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# VRANCEANU SURFACE IN $\mathbb{E}^4$ WITH POINTWISE 1- TYPE GAUSS MAP

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In this article we investigate Vranceanu rotation surfaces with pointwise 1-type Gauss map in Euclidean 4-space $\mathbb{E}^4$. We show that a Vranceanu rotation surface $M$ has harmonic Gauss map if and only if $M$ is a part of a plane. Further, we give necessary and sufficent conditions for Vranceanu rotation surface to have pointwise 1-type Gauss map.

**Key words :** Rotation surface, Gauss map, finite type, pointwise 1-type.

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## 1. INTRODUCTION

Since the late 1970's, the study of submanifolds of Euclidean space or pseudo-Euclidean space with the notion of finite type immersion has been extensively carried out. An isometric immersion $x : M \to \mathbb{E}^m$ of a submanifold $M$ in Euclidean $m-$space $\mathbb{E}^m$ is said to be of finite type if $x$ identified with the position vector field of $M$ in $\mathbb{E}^m$ can be expressed as a finite sum of eigenvectors of the Laplacian $\Delta$ of $M$, that is;

$$x = x_0 + \sum_{i=1}^{k} x_i$$

where $x_0$ is a constant map, $x_1, x_2, ..., x_k$ non-constant maps such that $\Delta x = \lambda_i x_i$, $\lambda_i \in \mathbb{R}$, $1 \leq i \leq k$. If $\lambda_1, \lambda_2, ..., \lambda_k$ are different, then $M$ is said to be of $k-$type. Similarly, a smooth map $\phi$ of an $n-$dimensional Riemannian manifold $M$ of $\mathbb{E}^m$ is said to be of finite type if $\phi$ is a finite sum of $\mathbb{E}^m$-valued eigenfunctions of $\Delta$ (of. [5], [6]). Granted, this notion of finite type immersion is naturally extended to the Gauss map $G$ on $M$ in Euclidean space [8]. Thus, if a submanifold $M$ of Euclidean space has 1-type Gauss map $G$, then $G$ satisfies $\Delta G = \lambda(G + C)$ for some $\lambda \in \mathbb{R}$ and some constant vector $C$ (of. [1], [2], [3], [10]). However, the Laplacian of the Gauss map of some typical well-known surfaces such as a helicoid, a catenoid and a right cone in Euclidean 3-space $\mathbb{E}^3$ take a somewhat different form; namely, $\Delta G = f(G + C)$ for some non-constant function $f$ and some constant vector $C$. Therefore, it is worth studying the class of solution surfaces satisfying such an equation. A submanifold $M$ of a Euclidean space $\mathbb{E}^m$ is said to have pointwise 1-type Gauss map if its Gauss map $G$ satisfies

$$\Delta G = f(G + C) \tag{1}$$

for some non-zero smooth function $f$ on $M$ and a constant vector $C$. A pointwise 1-type Gauss map is called proper if the function $f$ defined by (1) is non-constant. A submanifold with pointwise 1-type Gauss map is said to be of the first kind if the vector $C$ in (1) is zero vector. Otherwise, the pointwise 1-type Gauss map is said to be of the second kind ([7], [9], [11], [12]). In particular, if the Gauss map $G$ is a constant vector $-C$, we regard it of pointwise 1-type of the second kind. Granted, it is of the first kind as well in this case. In [9], one of the present authors characterized the minimal helicoid in terms of pointwise 1-type Gauss map of the first kind. Also, together with Chen, they proved that surfaces of revolution with pointwise 1-type Gauss map of the first kind coincides with surfaces of revolution with constant mean curvature.

Moreover, they characterized the rational surfaces of revolution with pointwise 1-type Gauss map [7]. In [18] D. W. Yoon studied Vranceanu rotation surfaces in Euclidean 4-space $\mathbb{E}^4$. He proved that the flat Vranceanu rotation surfaces with pointwise 1-type Gauss map is a Clifford torus, i.e. it is the product of two plane circles with the same radius. For more details see also [14, 17]. In this article investigate Vranceanu rotation surfaces with pointwise 1- type Gauss map in Euclidean 4-space $\mathbb{E}^4$. We show that a Vranceanu rotation surface $M$ has harmonic Gauss map if and only if $M$ is a part of a plane. Further, we give the complete classification of flat Vranceanu rotation surfaces with pointwise 1-type Gauss map. Finally, we give the necessary and sufficient conditions for non-flat Vranceanu rotation surface to have pointwise 1-type Gauss map.

