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ON THE OSCILLATORY BEHAVIOUR OF A CLASS OF NONLINEAR DELAY DIFFERENCE EQUATIONS OF SECOND ORDER[1]

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In this paper, sufficient conditions have been obtained for oscillation of all solutions of a class of nonlinear neutral delay difference equations of the form

$$\Delta(r(n)\Delta(y(n)+p(n)y(n-m)))+q(n)G(y(n-k))=0$$

under various ranges of $p(n)$. The nonlinear function $G, G \in C(\mathbb{R},\mathbb{R})$ is either sublinear or superlinear satisfying the condition $xG(x)>0$ for $x \neq 0$.

Key words : Nonlinear, Neutral difference equations, Oscillatory, Sublinear, Superlinear.

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1. Introduction

The study of oscillatory and asymptotic behaviour of all solutions of first and higher order neutral functional differential and difference equations is of great interest, since the last three decades. For the recent contribution, we refer the reader to ([3], [4], [5], [6], [9], [10]) and the references cited therein. It is observed that very few papers on oscillatory behaviour of solutions of higher order nonlinear neutral functional equations are available to meet our growing interest of research.

In [7], authors Parhi and Tripathy have considered a class of nonlinear neutral delay difference equation of the form

$$\Delta^m[y(n)+p(n)y(n-\tau)]+q(n)G(y(n-k))=0, \qquad (E)$$

where $m \geq 2$. They have studied (E) under the sublinear nature of G with different ranges of $p(n)$. However, the study of (E) with superlinear G is still unpublished. Meanwhile, in a very recent work [8], the author have predicted sufficient conditions for oscillation of all solutions of (E) with $m = 2$. But, the prototype of G is very much restricted for superlinear.

In this paper, authors have studied the second order nonlinear neutral delay difference equations of the form

$$\Delta[r(n)\Delta(y(n)+p(n)y(n-m))]+q(n)G(y(n-k))=0, n \geq 0, \qquad (1.1)$$

where Δ is the forward difference operator defined by $\Delta y(n) = y(n+1) - y(n), p, q, r,$ are real valued functions defined on $N(0) = \{0, 1, 2, 3, ...\}$ such that $q(n) \geq 0, r(n) > 0, G \in C(\mathbb{R}, \mathbb{R})$ is non decreasing and $xG(x) > 0$ for $x \neq 0$ and $m > 0, k \geq 0$ are integers. An attempt is made here to establish sufficient conditions under which every solution of equation (1.1) oscillates.

If $r(n) \equiv 1$ and $m = 2$, Eq.(1.1) is a particular case of (E). However, if $r(n) \not\equiv 1$, the Eq.(1.1) can not be viewed as the particular case of (E) when $m = 2$. Hence study of (1.1) is interesting. Study of oscillatory behaviour of solutions of (1.1), under the sublinear and superlinear nature of G is our prime interest.

By a solution of Eq.(1.1) we mean, a real valued function $y(n)$ defined on $N(-\rho) = \{-\rho, -\rho+1, ...\}$ which satisfies (1.1) for $n \geq 0$, where $\rho = max\{k, m\}$. If

$$y(n) = A_n, n = -\rho, -\rho+1, ..., 0 \qquad (1.2)$$

are given, then (1.1) admits a unique solution satisfying the initial condition (1.2). A solution $y(n)$ of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise, it is called nonoscillatory. In otherwords, a solution $y(n)$ of (1.1) is said to be oscillatory, if for every integer $N > 0$, there exists an $n \geq N$ such that $y(n)y(n+1) \leq 0$.

We need the following result for our use in the next section.

***Theorem* A** [2] — *If* $q(n) \geq 0$ *for* $n \geq 0$ *and*

$$\liminf_{n\to\infty} \sum_{s=n-k}^{n-1} q(s) > \left(\frac{k}{k+1}\right)^{k+1},$$

then $\Delta x(n)+q(n)x(n-k) \leq 0, n \geq 0$ *cannot have an eventually positive solution.*

2. Sublinear Oscillation

In this section, we have obtained the sufficient conditions for the oscillation of all solutions of Eq.(1.1) when G is sublinear. The following assumptions are useful in the sequel.

(A_1) $G(u)G(v) \geq G(uv)$ for $u > 0, v > 0$.

