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SYMPLECTIC MODULES OVER OVERRINGS OF POLYNOMIAL RINGS

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Let B be a commutative Noetherian ring of dimension d and let S be a set of all monic polynomials in $B[X]$. Let A be a subring of $S^{-1}B[X]$ which contains $B[X]$. Let P be a symplectic A-module of rank $2n \geq d$, $n > 0$. Then we prove that $ESp(A^2 \perp P, \langle , \rangle)$ acts transitively on $Um(A^2 \oplus P)$.

Key words : Projective module; unimodular element; cancellation problem.

1. INTRODUCTION

Let A be a commutative Noetherian ring. A $symplectic$ A-module is a pair $(P, \langle , \rangle)$, where P is a finitely generated projective A-module and $\langle , \rangle : P \times P \to A$ is a non-degenerate alternating bilinear form.

Let A be a ring of dimension d and P be a symplectic A-module of rank $2n$ with $2n \geq d$. Then Swan [11] has proved that $ESp(A^2 \perp P, \langle , \rangle)$ acts transitively on $Um(A^2 \oplus P)$. Further, the above result of Swan is extended to polynomial rings by Bhatwadekar and Laurent polynomial rings by Keshari in [2] and [6] respectively.

In this paper we prove the following result see (3.9). This generalizes a result of Bhatwadekar ([2], Theorem 4.8).

***Theorem* A** — *Let B be a ring of dimension d and let S be a multiplicative closed subset of $B[X]$ containing monic polynomials. Let A be a ring such that $B[X] \subseteq A \subseteq S^{-1}B[X]$. Let P be a symplectic A-module of rank $2n$ with $2n \geq d$, $n > 0$. Then $ESp(A^2 \perp P, \langle , \rangle)$ acts transitively on $Um(A^2 \oplus P)$.*

We prove the following result (3.13), which generalizes a result of Keshari ([6], Theorem A.7).

***Theorem* B** — *Let B be a ring of dimension d and $A = B[X_1, \ldots, X_m, Y, \frac{1}{g}]$, where $g \in B[Y]$ is monic. Let P be a symplectic A-module of rank $2n \geq d$. Then $ESp(A^2 \perp P, \langle , \rangle)$ acts transitively on $Um(A^2 \oplus P)$.*

The following result (4.3), generalizes a result of Keshari ([6], Theorem A.8).

***Theorem* C** — *Let B be a ring of dimension 2 and $A = B[X_1, \ldots, X_n, Y, \frac{1}{g}]$, where g is a monic polynomial in $B[Y]$. Let P be a projective A-module of rank 2 with trivial determinant. If A^2 is cancellative, then P is cancellative.*

2. Preliminaries

All rings considered in this paper are assumed to be commutative Noetherian and all modules are finitely generated.

Let A be a ring and let P be a projective A-module. Recall that $p \in P$ is called a unimodular element if there exists a $\phi \in P^* = \mathrm{Hom}_A(P, A)$, such that $\phi(p) = 1$. The set of all unimodular elements of P is denoted by $Um(P)$. A row $(a_1, \ldots, a_n) \in A^n$ is called a unimodular row if there exists $(b_1, \ldots, b_n) \in A^n$ such that $a_1b_1 + \ldots + a_nb_n = 1$. The set of all unimodular rows of length n is denoted by $Um_n(A)$.

Let n be a positive integer. Let $\mathrm{GL}_n(A)$ be the set of all $n \times n$ invertible matrices over A. $SL_n(A) := \{M \in \mathrm{GL}_n(A) | \det(M) = 1\}$. Let I be an ideal in A, we define $SL_n(A, I)$ to be kernel of the natural map $SL_n(A) \to SL_n(A/I)$. When $I = (a)$, we write $SL_n(A, a)$ for $SL_n(A, I)$.

We recall some preliminary facts about symplectic modules. For notations and terminology we follow [2]. For more detail, see ([2], Section 4).

Let A be a ring and let P be a finitely generated projective A-module. A bilinear form $\langle , \rangle : P \times P \to A$ is called $alternating$ if $\langle p, p \rangle = 0 \; \forall p \in P$. An alternating bilinear form $\langle , \rangle$ induces a homomorphism $\alpha : P \to P^*$ (defined as $\alpha(p)(q) = \langle p, q \rangle$) such that $\alpha + \alpha^* = 0$. An alternating form $\langle , \rangle$ on P is called non-$degenerate$ if the induced homomorphism from P to P^* is an isomorphism.

A $symplectic$ A-module is a pair $(P, \langle , \rangle)$, where P is a finitely generated projective A-module and $\langle , \rangle : P \times P \to A$ is a non-degenerate alternating bilinear form. If $(P, \langle , \rangle)$ is a symplectic A-module then the rank of P is even and P has trivial determinant.

If $(P, \langle , \rangle)$ and $(Q, \langle , \rangle)$ are two symplectic modules, then the non-degenerate alternating bilinear forms on P and Q will give rise (in a canonical manner) to a non-degenerate alternating bilinear form on $P \oplus Q$ and we denote the symplectic module thus obtained by $(P \perp Q, \langle , \rangle)$. The standard alternating form on A^2 (in the natural basis $\{e_1, e_2\}$) is given by $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

Two symplectic modules $(P, \langle , \rangle)$ and $(Q, \langle , \rangle)$ are isomorphic if there exists an isomorphism $\tau : P \to Q$ such that

$$\langle p_1, p_2 \rangle \;=\; \langle \tau(p_1), \tau(p_2) \rangle, \forall p_1, p_2 \in P.$$

An isometry of the symplectic module $(P, \langle , \rangle)$ is an automorphism of $(P, \langle , \rangle)$. We denote by $Sp(P, \langle , \rangle)$ the group of isometries of $(P, \langle , \rangle)$. It is easy to see that $SL_2(A) \subseteq Sp(A^2, \langle , \rangle)$.

Let $(P, \langle , \rangle)$ be a symplectic A-module and let $u, v \in P$ be such that $\langle u, v \rangle = 0$.

Let $a \in A$ and let $\tau_{(a,u,v)} : P \to P$ be a map defined by

$$\tau_{(a,u,v)}(p) = p + \langle p, v\rangle u + \langle p, u\rangle v + a\langle p, u\rangle u.$$

Then it is easy to see that $\tau_{(a,u,v)} \in Sp(P, \langle , \rangle)$. An isometry $\tau_{(a,u,v)}$ is called a $symplectic\ transvection$ if either u or v is unimodular. We denote by $ESp(P, \langle , \rangle)$ the subgroup of $Sp(P, \langle , \rangle)$, generated by symplectic transvections. $ESp(P, \langle , \rangle)$ is a normal subgroup of $Sp(P, \langle , \rangle)$.

