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HIGH-ORDER SCHWARZ-PICK LEMMA FOR THE SCHUR CLASS ON THE POLYDISC[1]

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In this paper, we give Schwarz-Pick Lemma for functions in the Schur class on the polydisc of $\mathbb{C}^n$, and generalize some early work of Schwarz-Pick Lemma for functions in the Schur class on the unit disk of $\mathbb{C}$ and functions in the Schur-Agler class on the polydisc of $\mathbb{C}^n$.

Key words : Schwarz-Pick estimate; holomorphic function; polydisc.

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1. INTRODUCTION

The class of holomorphic functions which are bounded by one on the unit disk $\mathbf{D} = \{z : |z| < 1\}$ is often called the Schur class on $\mathbf{D}$. For $\varphi(z)$ in the Schur class on $\mathbf{D}$, the classical Schwarz-Pick estimate is the inequality

$$|\varphi'(z)| \leq \frac{1 - |\varphi(z)|^2}{1 - |z|^2}, \ |z| < 1.$$

There were many results on Schwarz-Pick estimates for holomorphic function on $\mathbf{D}$, see [1-4]. The following estimate of higher order derivatives for $\varphi(z)$ in the Schur class on $\mathbf{D}$ was given by:

Theorem **A**[1, 4] — *If $\varphi(z)$ is in the Schur class on* ***D****, then*

$$|\varphi^{(m)}(z)| \leq \frac{m!(1 - |\varphi(z)|^2)}{(1 - |z|^2)^m}(1 + |z|)^{m-1}.$$

For notation, let $\mathbf{D}^n$ be the polydisc of $\mathbb{C}^n$. The Schur class on $\mathbf{D}^n$ is the set of holomorphic functions which are defined and bounded by one on $\mathbf{D}^n$. Anderson, Dritschel, and Rovnyak estimated derivatives of arbitrary order of functions in the Schur-Agler class on the polydisc of $\mathbb{C}^n$ in [5]. For $n = 1$ and $n = 2$, the Schur-Agler class coincides with the Schur class on $\mathbf{D}^n$. For $n > 2$, the Schur-Agler class is a proper subset of the Schur class on $\mathbf{D}^n$ (for example, see [5, 6]). Moreover, for the holomorphic functions on the unit ball and classical domain in several complex variables, we have presented the estimates for their high-order derivatives, see [7, 8]. In this paper, we will obtain estimates of higher order derivatives for functions in the Schur class on the polydisc of $\mathbb{C}^n$. Before we give the main results, we recall some commonly used notations for multi-indices.

A multi-index $v = (v_1, \cdots, v_n)$ consists of n nonnegative integers $v_i, \ 1 \leq i \leq n$. The degree of a multi-index is the sum $|v| = \sum_{i=1}^{n} v_i$. For vectors $z = (z_1, \cdots, z_n) \in \mathbb{C}^n$, $||z||_\infty = \max_{1 \leq i \leq n} |z_i|$, and the multi-indices can be used as exponents in a product $z^v = \prod_{i=1}^{n} z_i^{v_i}$; similarly, a_v represents the coefficient $a_{v_1, \cdots, v_n}$ of z^v in the Taylor expansion of a holomorphic function. Let $\mathcal{S}(\mathbf{D}^n)$ be the Schur class on $\mathbf{D}^n$. Then we have the following results.

***Theorem* 1.1** — *Let $\varphi(z) \in \mathcal{S}(\mathbf{D}^n)$. Then for multi index $m = (m_1, \cdots, m_n)$, such that $m_i > 0,\ i = 1, \cdots, n$,*

$$\begin{aligned} |\partial^m \varphi(z)| &\leq \prod_{i=1}^{n} m_i! \frac{1 - |\varphi(z)|^2}{(1 - |z_i|^2)^{m_i}} \prod_{i=1}^{n} (1 + |z_i|)^{m_i - 1} \\ &\leq |m|! \frac{1 - |\varphi(z)|^2}{(1 - ||z||_\infty^2)^{|m|}} (1 + ||z||_\infty)^{|m| - n}, \end{aligned}$$

where $\partial^m \varphi(z) = \frac{\partial^{|m|} \varphi(z)}{\partial z_1^{m_1} \cdots \partial z_n^{m_n}}$.