All functions under consideration are assumed to be smooth and the manifolds are connected unless otherwise stated.

## 2. Preliminaries

In the present section we recall definitions and results of [4]. Let $x : M \to \mathbb{E}^m$ be an immersion from an n-dimensional connected Riemannian manifold $M$ into an $m$=dimensional Euclidean space $\mathbb{E}^m$. We denote by $g$ the metric tensor of $\mathbb{E}^m$ as well as the induced metric on M. Let $\widetilde{\nabla}$ be the Levi-Civita connection of $\mathbb{E}^m$ and $\nabla$ the induced connection on $M$. Then the Gaussian and Weingarten formulas are given respectively by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{2}$$

$$\widetilde{\nabla}_X \xi = -A_\xi X + D_X \xi \tag{3}$$

where $X, Y$ are vector fields tangent to $M$ and $\xi$ normal to $M$. Moreover, $h$ is the second fundamental form, $D$ is the linear connection induced in the normal bundle $T^\perp M$, called normal connection and $A_\xi$ the shape operator in the direction of $\xi$ that is related with $h$ by

$$< h(X,Y), \xi >=< A_\xi X, Y > . \tag{4}$$

If we define a covariant differentiation $\overline{\nabla} h$ of the second fundamental form $h$ on the direct sum of the tangent bundle and the normal bundle $TM \oplus T^\perp M$ of $M$ by

$$(\overline{\nabla}_X h)(Y,Z) = D_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z),$$

for any vector fields $X, Y$ and $Z$ tangent to $M$. Then we have the Codazzi equation

$$(\overline{\nabla}_X h)(Y, Z) = (\overline{\nabla}_Y h)(X, Z). \tag{5}$$

We denote by $R$ the curvature tensor associated with $\nabla$ defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{6}$$

The equation of Gauss and Ricci are given by

$$\langle R(X, Y)Z, W\rangle = \langle h(X, W), h(Y, Z)\rangle - \langle h(X, Z), h(Y, W)\rangle, \tag{7}$$

$$\langle R^{\perp}(X, Y)\xi, \eta\rangle = \langle [A_\xi, A_\eta]X, Y\rangle \tag{8}$$

for vectors $X, Y, Z, W$ tangent to $M$ and $\xi, \eta$ normal to $M$.

Let us now define the Gauss map $G$ of a submanifold $M$ into $G(n, m)$ in $\wedge^n \mathbb{E}^m$, where $G(n, m)$ is the Grassmannian manifold consisting of all oriented $n$-planes through the origin of $\mathbb{E}^m$ and $\wedge^n \mathbb{E}^m$ is the vector space obtained by the exterior product of $n$ vectors in $\mathbb{E}^m$. In a natural way, we can identify $\wedge^n \mathbb{E}^m$ with some Euclidean space $\mathbb{E}^N$ where $N = \begin{pmatrix} m \\ n \end{pmatrix}$. Let $\{e_1, ..., e_n, e_{n+1}, ..., e_m\}$ be an adapted local orthonormal frame field in $\mathbb{E}^m$ such that $e_1, e_2, ..., e_n$, are tangent to $M$ and $e_{n+1}, e_{n+2}..., e_m$ normal to $M$. The map $G : M \to G(n, m)$ defined by $G(p) = (e_1 \wedge e_2 \wedge ... \wedge e_n)(p)$ is called the Gauss map of $M$ that is a smooth map which carries a point $p$ in $M$ into the oriented $n$-plane in $\mathbb{E}^m$ obtained from the parallel translation of the tangent space of $M$ at $p$ in $\mathbb{E}^m$.