(A_2) $G(-u) = -G(u)$, for $u \in \mathbb{R}$.

(A_3) There exists $\lambda > 0$ such that $G(u) + G(v) \geq \lambda G(u+v), u \in \mathbb{R}, v \in \mathbb{R}$.

(A_4) $\int_0^{\pm c} \frac{du}{G(u)} < \infty, c > 0$.

(A_5) $\liminf_{|x|\to 0} \frac{G(x)}{x} \geq \gamma > 0$.

(A_6) $\sum_{n=m}^{\infty} R(n) = \infty = \sum_{n=m}^{\infty} Q(n)$,

where $R(n) = min\{q(n), q(n-m)\}, Q(n) = min\left\{\frac{q(n)}{r(n+1)}, \frac{q(n-m)}{r(n-m+1)}\right\}$.

(A_7) $\sum_{n=0}^{\infty} S(n) = \infty$, where $S(n) = min\left\{\frac{q(n)}{r(n)}, \frac{q(n)}{r(n+1)}\right\}$.

(A_8) $\sum_{n=0}^{\infty} q(n) = \infty$.

(A_9) $\liminf_{n\to\infty} \sum_{s=n+m-k}^{n-1} \frac{q(s)}{r(s)} > \frac{b}{\gamma}\left(\frac{k-m}{k-m+1}\right)^{k-m+1}, b > 0$ and $k > m$.

***Theorem* 2.1** — *Let $0 \le p(n) < a < \infty$. Suppose that $(A_1)-(A_4)$, and (A_6) hold. Then every solution of Eq.(1.1) oscillates.*

PROOF : Let $y(n)$ be a nonoscillatoy solution of (1.1). Then there exists $n_1 > 0$ such that $y(n) > 0$ or < 0 for $n \ge n_1$. Assume that $y(n) > 0$ for $n \ge n_1$. Setting

$$z(n) = y(n) + p(n)y(n-m) \tag{2.1}$$

we have from (1.1)

$$\Delta(r(n)\Delta(z(n)) + q(n)G(y(n-k)) = 0, \tag{2.2}$$

that is,

$$\Delta(r(n)\Delta z(n)) = -q(n)G(y(n-k)) \le 0,$$

for $n \ge n_2 > n_1+\rho$. Hence $r(n)\Delta z(n)$ is non increasing. Consequently, $\Delta z(n) > 0$ or $\Delta z(n) < 0$ for $n \ge n_2$. If $\Delta z(n) < 0$, then Eq.(1.1) becomes

$$r(n+1)\Delta z(n+1) + q(n)G(y(n-k)) = r(n)\Delta z(n) < 0, \quad for \;\; n \ge n_2. \tag{2.3}$$

Using the above inequality (2.3), we get

$$r(n-m+1)\Delta z(n-m+1) + q(n-m)G(y(n-m-k)) < 0, \quad for \;\; n \ge n^*.$$

and hence for $n_3 > max\{n_2, n^*\}$,

$$0 > \Delta z(n+1) + \frac{q(n)}{r(n+1)}G(y(n-k)) + G(a)\Delta z(n-m+1) + G(a)\frac{q(n-m)}{r(n-m+1)}G(y(n-m-k)).$$

Using $(A_1), (A_3)$ and (A_6), the last inequality becomes

$$0 > \Delta z(n+1) + G(a)\Delta z(n-m+1) + \lambda Q(n)G(z(n-k)),$$

where $0 < z(n) \leq y(n) + ay(n-m)$. Thus

$$\lambda Q(n) + \frac{\Delta z(n+1)}{G(z(n-k))} + G(a)\frac{\Delta z(n-m+1)}{G(z(n-k))} < 0,$$

that is,

$$\lambda Q(n) + \int_{z(n+1)}^{z(n+2)} \frac{du}{G(u)} + G(a)\int_{z(n-m+1)}^{z(n-m+2)} \frac{dv}{G(v)} < 0,$$

where $z(n+2) < u < z(n+1)$ and $z(n-m+2) < v < z(n-m+1)$ and $n-k < n+1$. Hence for $n \geq n_3$,

$$\lambda \sum_{s=n_3}^{n} Q(s) + \sum_{s=n_3}^{n} \int_{z(s+1)}^{z(s+2)} \frac{du)}{G(u)} + G(a) \sum_{s=n_3}^{n} \int_{z(s-m+1)}^{z(s-m+2)} \frac{dv}{G(v)} < 0$$

that is,

$$\lambda \sum_{s=n_3}^{n} Q(s) + \int_{z(n_3+1)}^{z(n+2)} \frac{du}{G(u)} + G(a)\int_{z(n_3-m+1)}^{z(n-m+2)} \frac{dv}{G(v)} < 0.$$

