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ON THE GROWTH OF MEROMORPHIC FUNCTIONS WITH A RADIALLY DISTRIBUTED VALUE[1]

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The purpose of this paper is to investigate the growth of meromorphic functions with a radially distributed value. The paper is closely related to a more recent result due to Fang and Zalcman [M.L. Fang and L. Zalcman, On the value distribution of $f + a(f')^n$, Sci. China Ser. A-Math. **38**(2008), 279-285].

Key words : Meromorphic function, Growth, Angular characteristic function, Radially distributed value, Deficiency.

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1. Introduction

Let $f\colon \mathbb{C} \to \hat{\mathbb{C}}$ be a meromorphic function, where $\mathbb{C}$ is the whole complex plane and $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. We shall suppose that the reader is familiar with the fundamental concepts of Nevanlinna's value distribution theory (See [11]), and in particular with the most usual of its symbols:

$$m(r,f), \quad m\left(r, \frac{1}{f-a}\right), \quad N(r,f), \quad N\left(r, \frac{1}{f-a}\right), \quad T(r,f).$$

The deficiency of the finite value a is, by definition,

$$\delta(a,f) = \liminf_{r\to\infty} \frac{m\left(r, \frac{1}{f-a}\right)}{T(r,f)} = 1 - \limsup_{r\to\infty} \frac{N\left(r, \frac{1}{f-a}\right)}{T(r,f)} \tag{1.1}$$

and $\delta(\infty, f)$ is obtained by replacing in (1.1), $m\left(r, \frac{1}{f-a}\right)$ by $m(r,f)$, and $N\left(r, \frac{1}{f-a}\right)$ by $N(r,f)$, respectively. Meanwhile, the lower order μ and order λ are defined in turn as follows:

$$\mu := \mu(f) = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log r},$$

$$\lambda := \lambda(f) = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}.$$

For an unbounded subset Y of $\mathbb{C}$ and $a \in \hat{\mathbb{C}}$, we denote by $n(r, Y, f = a)$ the number of the zeros of $f - a$ counting multiplicities in $Y \cap \{z : |z| \le r\}$.

Let $P(X_1, X_2)$ be a polynomial in variables X_1 and X_2 with complex constant coefficients. We say that $P(f, f')$ has the *Picard property* if the only meromorphic function f in $\mathbb{C}$ such that $P(f, f')$ omits some finite complex number is the constant function. Furthermore, we say that $P(f, f')$ has the *weakly Picard property* if, for any function f transcendental meromorphic in $\mathbb{C}$, $P(f, f')$ assumes every finite complex number infinitely often. (See[8] and [2])

In the value distribution theory of meromorphic functions in $\mathbb{C}$, the following result [3,4,18], which was a conjecture of Hayman, is well-known.

***Theorem* A** — *Let f be a meromorphic function in $\mathbb{C}$. If for any $z \in \mathbb{C}$, $P(f, f')(z) = f(z)f'(z) \neq 1$, then f is a constant.* □

It is obvious from Theorem A that the growth of f is restricted. Here $P(f, f') = ff'$ has the *Picard property*. Now we suppose that $P(f, f') = ff'$ assumes the value 1 at infinitely many points, but most of these points are distributed in the vicinity of a finite number of rays. What can we say about the growth of f? In this direction, Yang and Yang [17] proved the following result.

***Theorem* B** — *Let f be an entire function of finite lower μ and order λ, and let $L_j : \arg z = \theta_j$ ($j = 1, 2, \ldots, q$; $0 \le \theta_1 < \theta_2 < \cdots < \theta_q < 2\pi$; $\theta_{q+1} = \theta_1 + 2\pi$) be a finite number of rays issuing from the origin. For any $\varepsilon > 0$, set $Y = \bigcup_{j=1}^{q} \{z : \theta_j + \varepsilon \le \arg z \le \theta_{j+1} - \varepsilon\}$. If f satisfies*

$$\limsup_{r\to\infty} \frac{\log n\,(r, Y, ff' = 1)}{\log r} \le \rho$$

for some positive number ρ, then

$$\lambda \le \max_{1\le j\le q} \left\{ \frac{\pi}{\theta_{j+1} - \theta_j}, \quad \rho \right\}.$$

□

Yang [16,17], in general, raised the following conjecture.