For any real function $f$ on $M$ the Laplacian of $f$ is defined by

$$\Delta f = -\sum_i (\widetilde{\nabla}_{e_i} \widetilde{\nabla}_{e_i} f - \widetilde{\nabla}_{\nabla_{e_i} e_i} f). \tag{9}$$

The mean curvature vector field $\overrightarrow{H}$ of $M$ is defined by

$$\overrightarrow{H} = \frac{1}{n} \sum_{i=1}^{n} h(e_i, e_i). \tag{10}$$

## 3. Vranceanu Surfaces with Pointwise 1-Type Gauss Map

Rotation surfaces were studied in [16] by Vranceanu as surfaces in $\mathbb{E}^4$ which are defined by the following parametrization

$$M : X(s,t) = (r(s)\cos s\cos t, r(s)\cos s\sin t, r(s)\sin s\cos t, r(s)\sin s\sin t) \tag{11}$$

We choose a moving frame Let $e_1, e_2, e_3, e_4$ such that $e_1, e_2$ are tangent to $M$ and $e_3, e_4$ are normal to $M$ as given in the following (see, [13, 15]):

$$\begin{aligned}
e_1 &= \frac{1}{r}\frac{\partial}{\partial t} \\
&= (-\cos s\sin t, \cos s\cos t, -\sin s\sin t, \sin s\cos t), \\
e_2 &= \frac{1}{A}\frac{\partial}{\partial s} \\
&= \frac{1}{A}(B\cos t, B\sin t, C\cos t, C\sin t), \\
e_3 &= \frac{1}{A}(-C\cos t, -C\sin t, B\cos t, B\sin t), \\
e_4 &= (-\sin s\sin t, \sin s\cos t, \cos s\sin t, -\cos s\cos t),
\end{aligned} \tag{12}$$

where

$$\begin{aligned}
A &= \sqrt{r^2(s) + (r'(s))^2}, \;\; B = r'(s)\cos s - r(s)\sin s, \\
C &= r'(s)\sin s + r(s)\cos s.
\end{aligned} \tag{13}$$

Furthermore, by covariant differentiation with respect to $e_1$ and $e_2$, a straightforward calculation gives:

$$\begin{aligned}
\tilde{\nabla}_{e_1} e_1 &= -\lambda(s)k(s)e_2 + \lambda(s)e_3, \\
\tilde{\nabla}_{e_2} e_2 &= \mu(s)e_3, \\
\tilde{\nabla}_{e_2} e_1 &= -\lambda(s)e_4, \\
\tilde{\nabla}_{e_1} e_2 &= \lambda(s)k(s)e_1 - \lambda(s)e_4, \\
\tilde{\nabla}_{e_1} e_3 &= -\lambda(s)e_1 - \lambda(s)k(s)e_4, \\
\tilde{\nabla}_{e_1} e_4 &= \lambda(s)e_2 + \lambda(s)k(s)e_3, \\
\tilde{\nabla}_{e_2} e_3 &= -\mu(s)e_2, \\
\tilde{\nabla}_{e_2} e_4 &= \lambda(s)e_1,
\end{aligned} \tag{14}$$

where

$$\begin{aligned} k &= \frac{r'(s)}{r(s)}, \\ \lambda &= \frac{1}{A} = \frac{1}{\sqrt{r^2(s) + (r'(s))^2}}, \\ \mu &= \frac{2(r'(s))^2 - r(s)r''(s) + r^2(s)}{(r^2 + (r')^2)^{3/2}}, \end{aligned} \tag{15}$$

are differentiable functions.

By the use of (4) the coefficients of the second fundamental form $h$ become

$$h_{ij}^k = < h(e_i, e_j), e_k > = < A_{e_k} e_i, e_j > \ 1 \leq i, j \leq 2, \ k = 3, 4. \tag{16}$$

Moreover, combining (2), (14) and (16) we get

$$\begin{aligned} &h_{11}^3 = \lambda, \quad h_{12}^3 = h_{21}^3 = 0, \\ &h_{22}^3 = \mu, \quad h_{11}^4 = h_{22}^4 = 0, \\ &h_{12}^4 = h_{21}^4 = -\lambda. \end{aligned} \tag{17}$$

By the use of (17) together with (14) we get the following result.