Since $lim_{n\to\infty} z(n)$ exists, then the last inequality implies that

$$\sum_{s=n_3}^{\infty} Q(s) < \infty,$$

a contradiction to (A_6). Let $\Delta z(n) > 0$ for $n \geq n_2$. Then $z(n)$ is nondecreasing on $[n_2, \infty)$ for $n \geq n_2$.Hence there exists a constant $\alpha > 0$ such that $z(n) > \alpha$ for $n \geq n^*$. Using and applying equation (2.2)we obtain

$$\Delta(r(n)\Delta z(n)) + G(a)\Delta(r(n-m)\Delta z(n-m)) + \lambda R(n)G(z(n-k)) \leq 0.$$

Thus for $n \geq n_3 > \max\{n_2, n^*\}$ the last inequality can be written as

$$\lambda R(n)G(\alpha) < -\Delta(r(n)\Delta z(n)) - G(a)\Delta(r(n-m)\Delta z(n-m)),$$

that is,

$$\sum_{n=n_3}^{\infty} R(n) < \infty,$$

a contradiction to (A_6).

Next, we assume that $y(n) < 0$ for $n \geq n_1$. Then setting $x(n) = -y(n) > 0$ for $n \geq n_1$ and using (A_2), Eq.(1.1) can be written as

$$\Delta[r(n)\Delta(x(n) + p(n)x(n-m))] + q(n)G(x(n-k)) = 0. \tag{2.4}$$

The above procedure is applicable to Eq.(2.4) to obtain the similar contradictions. Hence the theorem is proved.

Remark 2.2 : The prototype of G satisfying $(A_1), (A_2)$ and (A_3) is

$$G(u) = (\alpha + \beta|u|^{\lambda})|u|^{\mu} sgnu,$$

where $\alpha \geq 1, \beta \geq 1, \lambda \geq 0$ and $\mu \geq 0$.

Example 2.3 : Consider

$$\Delta[e^n\Delta(y(n) + (2 + (-1)^n)y(n-1))] + 2(1+e)e^n y^{\frac{1}{3}}(n-2) = 0, n \geq 0,$$

where $1 \leq p(n) = 2 + (-1)^n \leq 3, r(n) = e^n$ and $q(n) = 2(1+e)e^n$. Clearly, the above equation satisfies all the conditions of the Theorem 2.1 and hence it is oscillatory. In particular, $y(n) = (-1)^{3n}$ is such an oscillatory solution.

***Theorem* 2.4** — *Assume that* $-1 < b \leq p(n) \leq 0$ *and* $m < k$*. If* $(A_2), (A_4), (A_7)$ *and* (A_8) *hold, then Eq.(1.1) is oscillatory.*

PROOF : Proceeding as in Theorem 2.1, we get the inequality (2.3) when we consider the case $\Delta z(n) < 0$ and $z(n) > 0$ for $n \geq n_2$. Using the fact that $z(n) \leq y(n)$, the inequality (2.3), yields

$$\Delta z(n+1) + \frac{q(n)}{r(n+1)}G(z(n-k)) < 0, n \geq n_2,$$

that is ,

$$\Delta z(n+1) + S(n)G(z(n-k)) < 0$$

for $n \geq n_2$. Using the arguments of Theorem 2.1, it is easy to obtain a contradiction to (H_7). If $z(n) < 0$, for $n \geq n_2$. Then $y(n) < y(n-m)$ implies that $y(n)$ is bounded. Thus $z(n)$ is bounded, which contradicts to the hypothesis that $z(n) < 0$ and $\Delta z(n) < 0$ for $n \geq n_2$. Ultimately, $\Delta z(n) > 0$ for $n \geq n_2$, that is, $z(n)$