Yang's Conjecture : Let P be a property (or a set of properties) such that any entire (or meromorphic in $\mathbb{C}$) function satisfying P must be a constant. Suppose that f is an entire (or meromorphic in $\mathbb{C}$) function of finite lower order μ, and that $L_j : \arg z = \theta_j$ ($j = 1, 2, \ldots, q$; $0 \le \theta_1 < \theta_2 < \cdots < \theta_q < 2\pi$; $\theta_{q+1} = \theta_1 + 2\pi$) are a finite number of rays issuing from the origin. If f satisfies P in $\mathbb{C}\setminus \left(\bigcup_{j=1}^{q} L_j \right)$, then the order λ of f has the estimation

$$\lambda \le \max_{1\le j\le q} \left\{ \frac{\pi}{\theta_{j+1} - \theta_j}, \quad \theta_{q+1} = \theta_1 + 2\pi \right\}.$$

We note that in Theorem B, Yang and Yang chose $P(f, f') = ff'$, which has the *Picard property*, and the property P as $P(f, f') \ne 1$, and investigated the above conjecture. More recently, Fang and Zalcman [8] considered the value distribution of $f + a(f')^n$ in $\mathbb{C}$, where a is a non-zero finite complex number and $n \ge 2$ is a positive integer. Actually, they [8] gave an affirmative answer to a question posed by Ye [14], and obtained the following result.

***Theorem* C** — *Let f be a transcendental meromorphic function in $\mathbb{C}$, and let a be a non-zero finite complex number. Then, for any positive integer $n \ge 2$, the* *function* $P(f, f') = f + a(f')^n$ *assumes every finite complex number infinitely often.* □

In [8], Fang and Zalcman also pointed out that $P(f, f') = f + a(f')^n$ has the *weakly Picard property*, but not always the *Picard property*. Indeed, when $n \geq 3$, $P(f, f') = f + a(f')^n$ has the *Picard property*; but when $n \geq 2$, $P(f, f') = f + a(f')^n$ has the *weakly Picard property*.

Now Yang's Conjecture and Theorem C motivate us to ask the following question.

Question 1 : What can we say about the growth of f if, for any positive integer $n \geq 2$, $P(f, f') = f + a(f')^n$ assumes a value b at infinitely many points but most of these points are distributed in the vicinity of a finite number of rays?

In this paper, we shall investigate the above question. In fact, we shall prove the following result.

***Theorem* 1** — *Let f be a transcendental meromorphic function with $\delta(\infty, f') > 0$ in $\mathbb{C}$, and let $L_j : \arg z = \theta_j$ ($j = 1, 2, \ldots, q$) be a finite number of rays issuing from the origin such that*

$$-\pi \leq \theta_1 < \theta_2 < \cdots < \theta_q < \pi, \quad \theta_{q+1} = \theta_1 + 2\pi,$$

with $\omega = \max\{\pi/(\theta_{j+1} - \theta_j) : 1 \leq j \leq q\}$. *Set* $Y = \mathbb{C} \backslash \left(\bigcup_{j=1}^{q} L_j\right)$. *If f satisfies*

$$\limsup_{r\to\infty} \frac{\log n\,(r, Y, f + a(f')^n = b)}{\log r} \leq \rho \tag{1.2}$$

for some positive number ρ and a finite complex number b for some positive integer $n \geq 2$, then the order λ of f has the estimation $\lambda \leq \max\{\omega, \rho\}$. □