Remark 1.2 : This result reduces to Theorem A as $n = 1$.

More generally, denote $0! = 1$, we have the following corollary.

Corollary 1.3 — Let $\varphi(z) \in \mathcal{S}(\mathbf{D}^n)$. For multi index $m = (m_1, \cdots, m_n)$, such that for some integer $0 < k < n$, there are k indexes $m_i = 0$, without loss of generality, $m_1 = \cdots = m_k = 0$ and other $m_j > 0,\ k < j \leq n$. Then

$$\begin{aligned} |\partial^m \varphi(z)| &\leq \prod_{i=1}^{n} m_i! \frac{1 - |\varphi(z)|^2}{(1 - |z_i|^2)^{m_i}} \prod_{i=k+1}^{n} (1 + |z_i|)^{m_i - 1} \\ &\leq |m|! \frac{1 - |\varphi(z)|^2}{(1 - ||z||_\infty^2)^{|m|}} (1 + ||z||_\infty)^{|m| - n + k}, \end{aligned}$$

where $\partial^m \varphi(z) = \frac{\partial^{|m|} \varphi(z)}{\partial z_1^{m_1} \cdots \partial z_n^{m_n}}$.

Remark 1.4 : It is easy to see that Corollary 1.3 is more general than Theorem 7 of [5], since we consider the Schur class on $\mathbf{D}^n$ here while [5] only discussed the Schur-Agler class on $\mathbf{D}^n$.

From the above results, an explicit bound of Corollary 4.3 in [9] can be deduced.

Corollary 1.5 — Let $\varphi(z) \in \mathcal{S}(\mathbf{D}^n)$. Then for multi index $m = (m_1, \cdots, m_n)$, such that $m_i \geq 0,\ i = 1, \cdots, n$,

$$\sup_{z \in \mathbf{D}^n} \frac{|\partial^m \varphi(z)| \prod_{i=1}^{n} (1 - |z_i|^2)^{m_i}}{1 - |\varphi(z)|^2} \leq \prod_{i=1}^{n} m_i! 2^{|m| - n + k},$$

where $\partial^m \varphi(z) = \frac{\partial^{|m|}\varphi(z)}{\partial z_1^{m_1}\cdots\partial z_n^{m_n}}$ and k is the number of $m_i = 0$.

2. A LEMMA

Lemma 2.1 — Let $\varphi(z) \in \mathcal{S}(\mathbf{D}^n)$ and

$$\varphi(z) = \sum_{u=0}^{\infty} \sum_{|v|=u} a_v z^v. \tag{1}$$

Then we have $|a_v| \leq 1 - |a_0|^2$.

PROOF : For any $\zeta = (\zeta_1, \cdots, \zeta_n) \in \mathbf{D}_n$, then $\zeta\theta := (\zeta_1 e^{i\theta_1}, \cdots, \zeta_n e^{i\theta_n}) \in \mathbf{D}^n$, for $\theta_i \in \mathbb{R},\ i = 1, \cdots, n$. By the orthogonality we have

$$\begin{aligned} 1 &> \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} |\varphi(\zeta_1 e^{i\theta_1}, \cdots, \zeta_n e^{i\theta_n})|^2 d\theta_1 \cdots d\theta_n \\ &= \sum_{u=0}^{\infty} \sum_{|v|=u} |a_v|^2 |\zeta^v|^2. \end{aligned}$$

Let ζ be close to $\partial\mathbf{D}_n$, then

$$\sum_{u=0}^{\infty} \sum_{|v|=u} |a_v|^2 \leq 1. \tag{2}$$

On the other hand, let $\psi(z) = \frac{1}{k}\sum\limits_{l=1}^{k} \varphi(e^{\frac{l}{k}2\pi i} z)$, then $\psi(z) \in \mathcal{S}(\mathbf{D}^n)$, $\psi(0) = a_0$ and

