n \in J_{n,k}^{2^k+8}$.

$$\begin{aligned}
&\binom{n-1}{19} + \binom{n-1}{n-2^k-17} + 2^k - 1 - 2 \\
\equiv &\binom{2^{k+1} + 2^{r_1} + \cdots + 2^{r_l} + 4 + 1}{2^4 + 2 + 1} \\
&+ \binom{2^{k+1} + 2^{r_1} + \cdots + 2^{r_l} + 4 + 1}{2^k + 2^{r_1} + \cdots + 2^{r_l} - 1 - 8 - 2} + 1 \\
\equiv\ & 0 + 0 + 1 \equiv 1 \ mod\ 2.
\end{aligned}$$

By Lemma 2.1, x_n is indecomposable.

For $r_l = 4$, we take $x_n = [RP(2^{k+1} + 1, 2, 2, 2, 2, n - 2^{k+1} - 16, 0, 0)]$. From [15; Lemma 2.4], $x_n \in J_{n,k}^{2^k+8}$.

$$\begin{aligned} & \binom{2^{k+1} + 2^{r_1} + \cdots + 2^4 + 2^2 + 2 - 1}{2^{k+1} + 1} \\ & + \binom{2^{k+1} + 2^{r_1} + \cdots + 2^4 + 2^2 + 2 - 1}{2^{r_1} + \cdots + 2^4 + 2^2 + 2 - 2^4} \\ \equiv \quad & 1 + 0 \equiv 1 \; mod \; 2 \end{aligned}$$

By Lemma 2.1, x_n is indecomposable.

(c) $p = 2$.

If $p = 2$, then $r_l \geq 3$.

For $r_l > 4$, we take $x_n = [RP(19, n - 2^k - 17; 2^k - 1)]$. From [15; Lemma 2.4], $x_n \in J_{n,k}^{2^k+8}$.

$$\begin{aligned} & \binom{2^{k+1} + 2^{r_1} + \cdots + 2^{r_l} + 2 - 1}{2^4 + 2 + 1} \\ & + \binom{2^{k+1} + 2^{r_1} + \cdots + 2^{r_l} + 2 - 1}{2^k + 2^{r_1} + \cdots + 2^{r_l} - 1 - 8 - 4 - 2} + 1 \\ \equiv \quad & 0 + 0 + 1 \equiv 1 \; mod \; 2. \end{aligned}$$

By Lemma 2.1, x_n is indecomposable.

For $r_l = 4$ or 3, we take $x_n = [RP(2^{k+1} + 1, 2, 2, 2, 2, n - 2^{k+1} - 16, 0, 0)]$.

From [15; Lemma 2.4], $x_n \in J_{n,k}^{2^k+8}$.

$$\begin{aligned} & \binom{2^{k+1} + 2^{r_1} + \cdots + 2^{r_l} + 2 - 1}{2^{k+1} + 1} + \binom{2^{k+1} + 2^{r_1} + \cdots + 2^{r_l} + 2 - 1}{2^{r_1} + \cdots + 2^{r_l} + 2 - 2^4} \\ \equiv \quad & 1 + 0 \equiv 1 \; mod \; 2. \end{aligned}$$

By Lemma 2.1, x_n is indecomposable.

(2) If $S = \emptyset$, i.e. $n = 2^{k+1} + 2^{r_1} + 2^{r_2} + \cdots + 2^{r_s}, k = r_1 > r_2 > \cdots > r_s \geq 1, r_i = r_{i+1} + 1$ and $s \geq 2$.