Corollary 1 — Let f be a transcendental entire function and let $L_j : \arg z = \theta_j$ ($j = 1, 2, \ldots, q$) be a finite number of rays issuing from the origin such that

$$-\pi \leq \theta_1 < \theta_2 < \cdots < \theta_q < \pi, \quad \theta_{q+1} = \theta_1 + 2\pi,$$

with $\omega = \max\{\pi/(\theta_{j+1} - \theta_j) : 1 \leq j \leq q\}$. Set $Y = \mathbb{C} \backslash \left(\bigcup_{j=1}^{q} L_j\right)$. If f satisfies

$$\limsup_{r\to\infty} \frac{\log n\,(r, Y, f + a(f')^n = b)}{\log r} \leq \rho$$

for some positive number ρ and a finite complex number b for some positive integer $n \geq 2$, then the order λ of f has the estimation $\lambda \leq \max\{\omega, \rho\}$. $\square$

In order to prove Theorem 1, we require the following result, which is interesting on its own.

***Theorem* 2** — *Let f be a transcendental meromorphic function of finite lower order μ with $\delta(\infty, f') > 0$ in $\mathbb{C}$. For q pairs of real numbers $\{\alpha_j, \beta_j\}$ such that*

$$-\pi \leq \alpha_1 < \beta_1 \leq \alpha_2 < \beta_2 \leq \cdots \leq \alpha_q < \beta_q \leq \pi$$

with $\omega = \max\{\pi/(\beta_j - \alpha_j) : 1 \leq j \leq q\}$, suppose that, for some positive integer $n \geq 2$, we have

$$\limsup_{r\to\infty} \frac{\log n\,(r, Y, f + a(f')^n = b)}{\log r} \leq \rho \tag{1.3}$$

for some positive number ρ, a finite complex number b, where $Y = \bigcup_{j=1}^{q}\{z : \alpha_j \leq \arg z \leq \beta_j\}$, and suppose that

$$\sum_{j=1}^{q}(\alpha_{j+1} - \beta_j) < \frac{4}{\sigma} \arcsin\sqrt{\frac{\delta(\infty, f')}{2}}, \quad \alpha_{q+1} = \alpha_1 + 2\pi, \tag{1.4}$$

with $\sigma = \max\{\omega, \rho, \mu\}$. Then the order λ of f has the estimation $\lambda \leq \max\{\omega, \rho\}$. $\square$

Corollary 2 — Let f be a transcendental entire function of finite lower order μ. For q pairs of real numbers $\{\alpha_j, \beta_j\}$ such that

$$-\pi \leq \alpha_1 < \beta_1 \leq \alpha_2 < \beta_2 \leq \cdots \leq \alpha_q < \beta_q \leq \pi$$

with $\omega = \max\{\pi/(\beta_j - \alpha_j) : 1 \leq j \leq q\}$, suppose that, for some positive integer $n \geq 2$, we have

$$\limsup_{r\to\infty} \frac{\log n\,(r, Y, f + a(f')^n = b)}{\log r} \leq \rho$$

for some positive number ρ, a finite complex number b, where $Y = \bigcup_{j=1}^{q}\{z : \alpha_j \leq \arg z \leq \beta_j\}$, and suppose that

$$\sum_{j=1}^{q}(\alpha_{j+1} - \beta_j) < \frac{\pi}{\sigma}, \quad \alpha_{q+1} = \alpha_1 + 2\pi,$$

with $\sigma = \max\{\omega, \rho, \mu\}$. Then the order λ of f has the estimation $\lambda \leq \max \{\omega, \rho\}$. □

2. Some Lemmas

First of all, we need the Nevanlinna theory of meromorphic functions defined in an angular domain. Let f be a meromorphic function on the angular domain $\overline{\Omega}(\alpha, \beta) = \{z : \alpha \leq \arg z \leq \beta\}$, where $0 < \beta - \alpha \leq 2\pi$. Nevanlinna [13, 9] introduced the following notations.