$$\psi(z) = a_0 + \sum_{u=1}^{\infty} \sum_{|v|=uk} a_v z^v. \tag{3}$$

Furthermore, let $\phi(z) = \frac{\psi(z) - a_0}{1 - \bar{a}_0 \psi(z)}$. Obviously, $\phi(z) \in \mathcal{S}(\mathbf{D}^n)$ and $\phi(0) = 0$. From (3), we have

$$\phi(z) = \frac{\sum\limits_{u=1}^{\infty} \sum\limits_{|v|=uk} a_v z^v}{1 - \bar{a}_0 \psi(z)} = \frac{1}{1 - |a_0|^2} \sum_{|v|=k} a_v z^v + \sum_{l=2}^{\infty} \sum_{|v|=lk} d_v z^v.$$

Let $b_v = \frac{1}{1-|a_0|^2} a_v$, then from (2),

$$\sum_{|v|=k} |b_v|^2 = \sum_{|v|=k} |\frac{1}{1-|a_0|^2} a_v|^2 \leq 1.$$

Especially, we have $|a_v| \leq 1 - |a_0|^2$.

Remark 2.3 : It is easy to see that Lemma 2.1 generalizes Corollary 9 of [5] to the Schur class on the polydisc.

The estimate of Lemma 2.1 turns out to be sharp by the following example:

Example 2.3 : Let $f(z,w) = z^k w^l$, where $k, l \in \mathbb{Z}^+$. Obviously, it is holomorphic in $\mathbf{D}^2$, besides, $|z^k w^l| \leq 1$, so that $|f(z,w)| < 1$ in $\mathbf{D}^2$. Using Lemma 2.1, we have $|a_v| \leq 1 - |a_0|^2$. In this example $n = 2, |v| = k + l, a_0 = f(0,0) = 0$. Hence, we have $|a_v| \leq 1$. On the other hand, $a_{k,l} = 1$ in $f(z,w)$, which means that the estimate of Lemma 2.1 is sharp.

3. The Proof of Theorem 1.1

PROOF : Let $\tau(z) \in Aut(\mathbf{D}^n)$, where $Aut(\mathbf{D}^n)$ is the automorphic group of $\mathbf{D}^n$. By the representation of automorphism $\mathbf{D}^n$ [**?**], we have

$$\tau : (z_1, \cdots, z_n) \to (\tau_1(z), \cdots, \tau_n(z)),$$

$$\tau_i(z) = e^{i\theta_i} \frac{z_{p(i)} - \zeta_i}{1 - \bar{\zeta}_i z_i},\ i = 1, \cdots, n,$$

where $\theta_i \in \mathbb{R}$, p is permutations of $\{1, \cdots, n\}$ and $|\zeta_i| < 1$ for $i = 1, \cdots, n$.

Especially, let $p = id$, $\theta_i = 0,\ i = 1, \cdots, n$. Then

$$\tau_i(z) = \frac{z_i - \zeta_i}{1 - \bar{\zeta}_i z_i},\ i = 1, \cdots, n.$$

For $\varphi(z) \in \mathcal{S}(\mathbf{D}^n)$, we set

$$F(z) := \varphi(\tau^{-1}(z)) = \sum_{u=0}^{\infty} \sum_{|v|=u} c_v z^v,$$

then

$$\varphi(z) = F(\tau(z)) = \sum_{u=0}^{\infty} \sum_{|v|=u} c_v \prod_{i=1}^{n} (\frac{z_i - \zeta_i}{1 - \bar{\zeta}_i z_i})^{v_i}.$$

By computation, for multi index $m = (m_1, \cdots, m_n)$,

$$\frac{d^{m_i}}{dz_i^{m_i}} (\frac{z_i - \zeta_i}{1 - \bar{\zeta}_i z_i})^{v_i}|_{z_i = \zeta_i} = \begin{cases} 1, & m_i = v_i = 0, \\ 0, & m_i < v_i \text{ or } m_i > v_i = 0, \\ \frac{(\bar{\zeta}_i)^{m_i - v_i}}{(1-|\zeta_i|^2)^{m_i}} \frac{m_i!(m_i-1)!}{(m_i - v_i)!(v_i - 1)!}, & m_i \geq v_i > 0. \end{cases}$$