Let $(P, \langle , \rangle)$ be a symplectic A-module. Let $c, d \in A$, $q \in P$. If $u = (0, 1, 0)$ and $v = (0, c, q) \in A^2 \oplus P$, then $\tau_{(-c,u,v)}$ is a symplectic transvection of $(A^2 \perp P, \langle , \rangle)$ such that

$$\tau_{(-c,u,v)}((a, b, p)) = (a, b + ca + \langle p, q\rangle, p + aq).$$

Similarly, if $u = (1, 0, 0)$ and $v = (-d, 0, -q)$, then

$$\tau_{(d,u,v)}((a, b, p)) = (a + bd + \langle q, p\rangle, b, p + bq).$$

$E(A^2 \perp P, \langle , \rangle)$ denotes the subgroup of $Sp(A^2 \perp P, \langle , \rangle)$ generated by $\Theta_{(c,q)}$ and $\sigma_{(d,q)}$ for $c, d \in A$ and $q \in P$, where $\Theta_{(c,q)}$, $\sigma_{(d,q)}$ are defined as follows:

$$\Theta_{(c,q)}(a, b, p) = (a, b + ca + \langle p, q\rangle, p + aq)$$

$$\sigma_{(d,q)}(a, b, p) = (a + bd + \langle q, p\rangle, b, p + bq)$$

for $(a, b, p) \in A^2 \oplus P$.

Let B be a ring and let $A = B[X]$. Let F be a free A-module with a basis $\{e_1, \ldots, e_r\}$. Let $p(X) = \sum_{i=1}^{r} \gamma_i(X)e_i \in F$, where $\gamma_i(X) \in A$ for $1 \le i \le r$. Then, for $b \in B$, we denote by $p(bX)$ the element $\sum_{i=1}^{r} \gamma_i(bX)e_i$ of F.

We state the following result which is due to Bhatwadekar ([2], Theorem 4.8).

***Theorem* 2** — *Let R be a ring of dimension d. Let A be a polynomial ring in r (≥ 0) variables over R. Let $(P, \langle , \rangle)$ be a symplectic A-module of rank $2n > 0$. If $2n \ge d$ then $ESp(A^2 \perp P, \langle , \rangle)$ acts transitively on $Um(A^2 \oplus P)$.*

Remark 2.2 : We have observed that in the above theorem we can replace $ESp(A^2 \perp P, \langle,\rangle)$ by $E(A^2 \perp P, \langle,\rangle)$. Let I be an ideal in A, then clearly the natural map $E(A^2 \perp P, \langle,\rangle) \to E((A/I)^2 \perp P/IP, \langle,\rangle)$ is surjective.

The following result is due to Keshari ([6], Theorem A.7), which extends the above result of Bhatwadekar to Laurent polynomial ring.

***Theorem* 3** — *Let B be a ring of dimension d and $A = B[Y_1, \ldots, Y_{r'}, X_1^{\pm 1}, \ldots, X_r^{\pm 1}]$. Let $(P, \langle,\rangle)$ be a symplectic A-module of rank $2n \geq d$, then $ESp(A^2 \perp P, \langle,\rangle)$ acts transitively on $Um(A^2 \oplus P)$.*

3. Main Theorem

We begin this section by proving the following result of Bhatwadekar and Roy ([3], Lemma 4.1).

Lemma 3.1 — Let $B \subset C$ be rings of dimension d and $x \in B$ such that $B_x = C_x$. Then

(i) $B/(1+xb) = C/(1+xb)$ for all $b \in B$.

(ii) If I is an ideal of C such that $ht(I) \geq d$ and $I + xC = C$, then there exists an element $b \in B$ such that $1 + xb \in I$.

Proof : (i) Since $B_x = C_x$, by going modulo the ideal $(1+xb)B_x$ with $b \in B$, the proof follows.

(ii) Since $ht(I) = d$, without loss of generality we may assume that I is a maximal ideal of height d. By hypothesis, localizing C at $x(1 + xB)$, we have $B_{x(1+xB)} = C_{x(1+xB)}$. Since $\dim B_{x(1+xB)} < \dim B$, we have $\dim C_{x(1+xB)} < d$. Since I is maximal ideal of height d, we get $IC_{x(1+xB)} = C_{x(1+xB)}$. Therefore I contains an element of the form $x^m(1 + xa)^n$ for some $a \in B$ and for some positive integers m and n. Since $I + xC = C$ and I is a prime ideal, I contains an element of the form $1 + xb$ for some $b \in B$. This completes the proof of (ii).

The following result is a consequence of a theorem of Eisenbud-Evans as stated in $([9], p.1420)$. □

Lemma 3.2 — Let A be a ring and let P be a projective A-module of rank r. Let $(\alpha, a) \in (P^* \oplus A)$. Then there exists an element $\beta \in P^*$ such that ht$(I_a) \geq r$, where $I = (\alpha + a\beta)(P)$. In particular, if the ideal $(\alpha(P), a)$ has height $\geq r$, then ht$(I) \geq r$. Further, if $(\alpha(P), a)$ is an ideal of height $\geq r$ and I is a proper ideal of R, then ht$(I) = r$.

The following lemma is due to Suslin ([10], Lemma 2.1).

Lemma 3.3 — Let B be a ring and let $A = B[X, \frac{F_1}{g}, \ldots, \frac{F_r}{g}]$, where $F_i \in B[X]$ for $1 \leq i \leq r$ and $g \in B[X]$ is monic. Suppose $f_1, f_2 \in B[X]$ and $c \in (f_1, f_2)B[X] \cap B$. Then for any ideal I of A and $g, h \in A \cap B[X]$ with $h - g \in cI$, there exists $\Delta \in SL_2(A, I)$ such that $[f_1(g), f_2(g)]\Delta = [f_1(h), f_2(h)]$.

We now prove the following extension lemma, which for polynomial ring (i.e. $A = B[X]$) is proved in [7]. Our proof follows using the same argument as in the proof ([7], Lemma 1.1, Chap.3).

Lemma 3.4 — Let B be a ring and let $A = B[X, \frac{F_1}{g}, \ldots, \frac{F_r}{g}]$, where $F_i \in B[X]$ for $1 \leq i \leq r$ and $g \in B[X]$ is monic. If I is an ideal of A containing $1 + gH$ for some monic polynomial $H \in B[X]$ and J is an ideal of B such that $I + JA = A$, then $B \cap I + J = B$.

PROOF : Let $R = A/I \supseteq B/(B \cap I)$ and let $\bar{J}$ be the image of J in $B/(B \cap I)$. By hypothesis, we have $JR = R$. Since I contains a polynomial $1 + gH$, it is easy to see that R is integral over $B/(B \cap I)$. Therefore the "Going-Up" Theorem for integral extensions ([8], p.34) implies that $J = B/(B \cap I)$, therefore $B \cap I + J = B$. □

Definition 3.5 — Let A be a ring and s be a non-zerodivisor in A. Let $(P, \langle , \rangle)$ be a symplectic A-module of rank $2n$. A set $\{e_1, \ldots, e_n, f_1, \ldots, f_n\} \subset P$ is called an s-symplectic basis of P if the following holds:

(i) $\langle e_i, e_j \rangle = 0 = \langle f_i, f_j \rangle$ for $1 \leq i, j \leq n$.