is nondecreasing on $[n_2, \infty)$. Assume that $z(n) > 0$ for $n \geq n_3 > n_2$. Then it follows from (2.2) that

$$\Delta(r(n)z(n)) + q(n)G(z(n-k)) \leq 0,$$

due to $z(n) \leq y(n)$ for $n \geq n_4 > n_3 + k$. Furthermore, $z(n)$ is nondecreasing implies that we can find a constant $\alpha > 0$ and $n^* > 0$ such that $z(n) \geq \alpha$ for $n \geq n^*$. Hence summing the last inequality from n_5 to ∞, we obtain

$$\sum_{s=n_5}^{\infty} q(n) < \infty, \qquad n_5 > max\{n_4, n^*\},$$

a contradiction to (A_8). Suppose that $z(n) < 0$ for $n \geq n_2$. Then $y(n-k) > (\frac{1}{b})z(n+m-k)$. Consequently, Eq.(1.1) becomes

$$\Delta(r(n)\Delta z(n)) + q(n)\big(\frac{1}{b}z(n)\big) < 0$$

due to nondecreasing $z(n)$. It is easy to write the last inequality as

$$-\Delta z(n) + \frac{q(n)}{r(n)} G\left(\frac{1}{b}z(n+1)\right) < 0.$$

That is,

$$\Delta x(n) - \frac{1}{b}\frac{q(n)}{r(n)} G(x(n+1)) < 0,$$

where $x(n) = \frac{1}{b}z(n) > 0$. Hence

$$-\frac{1}{b}S(n) < -\frac{1}{b}\frac{q(n)}{r(n)} < -\frac{\Delta x(n)}{G(x(n+1))} = -\int_{x(n)}^{x(n+1)} \frac{du}{G(x(n+1))}.$$

For $x(n+1) < u < x(n)$, it is immediate to get

$$-\frac{1}{b}\sum_{n=n_2}^{N} S(n) < -\sum_{n=n_2}^{N} \int_{x(n)}^{x(n+1)} \frac{du}{G(u)} = -\int_{x(n_2)}^{x(N+1)} \frac{du}{G(u)},$$

that is,

$$-\frac{1}{b}\sum_{n=n_2}^{\infty} S(n) < -\lim_{N\to\infty} \int_{x(n_2)}^{x(N+1)} \frac{du}{G(u)} < \infty,$$

a contradiction.

The case $y(n) < 0$ for $n \geq n_1$ is similar. This completes the proof of the theorem.

***Theorem* 2.5** — *Let* $-\infty < -b \leq p(n) < -1, b > 0$. *If* $(A_2), (A_5), (A_8)$ *and* (A_9) *hold, then every bounded solution of (1.1) oscillates.*

PROOF : Let $y(n)$ be a bounded nonoscillatory solution of (1.1). So there exists $n_1 > 0$ such that $y(n) > 0$ for $n \geq n_1$. The case $y(n) < 0$ for $n \geq n_1$ can similarly be dealt with. Setting $z(n)$ as in (2.1) we get (2.2) and hence $\Delta(r(n)\Delta z(n)) \leq 0$ for $n \geq n_2 > n_1 + \rho$ implies that $\Delta z(n)$ is of one sign. Let $\Delta z(n) < 0$ for $n \geq n_3 > n_2$. Then we have two cases upon $z(n)$ namely $z(n) > 0$ or $z(n) < 0$ for $n \geq n_3$. Suppose the former holds. Hence $lim_{n\to\infty} z(n)$ exists. Following the Theorem 2.4, we have a contradiction to (A_8). On the otherhand, when the later holds, we have a contradiction to bounded $z(n)$ due to the fact that $z(n+m-k) \geq -by(n-k)$.