Hence, from above equation, if $|m| \geq |v|, m_i > 0,\ i = 1, \cdots, n$, we have

$$\begin{aligned} \frac{\partial^{|m|} \varphi(\zeta)}{\partial z_1^{m_1} \cdots \partial z_n^{m_n}} &= \sum_{u=0}^{|m|} \sum_{|v|=u} c_v \prod_{i=1}^{n} \frac{(\bar{\zeta}_i)^{m_i - v_i}}{(1 - |\zeta_i|^2)^{m_i}} \frac{m_i!(m_i - 1)!}{(m_i - v_i)!(v_i - 1)!} \\ &= \sum_{|v|=n, v_i > 0}^{|m|} c_v \prod_{i=1}^{n} \frac{(\bar{\zeta}_i)^{m_i - v_i}}{(1 - |\zeta_i|^2)^{m_i}} \frac{m_i!(m_i - 1)!}{(m_i - v_i)!(v_i - 1)!}, \end{aligned}$$

where $\sum\limits_{|v|=n, v_i>0}^{|m|}$ means that $|v| = \sum_{i=1}^{n} v_i$ sums from n to $|m|$ with every $v_i > 0$ for $i = 1, 2, \cdots, n$. Note that $c_0 = \varphi(\zeta)$, so by Lemma 2.1, we have

$$|c_v| \leq 1 - |c_0|^2 = 1 - |\varphi(\zeta)|^2.$$

Therefore

$$
\begin{aligned}
\left|\frac{\partial^{|m|}\varphi(\zeta)}{\partial z_1^{m_1}\cdots\partial z_n^{m_n}}\right| &\le \sum_{|v|=n,v_i>0}^{|m|}|c_v|\prod_{i=1}^{n}\frac{|\zeta_i|^{m_i-v_i}}{(1-|\zeta_i|^2)^{m_i}}\frac{m_i!(m_i-1)!}{(m_i-v_i)!(v_i-1)!}\\
&=\prod_{i=1}^{n}\frac{m_i!}{(1-|\zeta_i|^2)^{m_i}}\sum_{|v|=n,v_i>0}^{|m|}|c_v|\prod_{i=1}^{n}\frac{|\zeta_i|^{m_i-v_i}(m_i-1)!}{(m_i-v_i)!(v_i-1)!}\\
&\le\prod_{i=1}^{n}\frac{m_i!(1-|\varphi(\zeta)|^2)}{(1-|\zeta_i|^2)^{m_i}}\sum_{|v|=n,v_i>0}^{|m|}\prod_{i=1}^{n}\frac{|\zeta_i|^{m_i-v_i}(m_i-1)!}{(m_i-v_i)!(v_i-1)!}\\
&\le\prod_{i=1}^{n}\frac{m_i!(1-|\varphi(\zeta)|^2)}{(1-|\zeta_i|^2)^{m_i}}\prod_{i=1}^{n}\sum_{v_i=1}^{m_i}\frac{|\zeta_i|^{m_i-v_i}(m_i-1)!}{(m_i-v_i)!(v_i-1)!}\\
&=\prod_{i=1}^{n}\frac{m_i!(1-|\varphi(\zeta)|^2)}{(1-|\zeta_i|^2)^{m_i}}\prod_{i=1}^{n}\sum_{l_i=0}^{m_i-1}\frac{(m_i-1)!}{l_i!(m_i-l_i-1)!}|\zeta_i|^{l_i}\\
&=\prod_{i=1}^{n}\frac{m_i!(1-|\varphi(\zeta)|^2)}{(1-|\zeta_i|^2)^{m_i}}\prod_{i=1}^{n}\sum_{l_i=0}^{m_i-1}\begin{pmatrix} m_i-1\\ l_i\end{pmatrix}|\zeta_i|^{l_i}\\
&=\prod_{i=1}^{n}\frac{m_i!(1-|\varphi(\zeta)|^2)}{(1-|\zeta_i|^2)^{m_i}}\prod_{i=1}^{n}(1+|\zeta_i|)^{m_i-1}.
\end{aligned}
$$