(ii) $\langle e_i, f_i\rangle = s$ for $1 \le i \le n$ and $\langle e_i, f_j\rangle = 0$ for every $i \ne j$.

Remark 3.6 : Let P be a symplectic A-module of rank $2n$. If $\{e_1, \ldots, e_n, f_1, \ldots, f_n\} \subseteq P$ is s-symplectic basis of P then the module $F := \sum Ae_i + \sum Af_j$ is a free A-submodule of P and $sP \subseteq F$. For the proof see ([2], Lemma 4.2).

Lemma 3.7 — Let B be a reduced ring of dimension d and let $g \in B[X]$ be a monic polynomial. Assume that either

(i) $A = B[X, \frac{f_1}{g}, \ldots, \frac{f_n}{g}]$, where $f_i \in B[X]$ for $i = 1, \ldots, n$ or

(ii) $A = B[X_1, \ldots, X_m, X, \frac{1}{g}]$.

Let $(P, \langle,\rangle)$ be a symplectic A-module of rank $2r \ge d$, $r > 0$. Then P has an s-symplectic basis $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ such that $sP \subseteq F$, where $F := \sum Ae_i + \sum Af_j$ is a free A-submodule of P of rank $2r$.

PROOF : Let S be the set of all non-zerodivisors in B. Since rank of P is constant, $S^{-1}A$ is a PID if A is as in (i) else $S^{-1}A = k[X_1, \ldots, X_m, X, \frac{1}{g}]$ for some field k. Using [5], in both the cases $S^{-1}P$ is a free $S^{-1}A$-module of rank $2r$. Let $\tilde{p_1}$, $\tilde{q_1} \in S^{-1}P$ such that $\langle \tilde{p_1}, \tilde{q_1}\rangle = 1$, where $\langle,\rangle$ is a induced form on $S^{-1}P$. Write $S^{-1}P = \tilde{p_1}S^{-1}A \oplus \tilde{q_1}S^{-1}A \oplus Q$, where Q is a $S^{-1}A$-submodule of $S^{-1}P$ of rank $2r-2$. Apply the same argument to Q, by inductively, threre exists $\tilde{p_2}, \ldots, \tilde{p_r}$ and $\tilde{q_2}, \ldots, \tilde{q_r}$ such that $\langle \tilde{p_i}, \tilde{q_i}\rangle = 1$ and $\langle \tilde{p_i}, \tilde{q_j}\rangle = 0$ for $i \ne j$. Also, we have $\langle \tilde{p_1}, \tilde{q_i}\rangle = 0$ and $\langle \tilde{p_i}, \tilde{q_1}\rangle = 0$ for $2 \le i \le r$.

Choose $t \in S$ such that $\tilde{p_i} = \frac{e_i}{t}$, $\tilde{q_i} = \frac{f_i}{t}$, where e_i, $f_i \in P$ for $1 \le i \le r$. Let $s = t^2$.

Claim : $\{e_1, \ldots, e_r, f_1, \ldots, f_r\}$ is an s-symplectic basis of P.

Proof of Claim : We show that (i) $\langle e_i, e_j\rangle = 0 = \langle f_i, f_j\rangle$ for $1 \le i, j \le n$ and (ii) $\langle e_i, f_i\rangle = s$ and $\langle e_i, f_j\rangle = 0$ for every $i \ne j$.

For any i, j for $1 \le i, j \le r$, we have $\langle e_i, e_j\rangle = \langle t\tilde{p_i}, t\tilde{p_j}\rangle$. Since $s\langle \tilde{p_i}, \tilde{p_j}\rangle = 0$ in $S^{-1}A$, we have $\langle e_i, e_j\rangle = 0$. Similarly, we can show that $\langle f_i, f_j\rangle = 0$ for

$1 \leq j \leq r$ and $\langle e_i, f_j \rangle = 0$ for every $i \neq j$.

For any i, $1 \leq i \leq r$, we have

$$\langle e_i, f_i \rangle = \langle t\frac{e_i}{t}, t\frac{f_i}{t} \rangle = t^2 \langle \tilde{p_i}, \tilde{p_j} \rangle = s \langle \tilde{p_i}, \tilde{p_j} \rangle = s.$$

Note that the above expression is computed in $S^{-1}A$. This proves the claim. Now by (3.6), the proof follows. □

Now we prove the following result which is inspired by and the proof closely follows, Bhatwadekar's result ([2], Proposition 4.7).

Proposition 3.8 — Let B be a ring and let $s \in B$ be a non-zerodivisor. Let $A = B[X, \frac{F_1}{g}, \ldots, \frac{F_r}{g}]$, where $F_i \in B[X]$ and $g \in B[X]$ is monic and let $(P, \langle , \rangle)$ be a symplectic A-module of rank $2n$. Let $\{e_1, \ldots, e_n, f_1, \ldots, f_n\} \subseteq P$ be an s-symplectic basis of P. Let $(\alpha(X), \beta(X), p(X)) \in Um(A^2 \oplus P)$, where $\alpha(X), \beta(X) \in B[X]$, $p(X) \in \sum_{i=1}^n B[X]e_i + \sum_{j=1}^n B[X]f_j$ with $\alpha(X) \equiv 1$ modulo (sXg) and $\beta(X) = 1 + gH$ with $H \in B[X]$ is monic. Let $b, b' \in B$ be such that $b - b' \in sB$. Then there exists $\Psi \in SL_2(A, sX)ESp(A^2 \perp P, \langle , \rangle)$ such that $\Psi(\alpha(bX), \beta(bX), p(bX)) = (\alpha(b'X), \beta(b'X), p(b'x))$.

PROOF : Since $ESp(A^2 \perp P, \langle , \rangle)$ is a normal subgroup of $Sp(A^2 \perp P, \langle , \rangle)$, $G = SL_2(A, sX)ESp(A^2 \perp P, \langle , \rangle)$ is a group. Let J be the set of elements $c \in B$ having following property, $b - b' \in csB \Rightarrow \exists \Phi \in G$ such that $\Phi(\alpha(bX), \beta(bX), p(bX)) = (\alpha(b'X), \beta(b'X), p(b'x))$. Clearly J is non empty. Since G is a group, J is an ideal of B. We prove that $J = B$.

Let $t \in B \cap (\alpha(X)^2, \beta(X)^2)A$ and let $b, b' \in B$ be such that $b - b' \in tsB$. Since $p(X) \in F = \sum B[X]e_i + \sum B[X]f_j$, $p(bX) = p(b'X) - tsq(X)$ for some $q(X) \in F$.

Claim : $B \cap (\alpha(X)^2, \beta(X)^2)A = B \cap (\alpha(X)^2, \beta(X)^2)B[X]$.

