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# A REAL SEMINORM WITH SQUARE PROPERTY IS SUBMULTIPLICATIVE

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A seminorm with square property on a real associative algebra is submultiplicative.

**Key words** : Real seminorm; seminorm with square property; submultiplicative seminorm.

## 1. Introduction

It was proved in [3] that every complex seminorm with square property on a commutative algebra is submultiplicative, and it was posed the problem whether this result holds in a noncommutative algebra. This problem was answered in the particular case of Banach algebras [4] and fully resolved in [2], [5] and [7]. The result of [3] holds in the real case. But the results of [2], [4], [5] and [7] don't hold in the real case since we use the Hirschfeld-Zelazko Theorem [6], or its locally bounded version [2], which are not valid in the real case. Using a functional representation

Theorem [1, Theorem 1], we show that every real seminorm with square property is submultiplicative.

## 2. Preliminaries

Let $A$ be an associative algebra over the field $K = R$ or $C$. A seminorm on $A$ is a function $p : A \to [0, \infty)$ satisfying $p(a + b) \leq p(a) + p(b)$ for all $a, b$ in $A$ and $p(ka) = |k|p(a)$ for all $a$ in $A$ and $k$ in $K$. $p$ is a complex seminorm if $K$ = $C$ and $p$ is a real seminorm if $K$ = $R$. $p$ is submultiplicative if $p(ab) \leq p(a)p(b)$ for all $a, b$ in $A$. $p$ satisfies the square property if $p(a^2) = p(a)^2$ for all $a$ in $A$. Let $A$ be a real algebra and let a be any element of $A$. The spectrum $sp(a)$ of $a$ in $A$ is defined to be equal to the spectrum of $a$ as an element of the complexification of $A$. If $A$ is unital, then $sp(a) = \{s + it\epsilon C, (a - se)^2 + t^2e \notin A^{-1}\}$ for all $a$ in $A$, where $e$ is the unit of $A$ and $A^{-1}$ is the set of all invertible elements in $A$. Let $(A, ||.||)$ be a real normed algebra, the limit $r(a) = lim||a^n||^{1/n}$ exists for each $a$ in $A$. If $A$ is complete, then $r(a) = \sup\{|v|, v\epsilon sp(a)\}$ for every a in $A$.

## 3. Results

Let $(A, ||.||)$ be a real Banach algebra with unit such that $||a|| \leq mr(a)$ for some positive constant $m$ and all $a$ in $A$. Let $X(A)$ be the set of all nonzero multiplicative linear functionals from $A$ into the noncommutative algebra $H$ of quaternions. For a in $A$ and $x$ in $X(A)$, put $J(a)(x) = x(a)$. For $a$ in $A$, $J(a) : X(A) \to H$ is a map from $X(A)$ into $H$. $X(A)$ is endowed with the topology generated by $J(a), a\epsilon A$, that is the weakest topology such that all the functions $J(a), a\epsilon A$, are continuous. By [1, Theorem 1], $X(A)$ is a nonempty compact space and the map $J : A \to C(X(A), H), a \to J(a)$, is an isomorphism (into), where $C(X(A), H)$ is the real algebra of all continuous functions from $X(A)$ into $H$.

*Proposition* 3.1. — (1) An element a is invertible in $A$ if and only if $J(a)$ is invertible in $C(X(A), H)$, and $(2) sp(a) = sp(J(a))$ for all $a$ in $A$.

Proof : (1) The direct implication is obvious. Conversely, there exists $g$ in

$C(X(A), H)$ such that $J(a)g = gJ(a) = 1$ i.e., $x(a)g(x) = g(x)x(a) = 1$ for all $x$ in $X(A)$. LetT be a nonzero irreductible representation of $A$, by the proof of [1, Theorem 1] there exists $S : T(A) \to H$ an isomorphism (into). Since $SoT\epsilon X(A)$ and $0 \neq SoT(a) = S(T(a))$, it follows that $T(a) \neq 0$. If $aA \neq A$, there exists a maximal right ideal $M$ containing $aA$. Let $L$ be the canonical representation of $A$ on $A/M$ which is nonzero and irreductibte, also $L(a) = 0$ since $aA$ is included in $M$, contradiction. Then $aA = A$ and by the same $Aa = A$. There exist $b, c$ in $A$ such that $ab = ca = e$ ($e$ is the unit of $A$). We have $c = c(ab) = (ca)b = b$, so $a$ is invertible in $A$.