$$A_{\alpha,\beta}(r, f) = \frac{\omega}{\pi} \int_1^r \left(\frac{1}{t^\omega} - \frac{t^\omega}{r^{2\omega}} \right) \left\{ \log^+ \left| f\left(te^{i\alpha}\right) \right| + \log^+ \left| f\left(te^{i\beta}\right) \right| \right\} \frac{dt}{t},$$

$$B_{\alpha,\beta}(r, f) = \frac{2\omega}{\pi r^\omega} \int_\alpha^\beta \log^+ \left| f\left(re^{i\theta}\right) \right| \sin \omega (\theta - \alpha)\, d\theta,$$

$$C_{\alpha,\beta}(r, f) = 2 \sum_{1<|b_m|<r} \left(\frac{1}{|b_m|^\omega} - \frac{|b_m|^\omega}{r^{2\omega}} \right) \sin \omega (\theta_m - \alpha),$$

where $\omega = \pi / (\beta - \alpha)$ and $b_m = |b_m| e^{i\theta_m}$ are the poles of f in $\overline{\Omega}(\alpha, \beta)$ appearing according to their multiplicities. The function $C_{\alpha,\beta}(r, f)$ is called the angular counting function (counting multiplicities) of the poles of f in $\overline{\Omega}(\alpha, \beta)$, and $\overline{C}_{\alpha,\beta}(r, f)$ is called the angular reduced counting function (ignoring multiplicities) of the poles of f in $\overline{\Omega}(\alpha, \beta)$. Further, Nevanlinna's angular characteristic function $S_{\alpha,\beta}(r, f)$ is defined as follows:

$$S_{\alpha,\beta}(r, f) = A_{\alpha,\beta}(r, f) + B_{\alpha,\beta}(r, f) + C_{\alpha,\beta}(r, f).$$

Throughout the paper, we denote by $R(r, *)$ a quantity satisfying

$$R(r, *) = O\left\{\log\left(rT(r, *)\right)\right\}, \qquad \forall r \notin E,$$

where E denotes a set of positive real numbers with finite linear measure. It is not necessarily the same for every occurrence in the context.

Lemma 1 — [13, 17, 19]. Let f be meromorphic in $\mathbb{C}$. Then in $\overline{\Omega}(\alpha, \beta)$ for an arbitrary finite complex number a, we have

$$S_{\alpha,\beta}\left(r, \frac{1}{f - a}\right) = S_{\alpha,\beta}(r, f) + O(1),$$

and, for each positive integer k, we have

$$A_{\alpha,\beta}\left(r, \frac{f^{(k)}}{f}\right) + B_{\alpha,\beta}\left(r, \frac{f^{(k)}}{f}\right) = R(r, f). \qquad \square$$

Lemma 2 — [11, Theorem 3.1]. Let f be meromorphic in $\mathbb{C}$. Then, for each positive integer k, we have

$$T(r, f^{(k)}) \leq T(r, f) + k\overline{N}(r, f) + O\left(\log rT(r, f)\right) \qquad \forall r \notin E. \qquad \square$$

A little modification of the proof of Theorem 1 in [8] now implies the following important lemma, which will play a key role in the proofs of our results.

Lemma 3 — Let f be transcendental meromorphic in $\mathbb{C}$, let $a \neq 0$ and b be finite complex numbers, and let $n \geq 2$ be an integer. Then in $\overline{\Omega}\,(\alpha, \beta)$:

(i) if $n \geq 3$, then

$$B_{\alpha,\beta}(r, f') \leq \frac{8}{3} C_{\alpha,\beta}\left(r, \frac{1}{f + a(f')^n - b}\right) + R(r, f);$$

(ii) if $n = 2$, then

$$B_{\alpha,\beta}(r, f') \leq 2C_{\alpha,\beta}\left(r, \frac{1}{f + a(f')^n - b}\right) + R(r, f).$$