Proof of Claim : It is enough to prove $B \cap (\alpha(X)^2, \beta(X)^2)A \subset B \cap (\alpha(X)^2, \beta(X)^2)B[X]$. Let $t \in B \cap (\alpha(X)^2, \beta(X)^2)A$ then $t = a\alpha(X)^2 + b\beta(X)^2$, where

$a, b \in A$ and $\beta(X)^2 = 1 + gK$. Multiplying by gK to expression of t, we get the following

$$gKt = agK\alpha(X)^2 + bgK\beta(X)^2.$$

Adding both sides $-t$ and simplifying, we get

$$-t = -t(1 + gK) + agK\alpha(X)^2 + bgK\beta(X)^2.$$

Since $\beta(X)^2 = 1 + gK$, we get

$$t = -agK\alpha(X)^2 + (t - bgK)\beta(X)^2.$$

Applying this argument further if needed, we see that t belongs to the ideal of $B[X]$ generated by $\alpha(X)^2$ and $\beta(X)^2$. This proves the claim.

Now since $t \in (\alpha(X)^2, \beta(X)^2) \cap B$, t also belongs to the ideal of $B[X]$ generated by $\alpha(bX)^2$ and $\beta(bX)^2$. By ([2], Lemma 4.3), there exists $\psi \in ESp(A^2 \perp P, \langle , \rangle)$ such that $\psi(\alpha(bX), \beta(bX), p(bX)) = (\alpha(bX), \beta(bX), p(b'X))$. Since $bX - b'X \in tsXB[X]$, by (3.3), the element $(\alpha(bX), \beta(bX), p(b'X))$ can be transformed to $(\alpha(b'X), \beta(b'X), p(b'X))$, by an element of the $SL_2(A, sX)$, hence $t \in J$.

Since $\beta(X) = 1 + gH$ with $H \in B[X]$ monic, $B/(B \cap (\alpha(X)^2, \beta(X)^2)) \hookrightarrow A/(\alpha(X)^2, \beta(X)^2))$ is an integral extension. Since $(\alpha(X)^2, \beta(X)^2) + sA = A$, by (3.4), we have $B \cap (\alpha(X)^2, \beta(X)^2) + sB = B$. Therefore the above argument shows that $J + sB = B$.

Let m be a maximal ideal of B. If $s \in m$ then $m + J = B$. Now assume that $s \notin m$. To complete the proof, it is enough to show that $m + J = B$. Since $\alpha(X) \equiv 1$ modulo (sXg) and $(\alpha(X), \beta(X), p(X)) \in Um(A^2 \oplus P)$, we have $(\alpha(X), \beta(X), sXp(X)) \in Um(A^2 \oplus P)$. Let "$-$" denote reduction modulo the ideal $(mA, \beta(X))$. Note that $(\overline{\alpha(X)}, \overline{sXp(X)}) \in Um(\bar{A} \oplus \bar{P})$. Then it is easy to see that $g_1 \equiv 1$ modulo $(mA, \beta(X))$, where g_1 is of the form $\alpha(X) + \langle p_1', sXp(X)\rangle$ for some $p_1' \in P$. This shows that the ideals $(\alpha(X) + \langle p_1', sXp(X)\rangle, \beta(X))$ and mA are comaximal. Further $\alpha(X) +$

$\langle p_1', sXp(X)\rangle + d\beta(X) \in B[X]$ for some suitable $d \in A$. Clearly the ideals $(\alpha(X) + \langle p_1', sXp(X)\rangle + d\beta(X), \beta(X))$ and $(\alpha(X) + \langle p_1', sXp(X)\rangle, \beta(X))$ are equal. Therefore we can assume that $g_1 = \alpha(X) + \langle p', sXp(X)\rangle \in B[X]$ for some $p' \in P$ with $\langle p', sXp(X)\rangle \in B[X]$. Also we can take $q(X) = sXp' \in \sum B[X]e_i + \sum B[X]f_j$.

Let $p_1(X) = p(X) + \beta(X)q(X)$ and $\eta(X) = \alpha(X) + \langle q(X), p(X)\rangle$. Clearly $\eta(X) \in B[X]$ and $p_1(X) \in \sum B[X]e_i + \sum B[X]f_j$. Then $\eta(X) \equiv 1$ modulo (sX) and $mA + (\eta(X), \beta(X)) = A$. Moreover, for $b \in B$

$$\begin{aligned}\sigma_{(0,q(bX))}(\alpha(bX), \beta(bX), p(bX)) &= (\alpha(bX) + \langle q(bX), p(bX)\rangle, \beta(X),\\ &\quad p(bX) + \beta(bX)q(bX))\\ &= (\eta(bX), \beta(bX), p_1(bX)).\end{aligned}$$

Let $J_1 = B \cap (\eta(X)^2, \beta(X)^2)$. Since $mA + (\eta(X)^2, \beta(X)^2) = A$ and $\beta(X)^2$ is monic, as before we get $m + (\eta(X)^2, \beta(X)^2) \cap B = B$. Therefore $m + J_1 = B$. Let $t_1 \in J_1$. As before we see that $b, b' \in B$ such that $b - b' \in st_1B$ then there exists $\Phi \in G$ such that $\Phi(\eta(bX), \beta(bX), p_1(bX)) = (\eta(b'X), \beta(b'X), p_1(b'X))$. Therefore

$$\sigma_{(0,q(b'X))}{}^{-1} \circ \Phi \circ \sigma_{(0,q(bX))}(\alpha(bX), \beta(bX), p(bX)) = (\alpha(b'X), \beta(b'X), p(b'X)).$$

Hence $J_1 \subset J$ and thus $m + J = B$. Therefore $J = B$. This completes the proof. $\square$

Let B be a commutative Noetherian ring of dimension d. Let A be a ring such that $B[X] \subseteq A \subseteq S^{-1}B[X]$, where S denotes the multiplicatively closed set of monic polynomials in $B[X]$. Let P be any finitely generated projective A-module then P is regarded as a finitely generated projective D-module, where D is a subring of A generated by $B[X]$ and some finite number of elements of A. Moreover, we may assume that $D \subseteq B[X, \frac{1}{g}]$ for some $g \in S$. Therefore to prove Theorem A, we may assume that $(P, \langle , \rangle)$ is a symplectic $A = B[X, \frac{f_1}{g}, \ldots, \frac{f_l}{g}]$-module.

***Theorem* 3.9** — *Let B be a ring of dimension d and let $A = B[X, \frac{f_1}{g}, \ldots, \frac{f_l}{g}]$, where $g, f_i \in B[X]$ for $1 \leq i \leq l$ with g is monic. Let $(P, \langle , \rangle)$ be a symplectic A-module of rank $2r \geq d$, $r > 0$. Then $ESp(A^2 \perp P, \langle , \rangle)$ acts transitively on $Um(A^2 \oplus P)$.*

PROOF : Let $(a_1, a_2, p) \in Um(A^2 \oplus P)$. We want to show that there exists a $\Gamma \in ESp(A^2 \perp P, \langle , \rangle)$ such that $\Gamma(a_1, a_2, p) = (1, 0, 0)$. Without loss of generality we may assume that B is reduced. Let $\{e_1, \ldots, e_r, f_1, \ldots, f_r\} \subseteq P$ be s-symplectic basis of P and F be a A-submodule of P defined as in (3.7).