Assume that $\Delta z(n) > 0$ for $n \geq n_2$. If $z(n) > 0$, then there exists a constant $\alpha > 0$ and $n^* > 0$ such that $z(n) > \alpha$ for $n \geq n^*$. Thus $z(n) < y(n)$ implies that

$$\Delta(r(n)\Delta z(n)) + G(\alpha)q(n) < 0,$$

for $n \geq n_3 > max\{n_2, n^*\}$. Consequently,

$$G(\alpha)\sum_{n=n_3}^{\infty} q(n) = -\lim_{s\to\infty}\sum_{n=n_3}^{s-1} \Delta(r(s)\Delta z(s)) < \infty,$$

a contradiction to (A_8). Hence $z(n) < 0$ for $n \geq n_2$. In this case $\lim_{n\to\infty} z(n)$ exists. Let $\lim_{n\to\infty} z(n) = \beta, \beta \in [0,\infty)$. Due to $\Delta z(n) > 0$, Eq.(1.1) can be written as

$$-r(n)\Delta z(n) + q(n)G\bigg(-\frac{1}{b}z(n+m-k)\bigg) < 0,$$

where $y(n-k) \geq -\frac{1}{b}z(n+m-k)$. Setting $-\frac{1}{b}z(n) = x(n)$, the last inequality yields

$$r(n)\Delta x(n) + \frac{1}{b}q(n)G(x(n+m-k)) < 0$$

for $n \geq n_3 > n_2$ and hence using (A_5), we get

$$\Delta x(n) + \left(\frac{\gamma}{b}\right)\frac{q(n)}{r(n)}x(n+m-k) < 0$$

which has no positive solution due to (A_9), a contradiction to the fact that $x(n) > 0$ is a positive solution. Hence the proof of the theorem is complete.

3. Superlinear Oscillation

This section deals with the sufficient conditions for the oscillation of all solutions of equation (1.1) such that G is superlinear.

***Theorem* 3.1** — *Let* $0 \leq p(n) \leq a < 0$. *Assume that* $(A_1) - (A_3)$ *and* (A_6) *hold. If*

$$\sum_{n=0}^{\infty} \frac{1}{r(n)} = \infty,$$

then Eq.(1.1) is oscillatory.

Proof : Let $y(n)$ be a nonoscillatory solution of (1.1). Then there exists $n_1 > 0$ such that $y(n) > 0$ for $n \geq n_1$. The case $y(n) < 0$, for $n \geq n_1$ is similar which follows from the Eq.(2.4). Setting as in (2.1), we get (2.2). Hence $\Delta z(n) < 0$ or $\Delta z(n) > 0$ for $n \geq n_2 > n_1 + \rho$. Assume that $\Delta z(n) < 0$ for $n \geq n_2$. Then there exists a constant $C_1 < 0$ and for $n_3 > n_2$ such that $r(n)\Delta z(n) \leq r(n_3)\Delta z(n_3) = C_1 < 0$ for $n \geq n_3$. Consequently,

$$\begin{aligned} z(n) &\leq z(n_3) + r(n_3)\Delta z(n_3) \sum_{s=n_3}^{n-1} \frac{1}{r(s)} \\ &= z(n_3) + C_1 \sum_{s=n_3}^{n-1} \frac{1}{r(s)} \to -\infty \;\; as \;\; n \to \; \infty, \end{aligned}$$

a contradiction to the fact that $z(n) > 0$. Ultimately, $\Delta z(n) > 0$ for $n \geq n_2$. The rest of the proof follows from the Theorem 2.1. Hence or otherwise, the proof of the theorem is complete.

***Theorem* 3.2**[9] — *Assume that* $\liminf_{n\to\infty} \sum_{s=n-k}^{n-1} q(s) > 0$.
(A) Suppose there are a function $g \in C(\mathbb{R}, \mathbb{R}^+)$ *and a number* $\epsilon > 0$ *such that*

(a) $g(u)$ *is nondecreasing in* $\mathbb{R}^+$*,*

(b) $g(-u) = g(u)$ *and* $\lim_{u\to\infty} g(u) = 0$,

(c) $\int_0^\infty g(e^{-u})du < \infty$,

(d) $|\frac{G(u)}{u} - 1| \leq g(u), 0 < |u| < \epsilon$.

If

$$\sum_{n=0}^{\infty}\Big[\sum_{s=n}^{n+k} q(s)ln\Big(\sum_{s=n}^{n+k} q(s)\Big) - \sum_{s=n+1}^{n+k} q(s)ln\Big(\sum_{s=n+1}^{n+k} q(s)\Big)\Big] = \infty,$$

then every solution of $\Delta x(n) + q(n)G(x(n-k)) = 0$ *oscillates.*

Corollary 3.3 — If all the conditions of Theorem 3.2 are satisfied, then

$$\Delta x(n) + q(n)G(x(n-k)) \leq 0, n \geq 0$$

doesn't possess any eventually positive solution.