PROOF : Put

$$g = f + a(f')^n - b, \qquad \phi = \frac{g'}{g}. \tag{2.1}$$

Then $\phi \not\equiv 0$, for otherwise f must be a constant or a polynomial of degree 2, a contradiction. By the Nevanlinna's basic reasoning, Lemma 1, Lemma 2, and (2.1), we have

$$\left(A_{\alpha,\beta} + B_{\alpha,\beta}\right)(r, \phi) = R\,(r, g) = R\,(r, f)\,.$$

Now we rewrite (2.1) in the form

$$f'\left[1 + an(f')^{n-2} f''\right] = \phi\left[f + a(f')^n - b\right].$$

From this and Lemma 1, it thus follows that

$$\begin{aligned}
C_{\alpha,\beta}\left(r,\frac{1}{f'}\right) &+ C_{\alpha,\beta}\left(r,\frac{1}{(f')^{n-2}f''+\frac{1}{na}}\right)\\
&\leq C_{\alpha,\beta}\left(r,\frac{1}{\phi}\right)+C_{\alpha,\beta}\left(r,\frac{1}{f+a(f')^n-b}\right)\\
&\leq C_{\alpha,\beta}(r,\phi)+C_{\alpha,\beta}\left(r,\frac{1}{f+a(f')^n-b}\right)+R\,(r,f)\\
&\quad\text{(by lemma 1)}\\
&\leq \overline{C}_{\alpha,\beta}(r,f)+2C_{\alpha,\beta}\left(r,\frac{1}{f+a(f')^n-b}\right)+R\,(r,f)\\
&\quad\text{(by formula 2.1)}\\
&\leq \frac{1}{2}C_{\alpha,\beta}(r,f')+2C_{\alpha,\beta}\left(r,\frac{1}{f+a(f')^n-b}\right)+R\,(r,f)\,.(2.2)
\end{aligned}$$

On the other hand, by the Nevanlinna's basic reasoning, Lemma 1, Lemma 2, and (2.1) in [11, p.33], we deduce that

$$\begin{aligned}
&(A_{\alpha,\beta}+B_{\alpha,\beta})\left(r,\frac{1}{(f')^{n-1}}\right)+(A_{\alpha,\beta}+B_{\alpha,\beta})\left(r,\frac{1}{(f')^{n-2}f''+\frac{1}{na}}\right)\\
\leq\ &(A_{\alpha,\beta}+B_{\alpha,\beta})\left(r,\frac{1}{(f')^{n-2}f''}\right)+(A_{\alpha,\beta}+B_{\alpha,\beta})\\
&\left(r,\frac{f''}{f'}\right)+(A_{\alpha,\beta}+B_{\alpha,\beta})\left(r,\frac{1}{(f')^{n-2}f''+\frac{1}{na}}\right)\\
\leq\ &(A_{\alpha,\beta}+B_{\alpha,\beta})\left(r,\frac{1}{(f')^{n-2}f''}+\frac{1}{(f')^{n-2}f''+\frac{1}{na}}\right)+R\,(r,f)
\end{aligned}$$

$$\begin{aligned}
\leq\ &(A_{\alpha,\beta}+B_{\alpha,\beta})\left(r,\frac{\left((f')^{n-2}f''\right)'}{(f')^{n-2}f''}+\frac{\left((f')^{n-2}f''+\frac{1}{na}\right)'}{(f')^{n-2}f''+\frac{1}{na}}\right)\\
&+(A_{\alpha,\beta}+B_{\alpha,\beta})\left(r,\frac{1}{\left((f')^{n-2}f''\right)'}\right)+R\,(r,f)
\end{aligned}$$

$$\leq \quad S_{\alpha,\beta}\left(r,\left((f')^{n-2}f''\right)'\right)-C_{\alpha,\beta}\left(r,\frac{1}{\left((f')^{n-2}f''\right)'}\right)+R\left(r,f\right)$$