Since $s \in B$ is a non-zerodivisor, $A/(sXg)$ is a ring of dimension d. Moreover, since $2r \geq d$, By Bhatwadekar's result ([2], Theorem 4.8) and (2.2), there exists $\psi \in ESp(A^2 \perp P, \langle , \rangle)$ such that $\psi(a_1, a_2, p) \equiv (1, 0, 0)$ modulo (sXg). Hence we may assume that $(a_1, a_2, p) \equiv (1, 0, 0)$ modulo (sXg).

Since P is symplectic, we have $P \cong P^*$. Applying (3.2) to $(a_2, (a_1, p)) \in (A \oplus (A \oplus P)^*)$, there exists $(h, p_1) \in (A \oplus P)^*$ such that $ht(Aa_3 + I) \geq d + 1$, where $(a_3, p_2) = (a_1 + ha_2, p + a_2 p_1)$ and $I = p_2(P^*) = \langle P, p_2 \rangle$. Put $\alpha = a_3 + \langle p_1, p \rangle \in A$. Then

$$\sigma_{(h,p_1)}(a_1, a_2, p) = (a_1 + a_2 h + \langle p_1, p \rangle, a_2, p + a_2 p_1) = (\alpha, a_2, p_2).$$

Since $(a_3, a_2, p) \equiv (1, 0, 0)$ modulo $(sXg)A$ and $\alpha \equiv 1$ modulo $(sXg)A$. Moreover, since $\langle p_1, p_2 \rangle = \langle p_1, p \rangle \in I$, we have $(a_3, I)A = (\alpha, I)A$. Write $J = (\alpha, I)A$. Since J is an ideal of A of height $\geq d + 1$ and $J + gA = A$, we can apply (3.1), to the rings A, $B[X]$ and the ideal J. Furthermore, we can assume that J contains a polynomial of the form $1 + g(X)H$, say $\gamma(X)$ for some monic polynomial H in $B[X]$.

Let $\beta(X) = a_2 + \gamma(X)\gamma_1$ for some suitable $\gamma_1 \in A$ such that $\beta(X) \in B[X]$ and is of the form $1 + g(X)K$, for some monic polynomial $K \in B[X]$. Let $\gamma_1 \gamma(X) = \mu\alpha + \nu$ for some $\mu \in A$ and $\nu \in (sgX)I$. Since $I = \langle P, p_2 \rangle$, there exist $p_3 \in sgXP$ such that $\nu = -\langle p_3, p_2 \rangle = \langle p_2, p_3 \rangle$. Put $p_4 = p_2 + \alpha p_3$. Then

$$\Theta_{(\mu,p_3)}(\alpha, a_2, p_2) = (\alpha, a_2 + \mu\alpha + \langle p_2, p_3 \rangle, p_2 + \alpha p_3) = (\alpha, \beta(X), p_4).$$

Note that, $(\alpha, p_4) \equiv (1,0)$ modulo $(sXg)A$.

Since $sXgP \subset F$, write $p_4 = Xg(\sum_{i=1}^{r} h_i e_i + \sum_{j=1}^{r} k_j f_j)$ with $h_i, k_j \in A$ for $1 \leq i,j \leq r$. Let $h_1 = -\lambda g^{-k} + \tilde{h}_1$, where $\tilde{h}_1$ is an element of A such that the terms in $\tilde{h}_1$ have at most $k-1$ negative powers of g, and $\lambda \in B[X]$. Let $c_0 = (Xg)g^{-k}\lambda$. Then

$$\sigma_{(0,c_0 e_1)}(\alpha, \beta(X), p_4) = (\alpha + c_0\langle e_1, p_4\rangle, \beta(X), p_4 + \beta(X)c_0 e_1).$$

Note that if

$$p_4 + c_0\beta(X)e_1 = (Xg)((-\lambda g^{-k} + \tilde{h}_1 + \lambda g^{-k}\beta(X))e_1 + \sum_{i=2}^{r} h_i e_i + \sum_{j=1}^{r} k_j f_j),$$

then the coefficient of e_1 has less negative powers of g then h_1. Also note that $\alpha + c_0\langle e_1, p_4\rangle \equiv 1$ modulo $(sXg)A$. Hence by induction, applying such symplectic transformations one gets $\psi_1(\alpha, \beta(X), p_4) = (\alpha_1, \beta(X), p_5)$ with $p_5 \in \sum B[X]e_i + \sum B[X]f_j$. Now we write p_5 as $p_5(X)$. We still have $\alpha_1 \equiv 1$ modulo $(sXg)A$. Write $\Gamma_1 = \Psi_1\Theta_{(\mu,p_3)}\sigma_{(h,p_1)}$. Then $\Gamma_1(a_1, a_2, p) = (\alpha_1, \beta(X), p_5(x))$.

Since $\sigma_{(d,0)}(\alpha_1, \beta(X), p_5(X)) = (\alpha_1 + \beta(X)d, \beta(X), p_5(X))$, for $d \in A$, applying an appropriate such symplectic transformation say ψ_2, we may assume that $\psi_2(\alpha_1, \beta(X), p_5(X)) = (\alpha_2, \beta(X), p_5(X))$, where $\alpha_2 \in B[X]$ and $\alpha_2 \equiv 1$ modulo $(sXg)B[X]$. Now, we write α_2 as $\alpha_2(X)$. $(\alpha_2(X), \beta(X), p_5(X)) \in Um(A^2 \oplus P)$, with $\alpha_2(X), \beta(X) \in B[X]$, $p_5(X) \in \sum B[X]e_i + \sum B[X]f_j$, where $\beta(X) = 1 + gH$, $H \in B[X]$ is monic.

Since $\beta(X) = 1 + g(X)H$ with $H \in B[X]$ monic, $B/(B \cap (\alpha_2(X), \beta(X))) \hookrightarrow A/(\alpha_2(X), \beta(X))$ is an integral extension. Since $(\alpha_2(X), \beta(X)) + sA = A$, by (3.4), we have $B \cap (\alpha_2(X), \beta(X)) + sB = B$. Therefore there exists $c \in B$ such that $1 - sc \in B \cap (\alpha_2(X), \beta(X))$. Then, writing $b = 1$, $b' = 1 - sc$ and applying (3.8), $\exists \psi_3 \in SL_2(A, sX)ESp(A \perp P, \langle, \rangle)$ such that $\psi_3(\alpha_2(X), \beta(X), p_5(X)) = (\alpha_2(b'X), \beta(b'X), p_5(b'X))$. Since $\alpha_2(X) \equiv 1$ modulo (sX), we have $\alpha_2(b'X) \equiv 1$ modulo $(sb'X)$. Moreover, $b' = 1 - sc \in B \cap (\alpha_2(b'X), \beta(b'X))$, and we can even $b' \in B[X]\alpha_2(b'X) + B[X]\beta(b'X)$. Therefore $[\alpha_2(b'X), \beta(b'X)]$ is a unimodular row.