For $r_s \geq 2$, we take $x_n = [RP(2^{k+1}+1, \dfrac{2^{r_s+1}-8}{2}, \dfrac{2^{r_s+1}-8}{2}, n-2^{k+1}-2^{r_s+1}); 8]$. From [15; Lemma 2.4], $x_n \in J_{n,k}^{2^k+8}$.

$$\begin{aligned}
&\binom{n-1}{2^{k+1}+1} + 2\binom{n-1}{\frac{2^{r_s+1}-8}{2}} + \binom{n-1}{n-2^{k+1}-2^{r_s+1})} + 8 - 4 \\
\equiv\ &\binom{2^{k+1}+2^{r_1}+2^{r_2}+\cdots+2^{r_s}-1}{2^{k+1}+1} + \binom{2^{k+1}+2^{r_1}+2^{r_2}+\cdots+2^{r_s}-1}{2^{r_1}+2^{r_2}+\cdots+2^{r_s}-2^{r_s+1}} \\
\equiv\ &1 + \binom{2^{k+1}+2^{r_1}+2^{r_2}+\cdots+2^{r_{s-1}}+2^{r_s-1}+\cdots+2+1}{2^{r_1}+2^{r_2}+\cdots+2^{r_{s-2}}+2^{r_s}} \\
\equiv\ &1 + 0 \equiv 1 \ mod\ 2.
\end{aligned}$$

By Lemma 2.1, x_n is indecomposable.

For $r_s = 1$, we take $x_n = [RP(2^{k+1}+1, n-2^{k+1}-8; 8)]$. From [15; Lemma 2.4], $x_n \in J_{n,k}^{2^k+8}$.

$$\begin{aligned}
&\binom{2^{k+1}+2^{r_1}+2^{r_2}+\cdots+2-1}{2^{k+1}+1} + \binom{2^{k+1}+2^{r_1}+2^{r_2}+\cdots+2-1}{2^{r_1}+2^{r_2}+\cdots+2-8} \\
\equiv\ &1 + 0 \equiv 1 \ mod\ 2.
\end{aligned}$$

By Lemma 2.1, x_n is indecomposable. □

Proposition 3.4 — If $n \geq 2^k+9$ and $n \neq 2^u - 1$, then there exist indecomposable classes $x_n \in J_{n,k}^{2^k+8}$ for $k \geq 4$.

PROOF : Take x_n as in Lemma 3.1 for n odd, x_n as in Lemma 3.2 and Lemma 3.3 for $2^k+10 \leq n < 2^{k+2}$ even, and x_n as in [15; Lemma 3.1] for $n \geq 2^{k+2}$ even. Then, every x_n is indecomposable and $x_n \in J_{n,k}^{2^k+8}$. □

To complete the proof of the main theorem, we need the following two lemmas.

Lemma 3.5 — For $k \geq 4$ and $n = 2^k + 7$, there exist indecomposable classes $x_n \in J_{n,k}^{2^k+6}$.

PROOF : From [11; 5.1], there exists an indecomposable class $X \in J_{\frac{n-1}{2},k}^{2^{k-1}+3}$ (for $2^{k-1}+3 < 2^k$), so that X has a representative M admitting a $(Z_2)^k$-action with fixed point set F', where $\dim(F') = \frac{n-1}{2} - (2^{k-1}+3) = 0$.

Take $x_n = [P(1, M)]$ with the $(Z_2)^k$-action as in Lemma 3.1. The fixed point set of this action is $F = S^1 \times F' \times F'/T = P(1, F')$ with $\dim F = 1 = n - 2^k - 6$, then $x_n \in J_{*,k}^{2^k+6}$. By Lemma 2.2, x_n is indecomposable. □

Lemma 3.6 — For $k \geq 4$ and $n = 2^k + 8$, there exist indecomposable classes $x_n \in J_{n,k}^{2^k+6}$.