$$\text{(by lemma 1)}$$

$$\leq \quad S_{\alpha,\beta}\left(r,(f')^{n-2}f''\right)+\overline{C}_{\alpha,\beta}\left(r,f\right)-C_{\alpha,\beta}\left(r,\frac{1}{\left((f')^{n-2}f''\right)'}\right)+R\left(r,f\right)$$

$$\text{(by lemma 2, similarly)}$$

$$\leq \quad S_{\alpha,\beta}\left(r,(f')^{n-2}f''\right)+\frac{1}{2}C_{\alpha,\beta}\left(r,f'\right)$$

$$-C_{\alpha,\beta}\left(r,\frac{1}{\left((f')^{n-2}f''\right)'}\right)+R\left(r,f\right). \tag{2.3}$$

This, together with Lemma 1, yields

$$\begin{aligned}(n-1)S_{\alpha,\beta}\left(r,f'\right) \quad \leq \quad & \frac{1}{2}C_{\alpha,\beta}\left(r,f'\right)+(n-1)C_{\alpha,\beta}\left(r,\frac{1}{f'}\right)\\ & +C_{\alpha,\beta}\left(r,\frac{1}{(f')^{n-2}f''+\frac{1}{na}}\right)\\ & -C_{\alpha,\beta}\left(r,\frac{1}{((f')^{n-2}f'')'}\right)+R\left(r,f\right). \end{aligned} \tag{2.4}$$

If $n \geq 3$, then $n-1 \geq 2 > \frac{3}{2}$ and

$$\begin{aligned} & (n-1)C_{\alpha,\beta}\left(r,\frac{1}{f'}\right)+C_{\alpha,\beta}\left(r,\frac{1}{(f')^{n-2}f''+\frac{1}{na}}\right)-C_{\alpha,\beta}\left(r,\frac{1}{((f')^{n-2}f'')'}\right)\\ = \quad & C_{\alpha,\beta}\left(r,\frac{1}{(f')^{n-1}}\right)+C_{\alpha,\beta}\left(r,\frac{1}{(f')^{n-2}f''+\frac{1}{na}}\right)-C_{\alpha,\beta}\left(r,\frac{1}{((f')^{n-2}f'')'}\right)\\ \leq \quad & 2\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f'}\right)+\overline{C}_{\alpha,\beta}\left(r,\frac{1}{(f')^{n-2}f''+\frac{1}{na}}\right) \end{aligned}$$

This, together with (2.4), gives

$$\frac{3}{2}S_{\alpha,\beta}\left(r,f'\right) \leq \frac{1}{2}C_{\alpha,\beta}\left(r,f'\right) + 2\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f'}\right) + \overline{C}_{\alpha,\beta}\left(r,\frac{1}{(f')^{n-2}f'' + \frac{1}{na}}\right) + R\left(r,f\right).$$

Combining this with (2.2) gives

$$\frac{3}{2}S_{\alpha,\beta}\left(r,f'\right) \leq \frac{3}{2}C_{\alpha,\beta}\left(r,f'\right) + 4C_{\alpha,\beta}\left(r,\frac{1}{f+a(f')^n-b}\right) + R\left(r,f\right),$$

so that

$$B_{\alpha,\beta}(r,f') \leq \frac{8}{3}C_{\alpha,\beta}\left(r,\frac{1}{f+a(f')^n-b}\right) + R\left(r,f\right).$$

If $n=2$, then, by (2.2) and (2.4), we have

$$S_{\alpha,\beta}(r,f') \leq C_{\alpha,\beta}\left(r,f'\right) + 2C_{\alpha,\beta}\left(r,\frac{1}{f+a(f')^n-b}\right) + R\left(r,f\right),$$

so that

$$B_{\alpha,\beta}(r,f') \leq 2C_{\alpha,\beta}\left(r,\frac{1}{f+a(f')^n-b}\right) + R\left(r,f\right).$$

This completes the proof of Lemma 3. □

Lemma 4 [5] — Let f be transcendental meromorphic in $\mathbb{C}$. Then, for a positive integer k and a real number $\tau > 1$, we have

$$T(r,f) < K_{\tau,k}T(\tau r, f^{(k)}) + \log(\tau r) + O(1),$$

where $K_{\tau,k}$ is a positive number depending on only τ and k. □

For the proofs of the theorems, we moreover need some auxiliary results below concerning Pólya peaks and the spread relation.