Let $\psi_3 = \Delta^{-1}\phi$, where $\Delta \in SL_2(A, sX)$ and $\phi \in ESp(A \perp P, \langle,\rangle)$. Let $\Delta(\alpha_2(b'X), \beta(b'X)) = (\alpha_3(X), \beta_1(X))$. Then $\phi(\alpha_2(X), \beta(X), p_5(X)) = (\alpha_3(X), \beta_1(X), p_5(b'X))$. Since $\Delta \in SL_2(A, sX)$, $\alpha_3(X) \equiv 1$ modulo (sX) and $[\alpha_3(X), \beta_1(X)]$ is a unimodular row. Write $\Gamma_2 = \phi \circ \psi_2 \circ \Gamma_1$ then $\Gamma_2 \in ESp(A^2 \perp P, \langle,\rangle)$ and $\Gamma_2(a_1, a_2, p) = (\alpha_3(X), \beta_1(X), p_5(b'X))$ with $[\alpha_3(X), \beta_1(X)]$ a unimodular row. Therefore by ([2], Lemma 4.3), there exists $\Phi_1 \in ESp(A^2 \perp P, \langle,\rangle)$ such that $\Phi_1(\alpha_3(X), \beta_1(X), p_5(b'X)) = (\alpha_3(X), \beta_1(X), e_1)$. Since $\langle e_1, f_1\rangle = s$, we have $(\alpha_3(X), e_1) \in Um(A \oplus P)$. Therefore by ([2], Lemma 4.4), there exists a $\Phi_2 \in ESp(A^2 \perp P, \langle,\rangle)$ such that $\Phi_2(\alpha_3(X), \beta_1(X), e_1) = (1, 0, 0)$. Take $\Gamma = \Phi_2 \circ \Phi_1 \circ \Gamma_2$. □

The following lemma is due to Bass ([1], Lemma 4.2).

Lemma 3.10 — Let B be a ring and let $A = B[X]$. Let I be an ideal in A and $L_X(I) = \{b \in B : bX^n + b_{n-1}X^{n-1} + \ldots + b_0 \in I \text{ for some } b_{n-1}, \ldots, b_0 \in B\}$. Then $L_X(I)$ is an ideal and $ht(L_X(I)) \geq ht(I)$.

Corollary 3.11 — Let B be a ring of dimension d and let $A = B[X]$. If I be an ideal in A of height $\geq d+1$, then I contains a monic polynomial.

Now we prove the following lemma which we will use to prove the next result.

Lemma 3.12 — Let B be a ring of dimension d and $A = B[X_1, \ldots, X_n, Y, \frac{1}{g}]$, where g is a monic polynomial in $B[Y]$. Let I be an ideal in A of height $\geq d+1$. Then there exists an automorphism σ of A such that $\sigma(I)$ contains a polynomial of the form $1 + gH$, where H is monic polynomial in Y over $B[X_1, \ldots, X_n]$.

PROOF : We proceed by induction on n. First we prove the statement for $n = 0$. Since $ht(I) \geq d+1$, we can see that I contains a gK for some monic polynomial $K \in B[Y]$. Since $I + gA = A$, applying (3.1) to $B[Y]$ and A with $x = g$, we get $1 + gh \in I$ for some $h \in B[Y]$. Suitable combination of gK and $1 + gh$ will give the required element.

Assume $n \geq 1$. Consider $L_{X_1}(I)$. By (3.10), $ht(L_{X_1}(I)) \geq ht(I)$. Therefore by the induction hypothesis, the ideal $L_{X_1}(I)$ contains a polynomial of the form

$1+gL$ for some monic polynomial L in Y over $B[X_2,\ldots,X_n]$. Therefore there exists $F(X_1)\in I$ such that $F(X_1)=(1+gL)X_1^m+g_1X_1^{m-1}+\ldots+g_m$, where $g_i\in B[X_2,\ldots,X_n,Y,\frac{1}{g}]$ for $1\le i\le m-1$. Let $\phi : A\to A$ be a B-algebra automorphism defined by

$\phi(X_1)=X_1+g^s-g^{-s}$, for some large s,

$\phi(X_i)=X_i$ for $i=2$ to n and

$\phi(Y)=Y$.

Choosing s large enough, $\phi(I)$ contains a polynomial $\phi(F(X_1))$ which is of the form $\frac{1}{g^s}+\ldots+Lg^{s+1}$. Clearly $g^s\phi(F(X_1))\in\phi(I)$ is a required element. This complets the proof. □

For the Laurent polynomial ring (i.e. $g=Y$), the following result is due to Keshari ([6], Theorem A.7). The proof of the following result essentially follows Keshari's proof of ([6], Theorem A.7).

***Theorem* 3.13** — *Let B be a ring of dimension d and let $A=B[X_1,\ldots,X_n,Y,\frac{1}{g}]$, where $g\in B[Y]$ is monic. Let $(P,\langle,\rangle)$ be a symplectic A-modules of rank $2r\ge d$, $r>0$. Then $ESp(A^2\perp P,\langle,\rangle)$ acts transitively on $Um(A^2\oplus P)$.*

PROOF : Let $(a_1,a_2,p)\in Um(A^2\oplus P)$. We want to show that there exists a $\Gamma\in ESp(A^2\perp P,\langle,\rangle)$ such that $\Gamma(a_1,a_2,p)=(1,0,0)$.

Without loss of generality we can assume that B is reduced. Let $\{e_1,\ldots,e_r, f_1,\ldots,f_r\}\subseteq P$ be s-symplectic basis of P and F be a A-submodule of P defined as in (3.7). Let $R=B[X_1\ldots,X_n]$.

Let $F_1=\sum_{i=1}^r R[Y]e_i+\sum_{j=1}^r R[Y]f_j$. Let P be generated by $\mu_1,\cdots,\mu_l$ as an A-module such that (1) the set $\mu_1,\cdots,\mu_l$ contains $e_1,\cdots,e_r,f_1,\cdots,f_r$, (2) $s\mu_i\in F_1$ for $1\le i\le l$ and (3) $\langle\mu_i,\mu_j\rangle\in R[Y]$ for $1\le i,j\le l$. Let $M=\sum R[Y]\mu_i$. Since $R[Y,\frac{1}{g}]=A$, we have $MA=P$ and $sM\subset F_1$.

Since $s\in B$ is a non-zerodivisor, $B_1=B[Y,\frac{1}{g}]/(s(g-g(0)))$ is a ring of dimension d and $\bar{A}=A/s(g-g(0))A=B_1[X_1,\ldots,X_n]$. Moreover, $rk(\bar{P})\ge d$, where bar denote reduction modulo the ideal $s(g-g(0))$. Therefore by Bhatwadekar's result (2.1) and (2.2), there exists $\psi \in ESp(A^2 \perp P, \langle , \rangle)$ such that $\psi(a_1, a_2, p) \equiv (1,0,0)$ modulo $s(g-g(0))A$.