The proof follows from the Theorem 3.2.

Remark 3.4 : $\liminf_{n\to\infty} \sum_{s=n-k}^{n-1} q(s) > 0$ implies that $\sum_{n=0}^{\infty} q(s) = \infty$.

***Theorem* 3.5** — *Let* $-1 \leq b \leq p(n) \leq 0$. *Assume that* $(A), (A_2), (A_7)$ *and* (A_8) *hold. If*

$$(A_{11}) \qquad \text{liminf}_{n\to\infty} \sum_{t=n+m-k}^{n-1} S(t) > 0, k > m$$

and

$$(A_{12}) \qquad \sum_{n=0}^{\infty}\Big[\sum_{t=n}^{n+k-m} S(t)ln\Big(\sum_{t=n}^{n+k-m}\Big(-\frac{1}{b}\Big)S(t)\Big) - \sum_{t=n+1}^{n+k-m} S(t)ln\Big(\sum_{t=n+1}^{n+k-m}\Big(-\frac{1}{b}\Big)S(t)\Big)\Big] = \infty,$$

then every solution of Eq.(1.1)oscillates.

PROOF : Clearly, (A_{11}) implies (A_7). Let $y(n)$ be a nonoscillatory solution of (1.1) such that $y(n) > 0$ for $n \geq n_0$. Proceeding as in the proof of Theorem 2.1, we obtained the Eq. (2.2) and hence $\Delta(r(n)\Delta z(n)) \leq 0$ for $n \geq n_1 > n_0 + \rho$. Consequently, here arise four cases:

$$(i)\ \ z(n) > 0, \Delta z(n) < 0;$$
$$(ii)\ \ z(n) < 0, \Delta z(n) < 0;$$
$$(iii)\ \ z(n) > 0, \Delta z(n) > 0;$$
$$(iv)\ \ z(n) < 0, \Delta z(n) > 0.$$

The cases $(i), (ii)$ and (iii) follow from the proof of the Theorem 2.4. We consider the case (iv). Since $z(n) < 0$, then substituting $y(n-k) > (\frac{1}{b})z(n+m-k)$ for $n \geq n_2$, Eq.(1.1) can be written as

$$\Delta x(n) - \frac{1}{b}q(n)G(x(n+m-k)) \leq 0,$$

where $x(n) = \frac{1}{b}z(n) > 0$. In view of Theorem 3.2, Corollary 3.3 and (A_{12}), the last inequality has no positive solution, a contradiction. This completes the proof of the theorem.

***Theorem* 3.6** — *Let* $-\infty < b \leq p(n) < -1$. *Assume that* $m \geq k+1, (A_7)$ *and the following conditions*

$$(A_{13})\quad \int_0^\infty \frac{dx}{G(x)} < 0,$$
$$(A_{14})\quad \sum_{j=0}^{\infty} q(n_j) = \infty \ \ for\ \ every\ \ \{n_j\}\ \ of\ \ \{n\}$$

hold. Then every unbounded solution of Eq.(1.1) oscillates.

PROOF : Assume that $y(n)$ is an unbounded solution of (1.1) such that $y(n) > 0$ for $n \geq n_0$. The case $y(n) < 0$ for $n \geq n_0$ is similar. Here, we consider the four cases of the Theorem 3.5 of which the cases (i) and (iii) follows directly from the proof. Consider the case (ii). It is immediate to see that Eq.(1.1) reduces to

$$r(n+1)\Delta z(n+1) + q(n)G(\frac{1}{b}z(n+1)) < r(n)\Delta z(n) < 0,$$

that is,

$$\Delta z(n+1) + S(n)G(\frac{1}{b}z(n+1)) < 0,$$

for $n \geq n_2$. Hence

$$\begin{aligned} S(n) \quad &< -\frac{\Delta z(n+1)}{G(\frac{1}{b}z(n+1))} = -\int_{z(n+1)}^{z(n+2)} \frac{du}{G(\frac{1}{b}z(n+1)} \\ &< -\int_{z(n+1)}^{z(n+2)} \frac{du}{G(\frac{1}{b}u)} = -b\int_{z(n+1)}^{z(n+2)} \frac{d(\frac{1}{b}u)}{G(\frac{1}{b}u)}, \end{aligned}$$