*Lemma* 1 — Let $M$ a Vranceanu surface given with the surface patch (11). Then, we get

$$A_{e_3} = \begin{bmatrix} \lambda(s) & 0 \\ 0 & \mu(s) \end{bmatrix}, \ A_{e_4} = \begin{bmatrix} 0 & -\lambda(s) \\ -\lambda(s) & 0 \end{bmatrix}. \tag{18}$$

The Gauss curvature $K$ and the mean curvature vector field $\overrightarrow{H}$ are respectively given by

$$\begin{aligned} K &= \det A_{e_3} + \det A_{e_4} \\ &= \lambda(\mu - \lambda) \\ &= \frac{(r')^2 - rr''}{(r^2 + (r')^2)^2} \end{aligned} \tag{19}$$

and

$$\begin{aligned} \overrightarrow{H} &= \frac{1}{2}\sum_{i=1}^{2} h(e_i, e_i) \\ &= \frac{1}{2}(\lambda+\mu)e_3 \\ &= \frac{2r^2+3(r')^2-rr''}{2(r^2+(r')^2)^{3/2}}e_3. \end{aligned} \tag{20}$$

The Gauss map $G$ of the $M$ is defined by $G = e_1 \wedge e_2$. By using (14), (15) and straight-forward computation the Laplacian of the Gauss map can be expressed as

$$\begin{aligned} -\Delta G = & -(3\lambda^2(s)+\mu^2(s))e_1\wedge e_2 + \left(3\lambda^2(s)k(s)+\lambda'(s)\right)e_2\wedge e_4 \\ & +(\mu'(s)+\lambda(s)\mu(s)k(s)-2\lambda^2(s)k(s))e_1\wedge e_3 \\ & +2\left(\lambda(s)\mu(s)-\lambda^2(s)\right)e_3\wedge e_4. \end{aligned} \tag{21}$$

Suppose that the Gauss map of $M$ is harmonic, i.e. $\Delta G = \overrightarrow{0}$. Firstly, if $G = -C$, $M$ is part of a plane which is not a Vranceanu surface. Secondly, suppose the function $f$ identically zero. From (21) we get

$$3\lambda^2(s)+\mu^2(s) = 0.$$

It implies that $\lambda = 0$ and $\mu = 0$, a contradiction. So, this case cannot occur.

Thus we have

***Theorem* 1** *There is no Vranceanu surface with harmonic Gauss map.*

Now, we suppose that the Vranceanu surface given by the patch (11) is of pointwise 1-type Gauss map in $\mathbb{E}^4$. From (1) and (21)

$$\begin{aligned} f+f \langle C, e_1\wedge e_2 \rangle &= +3\lambda^2(s)+\mu^2(s), \\ f \langle C, e_1\wedge e_3 \rangle &= 2\lambda^2(s)\mu(s)-\mu'(s)-\lambda(s)\mu(s)k(s), \\ f \langle C, e_2\wedge e_4 \rangle &= -3\lambda^2(s)k(s)-\lambda'(s), \\ f \langle C, e_3\wedge e_4 \rangle &= -2\lambda(s)\mu(s)+2\lambda^2(s), \end{aligned} \tag{22}$$

where $f$ is a smooth non-zero function. Since $\lambda$ is non-zero everywhere on $M$, so is the function $f$. Then we obtain from (21)

$$\langle C, e_1 \wedge e_4 \rangle = 0,$$
$$\langle C, e_2 \wedge e_3 \rangle = 0. \tag{23}$$

Differentiating (23) and using (22) we get

$$f = 5\lambda^2(s)(k^2(s)+1) - \lambda(s)\mu(s)(2+k^2(s)) + k(s)(\lambda'(s)-\mu'(s)) + \mu^2(s). \tag{24}$$

By using the equations of Gauss (7), Codazzi (5) we get

$$\lambda'(s)k(s) + \lambda(s)k'(s) + \lambda^2(s)k^2(s) = \lambda^2(s) - \lambda(s)\mu(s), \tag{25}$$

and

$$\lambda'(s) = \lambda(s)k(s)\left(\mu(s) - 2\lambda(s)\right), \tag{26}$$

respectively.