Proof : Take $x_n = [RP(11; 2^k - 6)]$. Just as in Lemma 3.2, there exsits a $(Z_2)^2$-action on $RP(11)$ with fixed point set being four copies of $RP(2)$. Let $(Z_2)^0$ act as the identity on the rest of the base. By Lemma 2.3, $x_n \in J_{n,k}^{2^k+6}$.

$$\binom{2^k + 8 - 1}{8 + 2 + 1} + 2^k - 6 - 1 \equiv 1 \ mod\ 2.$$

By Lemma 2.1, x_n is indecomposable. □

4. Proof of the Main Theorem

Proof : We choose a system of generators x_n as follows:

(a) Let $x_2 = [RP(2)]$. Noticing that $\chi(x_2) = 1$, by [11; 5.1], we have $x_2 \in J_{*,k}^2$.

(b) For $3 \leq n \leq 2^k + 6$ and $n \neq 2^u - 1$, we can choose indecomposable classes x_n such that $\chi(x_n) = 0$ (otherwise, replacing x_n by $x_n + x_2^{n/2}$).

(c) Take x_n as in Lemma 3.5 for $n = 2^k + 7$ and x_n as in Lemma 3.6 for $n = 2^k + 8$. By Lemma 2.4, $x_{2^k+8} \notin J_{*,k}^{2^k+8}$.

(d) For $n \geq 2^k + 9$, let x_n be as in Proposition 3.4.

Since $J_{*,k}^{2^k+8}$ is an ideal of MO_*, to complete the proof it is necessary only to show that it contains the decomposable classes $x_{i_1} x_{i_2} \cdots x_{i_m}$ with $m \geq 2,\ 2 \leq i_1 \leq i_2 \leq \cdots \leq i_m \leq 2^k + 8$ and $i_1 + i_2 + \cdots + i_m \geq 2^k + 8$.

The argument is divided into two cases: Case 1. $i_1 + i_2 + \cdots + i_m > 2^k + 8$. Case 2. $i_1 + i_2 + \cdots + i_m = 2^k + 8$.

Case 1 — $i_1 + i_2 + \cdots + i_m > 2^k + 8$.

(1) The case $i_m > 2^k$.

(i) If $i_1+i_2+\cdots+i_{m-1}\geq 8$, from [11; 5.1] $x_{i_1}x_{i_2}\cdots x_{i_{m-1}}\in J^8_{*,k}$. From [12], $x_{i_m}\in J^{2^k}_{*,k}$ and $x_{i_1}x_{i_2}\cdots x_{i_m}\in J^8_{*,k}J^{2^k}_{*,k}\subset J^{2^k+8}_{*,k}$.

(ii) If $i_1+i_2+\cdots+i_{m-1}=7$, then $x_{i_1}x_{i_2}\cdots x_{i_{m-1}}\in J^4_{*,k}\bigcap J^7_{*,k}$ and $i_m>2^k+1$. If $i_m>2^k+4$, by [18; Theorem] $x_{i_m}\in J^{2^k+4}_{*,k}$ and $x_{i_1}x_{i_2}\cdots x_{i_m}\in J^4_{*,k}J^{2^k+4}_{*,k}\subset J^{2^k+8}_{*,k}$.

If $i_m\leq 2^k+4$, from (b), $\chi(x_{i_m})=0$. By [14; Theorem 1], $x_{i_m}\in J^{2^k+1}_{*,k}$. So $x_{i_1}x_{i_2}\cdots x_{i_m}\in J^7_{*,k}J^{2^k+1}_{*,k}\subset J^{2^k+8}_{*,k}$.

(iii) If $i_1+i_2+\cdots+i_{m-1}=6$, then $i_m>2^k+2$. From [16], $x_{i_m}\in J^{2^k+2}_{*,k}$. By [11; 5.1], $x_{i_1}x_{i_2}\cdots x_{i_{m-1}}\in J^6_{*,k}$. Then $x_{i_1}x_{i_2}\cdots x_{i_m}\in J^6_{*,k}J^{2^k+2}_{*,k}\subset J^{2^k+8}_{*,k}$.