where $z(n+2) < u < z(n+1)$. Thus

$$\sum_{n=n_2}^{N} S(n) < -b\sum_{n=n_2}^{N} \int_{z(n+1)}^{z(n+2)} \frac{d(\frac{1}{b}u)}{G(\frac{1}{b}u)} = -b\int_{z(n_2+1)}^{z(N+2)} \frac{d(\frac{1}{b}u)}{G(\frac{1}{b}u)}$$

implies that

$$\sum_{n=n_2}^{\infty} S(n) < -b\lim_{N\to\infty}\int_{z(n_2+1)}^{z(N+2)} \frac{d(\frac{1}{b}u)}{G(\frac{1}{b}u)} < \infty,$$

a contradiction to (A_7). Hence we conclude that case (iv) holds. Since $y(n)$ is unbounded, there exists a sequence $\{n_j\}$ of $\{n\}$ such that $y(n_j) \to \infty \ as \ j \to \infty$. Hence for every $M > 0$, it is possible to find $n_2 > n_1$ such that $n_j \geq n_2$ implies $y(n_j) > M$. Consequently, Eq.(1.1) yields

$$\sum_{n_j=n_3}^{t-1} q(n_j)G(y(n_j-k)) = -\sum_{n_j=n_3}^{t-1} \Delta^2 z(n_j) = -\Delta z(t) + \Delta z(n_3),$$

where $n_3 > n_2 + k$. Thus

$$G(M)\sum_{n_j=n_3}^{\infty} q(n_j) < \sum_{n_j=n_3}^{\infty} q(n_j)G(y(n_j-k)) < \infty,$$

a contradiction to (A_{14}). This completes the proof of the theorem.

Remark 3.7 : The prototype of G satisfying the Theorem 3.5 and Theorem 3.6 may be of the form

$$G(u) = \begin{cases} u[1+(a+ln^2|u|)^{-1}], & u \neq 0; \\ 0, & u = 0. \end{cases}$$

$$g(u) = \begin{cases} a, & |u| > 1; \\ (a+ln^2|u|)^{-1}, & 0 < |u| \leq 1; \\ 0, & u = 0. \end{cases}$$

Remark 3.8 : Theorem 3.5 and Theorem 3.6 can be seen by using the assumption (A_{10}). It seems that every solution of Eq.(1.1) oscillates or tends to zero as $n \to \infty$ due to (A_{10}). However, this concept is not the part of the object of this paper. Hence with out (A_{10}), G is highly restricted to show that Eq.(1.1) is oscillatory. In Theorems 3.1 and 3.7, the nonlinear function G is more general than the Theorems 3.5 and 3.6. We note that, Theorem 3.1 can be proved by using the similar assumptions of (A_{11}) and (A_{12}).

4. DISCUSSIONS

It seems that results cited in this work may be extended to a class of nonlinear neutral forced difference equations of the form

$$\Delta(r(n)\Delta(y(n) + p(n)y(n-m))) + q(n)G(y(n-k)) = f(t)$$

under various ranges of $p(n)$ with suitable forcing functions. The nonlinear function $G, G \in C(\mathbb{R}, \mathbb{R})$ is either sublinear or superlinear satisfying the condition $xG(x) > 0$ for $x \neq 0$.

It would be interesting to note that these results may be extended to the difference equations with positive and negative coefficients of the form

$$\Delta(r(n)\Delta(y(n) + p(n)y(n-m))) + q_1(n)G(y(n-\tau)) - q_2(n)H(y(n-\sigma)) = 0,$$

and to the higher order difference equations of the form

$$\Delta^l(r(n)\Delta^l(y(n) + p(n)y(n-m))) + q_1(n)G(y(n-\tau)) - q_2(n) \\ H(y(n-\sigma)) = 0, l \geq 1,$$

where p, q_1, q_2 are real valued functions such that $q_1(n), q_2(n) \geq 0$ for $n \in N(0) = \{0, 1, 2, ...\}$, G, H are nondecreasing functions satisfying $xG(x) > 0, xH(x) > 0$ for $x \neq 0$ and m, τ, σ are positive integers.

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