Further, substituting (26) into (25) we obtain

$$(\lambda(s) - \mu(s))(k^2(s) + 1) - k'(s) = 0. \tag{27}$$

We now suppose that the Vranceanu surface $M$ has the Gauss map with pointwise 1-type of the first kind, i.e., $C$ is zero. Then, (22) implies $\lambda = \mu$ on $M$ and thus $M$ is flat. Combining (15) and (19), we get $k$ is constant. By using (26) with $\lambda = \mu$, we get $\lambda'(s) = -k\lambda^2(s)$ and thus $\lambda(s) = \frac{1}{ks+a}$ for some constant $a$. Since the Gauss map is of pointwise 1-type of the first kind, the mean curvature $||\overrightarrow{H}||$ is constant and $\lambda$ is constant (for details, see [12]). Therefore, $k = 0$ and $r$ is a constant function. Hence, the Vranceanu surface $M$ is part of Clifford torus.

We now consider the Gauss map $G$ of $M$ is of pointwise 1-type of the second kind, i.e., $C \neq 0$. We distinguish the following cases:

*Case* I: $M$ is flat, or, equivalently, $\lambda(s) = \mu(s)$ for all $s$.

*Case* II: $M$ is non flat, or, equivalently, $\lambda \neq \mu$.

Let us consider these in turn:

*Case* I: Suppose that $M$ is flat. Thus, by using (27), we get $k'(s) = 0$. So, from the first equation of (15) we get $r(s) = \alpha \exp(\beta s)$, for some constants $\alpha \neq 0$ and

$\beta$. By applying homothetic transformation if necessary we may assume that $\alpha = 1$. Thus, using (15) we get

$$\begin{aligned} r(s) &= e^{(\beta s)}, \\ \lambda(s) &= \mu(s) = \frac{-e^{(\beta s)}}{\sqrt{1+k^2}}. \end{aligned} \tag{28}$$

Consequently, substituting (28) into (24) we obtain

$$\begin{aligned} f &= 4\lambda^2(s)(k^2+1) \\ &= 4e^{(-2\beta s)}. \end{aligned} \tag{29}$$

Thus, summing up the following theorem is proved.

***Theorem* 2** — *Let $M$ be a flat Vranceanu surface given with the parametrization (11). If $M$ has pointwise 1-type Gauss map of the second kind, then $M$ is represented by the function $f = 4e^{(-2\beta s)}$, where $\beta$ is a real constant.*

*Case* II: Suppose that $M$ is a non flat surface. Then, the open subset $\mathbf{M_0} = \{\mathbf{p} \in \mathbf{M} | \mathbf{K}(\mathbf{p}) \neq \mathbf{0}\}$ is not empty. Then, the functions $\lambda \neq \mu$ everywhere on $\mathbf{M_0}$. Substituting (15) into (27), we get

$$\left(r(s)r''(s) - (r'(s)^2)\right)\left(1 - (r^2(s) + (r'(s))^2)^{1/2}\right) = 0 \tag{30}$$

on $\mathbf{M_0}$. On $\mathbf{M_0}$, by (19) $r(s)r''(s) - (r'(s)^2 \neq 0$. So, from the equation (30) we get

$$r^2(s) + (r'(s))^2 = 1. \tag{31}$$

Thus, differentiating (31) with respect to $s$ we get $r'(s)(r(s) + r''(s)) = 0$ $\mathbf{M_0}$. Suppose $(r + r'')(p) \neq 0$ at $p \in \mathbf{M_0}$. On a component $\mathbf{O}$ containing $p$ in $\mathbf{M_0}$, $r' = 0$. Thus, $k = 0$ on $\mathbf{O}$. If we make use of (27), $\lambda = \mu$ on $\mathbf{O}$, which is a contradiction. Therefore, we have

$$r(s) + r''(s) = 0$$

on $\mathbf{M_0}$, which has non-trivial solutions

$$r(s) = A\sin(s + s_o).$$

for some constants $A \neq 0$ and $s_0$. By connectedness of $M$ and the continuity of $K$, $M = \mathbf{M_0}$ that means $M$ does not have a flat point.

Thus, the following theorem is proved.

***Theorem* 3** — *Let $M$ be a non flat Vranceanu surface given with the parametrization (11). If $M$ has pointwise 1-type Gauss map of the second kind, then $M$ is given up to homothety by*

$$X(s,t) = \left(\frac{1}{2}\sin s\cos t, \frac{1}{2}\sin s\sin t, \sin^2 s\cos t, \sin^2 s\sin t\right).$$

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