Replacing (a_1, a_2, p) by $\psi(a_1, a_2, p)$, we may assume that $(a_1, a_2, p) \equiv (1,0,0)$ modulo $s(g-g(0))A$. Since P is symplectic, we have $P \cong P^*$, therefore applying (3.2) to $(a_2, (a_1, p)) \in (A \oplus (A \oplus P)^*)$, there exists $(h, p_1) \in (A \oplus P)^*$ such that $ht(Aa_3 + I) \geq d+1$, where $(a_3, p_2) = (a_1 + ha_2, p + a_2 p_1)$ and $I = p_2(P^*) = \langle P, p_2 \rangle$. Put $\alpha = a_3 + \langle p_1, p \rangle \in A$. Then

$$\sigma_{(h,p_1)}(a_1, a_2, p) = (a_1 + a_2 h + \langle p_1, p \rangle, a_2, p + a_2 p_1) = (\alpha, a_2, p_2).$$

Since $(a_3, a_2, p) \equiv (1,0,0)$ modulo $s(g-g(0))A$, we have $\alpha \equiv 1$ modulo $s(g-g(0))A$. Moreover, since $\langle p_1, p_2 \rangle = \langle p_1, p \rangle \in I$, we have $(a_3, I)A = (\alpha, I)A$. Write $J = (\alpha, I)A$. Then J is an ideal of A of height $\geq d+1$. By (3.12), after a suitable change of variable, we can assume that J contains a polynomial of the form $1 + gH$, say $\gamma(Y)$, for some monic polynomial $H \in R[Y]$.

Let $\beta(Y) = a_2 + \gamma(Y)\gamma_1$ for some suitable $\gamma_1 \in A$ such that $\beta(Y) \in R[Y]$ and is of the form $1 + gK$, for some monic polynomial $K \in R[Y]$. Let $\gamma_1\gamma(Y) = \mu\alpha + \nu$ for some $\mu \in A$ and $\nu \in s(g-g(0))I$. Since $I = \langle P, p_2 \rangle$, there exist $p_3 \in s(g-g(0))P$ such that $\nu = -\langle p_3, p_2 \rangle = \langle p_2, p_3 \rangle$. Put $p_4 = p_2 + \alpha p_3$. Then

$$\Theta_{(\mu,p_3)}(\alpha, a_2, p_2) = (\alpha, a_2 + \mu\alpha + \langle p_2, p_3 \rangle, p_2 + \alpha p_3) = (\alpha, \beta(Y), p_4).$$

Note that, $(\alpha, p_4) \equiv (1,0)$ modulo $s(g-g(0))A$.

Since $s(g-g(0))P \subset F$, write $p_4 = (g-g(0))(\sum_{i=1}^r h_i e_i + \sum_{j=1}^r k_j f_j)$ for some $h_i, k_j \in A$. Let $h_1 = -\lambda g^{-k} + \tilde{h}_1$, where $\tilde{h}_1$ is an element of A such that the terms in $\tilde{h}_1$ have at most $k-1$ negative powers of g and $\lambda \in R[Y]$. Let $c_0 = (g-g(0))g^{-k}\lambda$. Then

$$\sigma_{(0,c_0e_1)}(\alpha, \beta(Y), p_4) = (\alpha + c_0\langle e_1, p_4 \rangle, \beta(Y), p_4 + \beta(Y)c_0e_1).$$

Note that if

$$p_4+c_0\beta(Y)e_1=(g-g(0))((-\lambda g^{-k}+\tilde{h}_1+\lambda g^{-k}\beta(Y))e_1+\sum_{i=2}^{r}h_ie_i+\sum_{j=1}^{r}k_jf_j),$$

then the coefficient of e_1 has less negative powers of g then h_1. Also note that $\alpha+c_0\langle e_1,p_4\rangle\equiv 1$ modulo $s(g-g(0))A$. Hence by induction, applying such symplectic transformations one gets $\psi_1(\alpha,\beta(Y),p_4)=(\alpha_1,\beta(Y),p_5)$ with $p_5\in F_1$. Now we write p_5 as $p_5(Y)$. We still have $\alpha_1\equiv 1$ modulo $s(g-g(0))A$. Write $\Gamma_1=\Psi_1\Theta_{(\mu,p_3)}\sigma_{(h,p_1)}$. Then $\Gamma_1(a_1,a_2,p)=(\alpha_1,\beta(Y),p_5(Y))$.

Since $\sigma_{(d,0)}(\alpha_1,\beta(Y),p_5(Y))=(\alpha_1+\beta(Y)d,\beta(Y),p_5(Y))$, for $d\in A$, applying an appropriate such transformations, say ψ_2, we may assume that $\psi_2(\alpha_1,\beta(Y),p_5(Y))=(\alpha_2,\beta(Y),p_5(Y))$ with $\alpha_2\in R[Y]$ and $\alpha_2\equiv 1$ modulo $s(g-g(0))R[Y]$. Now, we write α_2 as $\alpha_2(Y)$. Since $\beta(Y)$ is of the form $1+gK$, where $K\in R[Y]$, $(\alpha_2(Y),\beta(Y),p_5(Y))\in Um(R[Y]^2\perp F_1,\langle,\rangle)$. Since $\alpha_2\equiv 1$ modulo $s(g-g(0))R[Y]$, we have $\alpha_2(Y)\equiv 1$ modulo $sYR[Y]$.

Since $\beta(Y)=1+gK$, where $K\in R[Y]$ monic, $R/(R\cap(\alpha_2(Y),\beta(Y)))\hookrightarrow A/(\alpha_2(Y),\beta(Y))$ is an integral extension. Since $(\alpha_2(Y),\beta(Y))+sA=A$, by (3.4), we have $R\cap(\alpha_2(Y),\beta(Y))+sR=R$. Therefore there exists $c\in R$ such that $1-sc\in R\cap(\alpha_2(Y),\beta(Y))$. Recall that $sM\subset F_1$. Therefore, writing $b=1,b'=1-sc$ and applying ([2], Proposition 4.7), there exists $\psi_3\in SL_2(R[Y],sY)ESp(R[Y]^2\perp M,\langle,\rangle)$ such that $\psi_3(\alpha_2(Y),\beta(Y),p_5(Y))=(\alpha_2(b'Y),\beta(b'Y),p_5(b'Y))$. Since $\alpha_2(Y)\equiv 1$ modulo $sYR[Y]$, we have $\alpha_2(b'Y)\equiv 1$ modulo $(sb'Y)R[Y]$. Moreover $b'=1-cs\in B\cap(\alpha_2(b'Y),\beta(b'Y))$. Therefore $[\alpha_2(b'Y),\beta(b'Y)]$ is a unimodular row.