Lemma 5 [7, 15, 19] — Let f be a transcendental meromorphic function of finite lower order μ and order λ $(0 < \lambda \leq \infty)$ in $\mathbb{C}$. Then, for an arbitrary positive number σ satisfying $\mu \leq \sigma \leq l$ and any set E of finite linear measure, there exist Pólya peaks $\{r_n\}$ satisfying:

(i) $r_n \notin E$, $\lim_{n\to\infty}(r_n/n) = \infty$;

(ii) $\liminf_{n\to\infty} (\log T(r_n, f)) / \log r_n \geq \sigma$;

(iii) $T(t, f) < (1 + o(1)) (t/r_n)^{\sigma} T(r_n, f)$, $\quad t \in [r_n/n, nr_n]$.

A sequence of $\{r_n\}$ satisfying (i), (ii) and (iii) in Lemma 5 is called a Pólya peak of order σ of f outside E. Given a positive function $\Lambda = \Lambda(r)$ on $(0, \infty)$ with $\Lambda \to 0$ as $r \to \infty$, we define

$$D_{\Lambda}(r, a) = \left\{ \theta \in [-\pi, \pi) \left| \log^{+} \frac{1}{|f(re^{i\theta}) - a|} > \Lambda(r) T(r, f) \right. \right\}$$

and

$$D_{\Lambda}(r, \infty) = \left\{ \theta \in [-\pi, \pi) \left| \log^{+} |f(re^{i\theta})| > \Lambda(r) T(r, f) \right. \right\}. \qquad \square$$

Lemma 6 [1] — Let f be a transcendental meromorphic function of finite lower order μ and order λ $(0 < \lambda \leq \infty)$ in $\mathbb{C}$. Suppose that $\delta = \delta(a, f) > 0$ for some $a \in \hat{\mathbb{C}}$. Then for an arbitrary Pólya peak $\{r_n\}$ of order σ $(\mu \leq \sigma \leq \lambda)$ and an arbitrary positive function $\Lambda = \Lambda(r)$ with $\Lambda \to 0$ as $r \to \infty$, we have

$$\liminf_{n\to\infty} \text{meas} D_{\Lambda}(r_n, a) \geq \min\left\{ 2\pi, \ \frac{4}{\sigma} \arcsin \sqrt{\frac{\delta}{2}} \right\}. \qquad \square$$

Finally, we require the following results due to Edrei, Hayman and Miles.

Lemma 7 [6] — Let f be a transcendental meromorphic function with $\delta = \delta(\infty, f) > 0$ in $\mathbb{C}$. Then, given $\epsilon > 0$, we have

$$\text{meas} E(r, f) > \frac{1}{T^{\epsilon}(r, f)[\log r]^{1+\epsilon}}, \qquad \forall r \notin F,$$

where

$$E(r, f) = \left\{ \theta \in [-\pi, \pi) : \log^{+} |f(re^{i\theta})| > \frac{\delta}{4} T(r, f) \right\}$$

and F is a set of positive real numbers with finite logarithmic measure (i.e., $\int_F \frac{dt}{t} < \infty$) depending on ϵ only. $\square$