(2) $s + it\epsilon sp(a)$ iff $(a - se)^2 + t^2e \notin A^{-1}$

iff $J((a - se)^2 + t^2e) \notin C(X(A), H)^{-1}$ by (1)

iff $(J(a) - sJ(e))^2 + t^2J(e) \notin C(X(A), H)^{-1}$

iff $s + it\epsilon sp(J(a))$.

***Theorem* 3.2** — *Let A be a real associative algebra. Then every seminorm with square property on A is submultiplicative.*

PROOF : If $A$ is commutative, see [3, Theorem 1]. If $A$ is noncommutative, let $p$ be a seminorm with square property on $A$. By [5] or [7], there exists $m > 0$ such that $p(ab) \leq mp(a)p(b)$ for all $a, b$ in $A$. $Ker(p)$ is a two sided ideal in $A$, the norm $|.|$ on the quotient algebra $A/Ker(p)$, defined by $|a + Ker(p)| = p(a)$ is a norm with square property on $A/Ker(p)$. Define $||a + Ker(p)||$ = $m|a + Ker(p)|$ for all $a$ in $A$. Let $a, b$ in $A$, $||ab + Ker(p)||$ = $m|ab + Ker(p)| \leq m^2$ $|a + Ker(p)|$ $|b + Ker(p)|$ = $||a + Ker(p)||$ $||b + Ker(p)||$. $(A/Ker(p), ||.||)$ is a real normed algebra. Let $a$ in $A$, $||a^2 + Ker(p)||$ = $m|a^2 + Ker(p)|$ = $m|a + Ker(p)|^2$ = $m^{-1}$ $(m|a + Ker(p)|)^2 = m^{-1}$ $||a + Ker(p)||^2$. The completion $B$ of $(A/Ker(p), ||.||)$ satisfies also the property $||b^2||$ = $m^{-1}$ $||b||^2$ for all $b$ in $B$, and consequently $||b^{2^n}||^{2^{-n}}$ = $m^{2^{-n}-1}||b||$ for all $b$ in $B$ and $n$ in $N^*$, then $r(b)$ = $m^{-1}$ $||b||$ i.e., $||b|| = mr(b)$. We consider two cases.

$B$ is unital: By [1, Theorem 1], $X(B)$ is a nonempty compact space and the map $J : B \to C(X(B), H)$ is an isomorphism (into). $C(X(B), H)$ is a real Banach algebra with unit under the supnorm $||.||_s$. By Proposition 3.1, $r(b) = r(J(b))$ for all $b$ in $B$. Let $b$ in $B$, $||b||$ = $mr(b) = mr(J(b))$ = $m||J(b)||_s$ since the supnorm satisfies the square property. Then $||J(b)||_s$ = $m^{-1}||b|| = |b|$ for all $b$ in $A/Ker(p)$, so $|.|$ is submultiplicative on $A/Ker(p)$ i.e., $p$ is submultiplicative.

$B$ is not unital: Let $B_1$ be the algebra obtained from $B$ by adjoining the unit. By the same proof of [6, Lemma 2], there exists a norm $N$ on $B_1$ such that

(i) $(B_1, N)$ is a real Banach algebra with unit

(ii) $N(b) \leq m^3 r_{B_1}(b)$ for all $b$ in $B_1$

(iii) $N$ and $||.||$ are equivalent on $B$.

By [1, Theorem 1], $X(B_1)$ is a nonempty compact space and the map $J : B_1 \to C(X(B_1), H)$ is an isomorphism (into). Let $b$ in $B$, $||b|| = mr_B(b)$ = $mr_{B_1}(b)$ by (iii)

= $mr(J(b))$ by Proposition 3.1

= $m||J(b)||_s$ by the square property of the supnorm.

Then $||J(b)||_s = m^{-1}||b|| = |b|$ for all $b$ in $A/Ker(p)$, so $|.|$ is submultiplicative on $A/Ker(p)$ i.e., $p$ is submultiplicative.

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