(iv) If $i_1+i_2+\cdots+i_{m-1}=5$, then $x_{i_1}x_{i_2}\cdots x_{i_{m-1}}\in J^4_{*,k}\bigcap J^5_{*,k}$ and $i_m>2^k+3$. If $i_m>2^k+4$, by [18; Theorem], $x_{i_m}\in J^{2^k+4}_{*,k}$ and $x_{i_1}x_{i_2}\cdots x_{i_m}\in J^4_{*,k}J^{2^k+4}_{*,k}\subset J^{2^k+8}_{*,k}$.

If $i_m=2^k+4$, from (b), $\chi(x_{i_m})=0$.

By [15; Proposition 4.3], $x_{i_m}\in J^{2^k+3}_{*,k}$. Then $x_{i_1}x_{i_2}\cdots x_{i_m}\in J^5_{*,k}J^{2^k+3}_{*,k}\subset J^{2^k+8}_{*,k}$.

(v) If $i_1+i_2+\cdots+i_{m-1}=4$, then $i_m>2^k+4$. From [18; Theorem], $x_{i_m}\in J^{2^k+4}_{*,k}$. By [11; 5.1], $x_{i_1}x_{i_2}\cdots x_{i_{m-1}}\in J^4_{*,k}$. Then $x_{i_1}x_{i_2}\cdots x_{i_m}\in J^4_{*,k}J^{2^k+4}_{*,k}\subset J^{2^k+8}_{*,k}$.

(vi) If $i_1+i_2+\cdots+i_{m-1}=2$, then $m=2, i_1=2, x_2\in J^2_{*,k}$ and $2^k+6<i_2\leq 2^k+8$. From (c), $x_{i_2}\in J^{2^k+6}_{*,k}$. Then $x_2x_{i_2}\in J^2_{*,k}J^{2^k+6}_{*,k}\subset J^{2^k+8}_{*,k}$.

(2) The case $i_m\leq 2^k$ and there exists some $l(1\leq l\leq m)$ such that $8\leq i_l<2^k$.

From [11; 5.1], $x_{i_l}\in J^{i_l}_{*,k}$. Since $i_1+i_2+\cdots+i_m>2^k+8$, $i_1+i_2+\cdots+i_{l-1}+i_{l+1}+\cdots+i_m>2^k+8-i_l$. Because $2^k+8-i_l\leq 2^k$, from [11; 5.1] and [12], we have $x_{i_1}x_{i_2}\cdots x_{i_{l-1}}x_{i_{l+1}}\cdots x_{i_m}\in J^{2^k+8-i_l}_{*,k}$. Then $x_{i_1}x_{i_2}\cdots x_{i_m}\in J^{2^k+8-i_l}_{*,k}J^{i_l}_{*,k}\subset J^{2^k+8}_{*,k}$.

(3) The case $i_m\leq 2^k$, with no $l(1\leq l\leq m)$ satisfying $8\leq i_l<2^k$.

In this case, $i_j=2^k$, or $i_j\leq 6(1\leq j\leq m)$.

(i) If $i_m = 2^k$, then $i_1 + i_2 + \cdots + i_{m-1} \geq 9$ and $x_{i_m} \in J_{*,k}^{2^k-2} \bigcap J_{*,k}^{2^k-1}$. If there are odd numbers in $i_1, \cdots, i_{m-1}$, then $\chi(x_{i_1}x_{i_2}\cdots x_{i_{m-1}}) = 0$ and $x_{i_1}x_{i_2}\cdots x_{i_{m-1}} \in J_{*,k}^9$. So $x_{i_1}x_{i_2}\cdots x_{i_m} \in J_{*,k}^9 J_{*,k}^{2^k-l} \subset J_{*,k}^{2^k+8}$. If $i_1, \cdots, i_{m-1}$ are all even, $i_1 + i_2 + \cdots + i_{m-1} \geq 10$ and $x_{i_1}x_{i_2}\cdots x_{i_{m-1}} \in J_{*,k}^{10}$. Then $x_{i_1}x_{i_2}\cdots x_{i_m} \in J_{*,k}^{10} J_{*,k}^{2^k-2} \subset J_{*,k}^{2^k+8}$.