At last, replace ζ with z, we prove Theorem 1.1. □

4. THE PROOF OF COROLLARY 1.3

PROOF : If $|m|\ge|v|, m_i\ge 0,\ i=1,\cdots,n,$ without loss of generality, we assume $m_1=0, m_i>0, i=2,\cdots,n$. Then

$$
\begin{aligned}
\frac{\partial^{|m|}\varphi(\zeta)}{\partial z_2^{m_2}\cdots\partial z_n^{m_n}} &=\sum_{u=0}^{|m|}\sum_{|v|=u,v_1=0}c_v\prod_{i=2}^{n}\frac{(\bar{\zeta}_i)^{m_i-v_i}}{(1-|\zeta_i|^2)^{m_i}}\frac{m_i!(m_i-1)!}{(m_i-v_i)!(v_i-1)!}\\
&=\sum_{|v|=n-1,v_1=0,v_i>0(i\neq 1)}^{|m|}c_v\prod_{i=2}^{n}\frac{(\bar{\zeta}_i)^{m_i-v_i}}{(1-|\zeta_i|^2)^{m_i}}\frac{m_i!(m_i-1)!}{(m_i-v_i)!(v_i-1)!}.
\end{aligned}
\tag{4}
$$

Using Lemma 2.1 again, we have for $|v| > 0$,

$$|c_v| \leq 1 - |c_0|^2 = 1 - |\varphi(\zeta)|^2.$$

Therefore, by the similar computation with (4),

$$\begin{aligned}
|\frac{\partial^{|m|}\varphi(\zeta)}{\partial z_2^{m_2}\cdots\partial z_n^{m_n}}| &\leq \sum_{|v|=n-1, v_1=0, v_i>0(i\neq 1)}^{|m|} |c_v| \prod_{i=2}^{n} \frac{|\zeta_i|^{m_i-v_i}}{(1-|\zeta_i|^2)^{m_i}} \frac{m_i!(m_i-1)!}{(m_i-v_i)!(v_i-1)!} \\
&\leq \prod_{i=2}^{n} \frac{m_i!(1-|\varphi(\zeta)|^2)}{(1-|\zeta_i|^2)^{m_i}} \prod_{i=2}^{n} \sum_{v_i=1}^{m_i} \frac{|\zeta_i|^{m_i-v_i}(m_i-1)!}{(m_i-v_i)!(v_i-1)!} \\
&= \prod_{i=2}^{n} \frac{m_i!(1-|\varphi(\zeta)|^2)}{(1-|\zeta_i|^2)^{m_i}} \prod_{i=2}^{n} \sum_{l_i=0}^{m_i-1} \frac{(m_i-1)!}{l_i!(m_i-l_i-1)!} |\zeta_i|^{l_i} \\
&= \prod_{i=2}^{n} \frac{m_i!(1-|\varphi(\zeta)|^2)}{(1-|\zeta_i|^2)^{m_i}} \prod_{i=2}^{n} \sum_{l_i=0}^{m_i-1} \begin{pmatrix} m_i-1 \\ l_i \end{pmatrix} |\zeta_i|^{l_i} \\
&= \prod_{i=2}^{n} \frac{m_i!(1-|\varphi(\zeta)|^2)}{(1-|\zeta_i|^2)^{m_i}} \prod_{i=2}^{n} (1+|\zeta_i|)^{m_i-1}.
\end{aligned} \tag{5}$$

At last, replace ζ with z.

For multi index $m = (m_1, \cdots, m_n)$, such that there are k indexes $m_i = 0$, without loss of generality, we assume $m_1 = \cdots = m_k = 0$ and other $m_j > 0,\ k < j \leq n$. Denote $0! = 1$. From (5), by induction,

$$|\partial^m \varphi(z)| \leq \prod_{i=1}^{n} \frac{m_i!(1-|\varphi(\zeta)|^2)}{(1-|\zeta_i|^2)^{m_i}} \prod_{i=k+1}^{n} (1+|\zeta_i|)^{m_i-1},$$

where $\partial^m\varphi(z) = \frac{\partial^{|m|}\varphi(z)}{\partial z_1^{m_1}\cdots\partial z_n^{m_n}}$. The proof of Corollary 1.3 is completed. □

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