Let $\psi_3=\Delta^{-1}\phi$, where $\Delta\in SL_2(R[Y],sY)$ and $\phi\in E(R[Y]^2\perp M,\langle,\rangle)$. Let $\Delta(\alpha_2(b'Y),\beta(b'Y))=(\alpha_3(Y),\beta_1(Y))$. Then

$$\phi(\alpha_2(b'Y),\beta(b'Y),p_5(Y))=(\alpha_3(Y),\beta_1(Y),p_5(b'Y)).$$

Since $\Delta\in SL_2(R[Y],sY)$, we have $\alpha_3(Y)\equiv 1$ modulo $(sY)R[Y]$ and $[\alpha_3(Y),\beta_1(Y)]$ is a unimodular row. Write $\Gamma_2=(\phi\otimes A)\psi_2\Gamma_1$ then $\Gamma_2\in ESp(A^2\perp$

$P, \langle , \rangle)$ and $\Gamma_2(a_1, a_2, p) = (\alpha_3(Y), \beta_1(Y), p_5(b'Y))$ with $[\alpha_3(Y), \beta_1(Y)]$ a unimodular row. Therefore by ([2], Lemma 4.3), there exists $\Phi_1 \in ESp(A^2 \perp P, \langle , \rangle)$ such that $\Phi_1(\alpha_3(Y), \beta_1(Y), p_5(b'Y)) = (\alpha_3(Y), \beta_1(Y), e_1)$. Since $\langle e_1, f_1 \rangle = s$, we have $(\alpha_3(Y), e_1)$ is an element of $Um(A \oplus P)$. Therefore by ([2], Lemma 4.4), there exist a $\Phi_2 \in ESp(A^2 \perp P, \langle , \rangle)$ such that $\Phi_2(\alpha_3(X), \beta_1(X), e_1) = (1, 0, 0)$. Let $\Gamma = \Phi_2 \circ \Phi_1 \circ \Gamma_2$. □

4. Projective Modules Over Overrings of Two-Dimensional Rings

We begin by stating the following result which is similar to ([2], Lemma 5.2) and whose proof is essentially contained in ([2], Lemma 5.2).

Lemma 4.1 — Let B be a ring of dimension 2 and let $A = B[X_1, \ldots, X_n, Y, \frac{1}{g}]$, where g is a monic polynomial in $B[Y]$. Let P and Q be two projective A-modules of rank 2 having trivial determinant such that $A^n \oplus P \cong A^n \oplus Q$. Let ω_P and ω_Q be generators of $\wedge^2(P)$ and $\wedge^2(Q)$ respectively. If A^2 is cancellative then there exist co-maximal ideals I and I_1 of A with ht$(I) = 2$ and ht$(I_1) = 2$ or $I_1 = A$, there exist surjections $\alpha : P \twoheadrightarrow I$ and $\beta : Q \twoheadrightarrow I \cap I_1$ and there exist $\theta : A^2 \twoheadrightarrow I_1$ and isomorphisms $\delta : Q/IQ \to P/IP$ and $\delta_1 : Q/I_1Q \to (A/I_1)^2$ such that

(i) $(\alpha \otimes A/I)\delta = \beta \otimes A/I$.

(ii) $(\theta \otimes A/I_1)\delta_1 = \beta \otimes A/I_1$.

(iii) $\wedge^2(\delta)(\omega_Q \otimes A/I) = \omega_P \otimes A/I$.

(iv) $\wedge^2(\delta_1)(\omega_Q \otimes A/I_1) = (e_1 \wedge e_2) \otimes A/I_1$, where $e_1 = (1, 0)$, $e_2 = (0, 1)$.

Now we state the following result whose proof is implicit in ([4], Theorem 7.2) and ([2], Proposition 5.3).

Proposition 4.1 — Let A be a ring and let P be a projective A-module of rank 2 with trivial determinant. Let ω_P be a generator of $\wedge^2(P)$. Let $\alpha : P \twoheadrightarrow I$ and $\theta : A^2 \twoheadrightarrow I_1$ be surjections, where I and I_1 are ideals of height 2 which are co maximal. Then there exists a projective A-module P_1 of rank 2 with trivial determinant, a generator ω_1 of $\wedge^2(P_1)$, a surjection $\beta_1 : P_1 \twoheadrightarrow I \cap I_1$ and maps $\lambda_1 : P_1 \to P$ and $\lambda_2 : P_1 \to A^2$ such that

(i) $\alpha\lambda_1 = \beta_1 = \theta\lambda_2$.

(ii) $\wedge^2\lambda_1(\omega_1) = u\omega_P$ with $u - 1 \in I$.

(iii) $\wedge^2\lambda_2(\omega_1) = \nu(e_1 \wedge e_2)$ with $\nu - 1 \in I_1$, where $e_1 = (1,0)$, $e_2 = (0,1)$.

(iv) $(A^2, (e_1 \wedge e_2)) \perp (P, \omega_P) \cong (A^2, e_1 \wedge e_2) \perp (P_1, \omega_1)$.

For the Laurent polynomial ring (i.e. $g = Y$), the following result is due to Keshari ([6], Theorem A.7). The proof closely follows ([6], Theorem A.7).

***Theorem* 4.3** — *Let B be a ring of dimension 2 and $A = B[X_1, \ldots, X_n, Y, \frac{1}{g}]$, where g is a monic polynomial in $B[Y]$. Let P be a projective A-module of rank 2 with trivial determinant. If A^2 is cancellative, then P is cancellative.*

PROOF : Let Q be a projective A-module of rank 2 such that $A \oplus P \cong A \oplus Q$. We want to show that $P \cong Q$. Choose I, I_1, β, θ, δ and δ_1 as in (4.1). Choose corresponding $P_1, \omega_1, \beta_1, \lambda_1$ and λ_2 as in (4.2). By (3.13), the symplectic modules (P, ω_P) and (P_1, ω_1) are isomorphic.

Let $K = I \cap I_1$. Then we have surjections $\beta_1 : P_1 \twoheadrightarrow K$ and $\beta : Q \twoheadrightarrow K$. Let $\tilde{\Gamma} : P_1/KP_1 \tilde{\to} Q/KQ$ be an isomorphism defined as $\tilde{\Gamma} \otimes A/I = \delta^{-1}(\lambda_1 \otimes A/I)$ and $\tilde{\Gamma} \otimes A/I_1 = \delta_1^{-1}(\lambda_2 \otimes A/I_2)$. Then it is easy to see that $\wedge^2(\tilde{\Gamma})(\omega_1 \otimes A/K) = \omega_Q \otimes A/K$ and $(\beta \otimes A/K)\tilde{\Gamma} = \beta_1 \otimes A/K$. Therefore, by ([2], Lemma 3.5), the symplectic modules (P_1, ω_1) and (Q, ω_Q) are isomorphic. Hence there exists an isomorphism $\Delta : P \tilde{\to} Q$ such that $\wedge^2(\Delta)(\omega_P) = \omega_Q$. $\square$

Proposition 4.4 — Let B be a smooth affine domain of dimension 2 over an algebraically closed field k of characteristic 0. Let $A = B[X_1, \ldots, X_n, Y, \frac{1}{f}]$, where f is a monic polynomial in $B[Y]$. Then A^2 is cancellative.

PROOF : Take f instead of Y in the proof of ([6], Proposition A.9). $\square$

As an application of (4.3), we get the following result, which is proved in ([5], Corollary 4.4).

Corollary 4.5 — Let B be a smooth affine domain of dimension 2 over an algebraically closed field k of characteristic 0. Let $A = B[X_1, \ldots, X_n, Y, \frac{1}{f}]$, where f is a monic polynomial in $B[Y]$. Then every projective A-module of rank 2 with trivial determinant is cancellative.

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