(ii) If $i_m \leq 6$ and there exists some t such that $i_t \neq 2$, then $x_{i_m} \in J_{*,k}^{i_m}$ and $i_1+i_2+\cdots+i_{m-1} > 2^k+8-i_m$. From [15],16] and [18], $x_{i_1}x_{i_2}\cdots x_{i_{m-1}} \in J_{*,k}^{2^k+8-i_m}$. Then $x_{i_1}x_{i_2}\cdots x_{i_m} \in J_{*,k}^{2^k+8-i_m} J_{*,k}^{i_m} \subset J_{*,k}^{2^k+8}$.

(iii) If $i_1 = i_2 = \cdots = i_m = 2$, then $2(m-4) > 2^k$. From [11; 5.1] and [12], $x_2^m = x_2^4 x_2^{m-4} \in J_{*,k}^8 J_{*,k}^{2^k} \subset J_{*,k}^{2^k+8}$.

Case 2 — $i_1 + i_2 + \cdots + i_m = 2^k + 8$.

If $i_m < 2^k$, from [11; 5.1], $x_{i_r} \in J_{*,k}^{i_r}$ $(r = 1, 2, \cdots, m)$ and

$$x_{i_1}x_{i_2}\cdots x_{i_m} \in J_{*,k}^{i_1} J_{*,k}^{i_2} \cdots J_{*,k}^{i_m} \subset J_{*,k}^{2^k+8}.$$

If $i_m \geq 2^k$, then $s_{(i_1,i_2,\cdots,i_m)}[x_{i_1}x_{i_2}\cdots x_{i_m}] \neq 0$. By Lemma 2.5, the decomposable classes $x_2x_{2^k+6}, x_4x_{2^k+4}, {x_2}^2x_{2^k+4}, x_5x_{2^k+3}, x_6x_{2^k+2}, x_2x_4x_{2^k+2}, {x_2}^3x_{2^k+2}, x_2x_5x_{2^k+1}, x_8x_{2^k}, x_2x_6x_{2^k}, {x_4}^2x_{2^k}, {x_2}^2x_4x_{2^k}, {x_2}^4x_{2^k}$ are not in $J_{*,k}^{2^k+8}$. For some linear combination of the above classes such as

$$x_{i_{1,1}}x_{i_{2,1}}\cdots x_{i_{m_1,1}} + x_{i_{1,2}}x_{i_{2,2}}\cdots x_{i_{m_2,2}} + \cdots + x_{i_{1,j}}x_{i_{2,j}}\cdots x_{i_{m_j,j}},$$

we can find

$$(i_1, i_2, \cdots, i_{n_j}) \in \{(i_{1,1}, i_{2,1}\cdots i_{m_1,1}), (i_{1,2}, i_{2,2}\cdots i_{m_2,2}), \cdots, (i_{1,j}, i_{2,j}, \cdots i_{m_j,j})\}$$

such that $(i_1, i_2, \cdots, i_{n_j})$ is not the refinement of other elements in

$$\{(i_{1,1}, i_{2,1}\cdots i_{m_1,1}), (i_{1,2}, i_{2,2}\cdots i_{m_2,2}), \cdots (i_{1,j}, i_{2,j}, \cdots i_{m_j,j})\}.$$

Then, from [23],

$$s_{(i_1,i_2,\cdots,i_{n_j})}[x_{i_{1,1}}x_{i_{2,1}}\cdots x_{i_{m_1,1}}+x_{i_{1,2}}x_{i_{2,2}}\cdots x_{i_{m_2,2}}+\cdots+x_{i_{1,j}}x_{i_{2,j}}\cdots x_{i_{m_j,j}}] \neq 0.$$

By Lemma 2.5, any linear combination of the above classes are not in $J_{*,k}^{2^k+8}$. □

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