Lemma 8 [12] — Let f be a transcendental meromorphic function in $\mathbb{C}$. Then for each $K > 1$ there exists a set $M(K)$ of the lower logarithmic density at least $d(K) = 1 - (2e^{K-1} - 1)^{-1} > 0$, that is,

$$\underline{\log \text{dens}} M(K) := \liminf_{r\to\infty} \frac{1}{\log r} \int_{M(K)\cap[1,r]} \frac{dt}{t} \geq d(K),$$

such that, for every $p \in \mathbb{N}$, we have

$$\limsup_{r\to\infty, r\in M(K)} \frac{T(r,f)}{T\left(r, f^{(p)}\right)} \le 3eK. \tag{2.5}$$

□

3. Proofs of Theorems 1 and 2

We shall prove Theorems 1 and 2 by a method whose idea comes from Zheng [19].

3.1 *Proof of Theorem 2* Assume towards a contradiction that Theorem 2 does not hold, i.e., $\lambda(f) > \max\{\omega, \rho\}$. We need to treat two cases separately.

Case 1: $\lambda(f) > \mu(f)$. Then $\lambda(f') = \lambda(f) > \sigma \ge \mu(f) = \mu(f')$. From (1.3), we have

$$n\left(r, \overline{\Omega}(\alpha_j, \beta_j), f + a(f')^n = b\right) \le r^{\rho+\epsilon}, \qquad j = 1, 2, \ldots, q \tag{3.1}$$

for arbitrarily small $\epsilon > 0$ and sufficiently large $r \ge r_0$.

Let ξ_m be the zeros of $f + a(f')^n - b$ on $\overline{\Omega}(\alpha_j, \beta_j)$ appearing according to their multiplicities, and set $\omega_j = \pi/(\beta_j - \alpha_j)$. By the definition of $C_{\alpha,\beta}(r,*)$, we deduce that

$$\begin{aligned}
& C_{\alpha_j,\beta_j}\left(r, \frac{1}{f + a(f')^n - b}\right) \le 2 \sum_{1<|\xi_m|<r} \frac{1}{|\xi_m|^{\omega_j}} \\
= \;& 2\int_1^r \frac{dn\left(t, \overline{\Omega}(\alpha_j, \beta_j), f + a(f')^n = b\right)}{t^{\omega_j}} \\
\le \;& 2\frac{n\left(r, \overline{\Omega}(\alpha_j, \beta_j), f + a(f')^n = b\right)}{r^{\omega_j}} + 2\omega_j \int_1^r \frac{n\left(t, \overline{\Omega}(\alpha_j, \beta_j), f + a(f')^n = b\right)}{t^{\omega_j+1}} dt \\
\le \;& 2r^{\rho+\epsilon-\omega_j} + O(1) + 2\omega_j \int_{r_0}^r \frac{t^{\rho+\epsilon}}{t^{\omega_j+1}} dt \\
\le \;& K_{j,\varepsilon} r^{\rho+\epsilon-\omega_j} + O(\log r),
\end{aligned} \tag{3.2}$$

where $K_{j,\varepsilon}$ is a positive number depending on only j and ε, which is not necessarily the same for every occurrence in the context.

From Lemma 3, we have, for $n \ge 3$,

$$B_{\alpha_j,\beta_j}\left(r, f'\right) \le \frac{8}{3} C_{\alpha_j,\beta_j}\left(r, \frac{1}{f + a(f')^n - b}\right) + R\left(r, f\right), \tag{3.3}$$

and for $n = 2$,

$$B_{\alpha_j,\beta_j}\left(r, f'\right) \leq 2C_{\alpha_j,\beta_j}\left(r, \frac{1}{f + a(f')^n - b}\right) + R\left(r, f\right). \tag{3.4}$$

Thus, it follows by (3.2), (3.3), and (3.4) that, for $n \geq 2$,

$$B_{\alpha_j,\beta_j}(r, f') \leq K_{j,\varepsilon} r^{\rho+\epsilon-\omega_j} + O\left(\log rT(r, f)\right), \qquad \forall r \notin E, \tag{3.5}$$

where the exceptional set E associated with $R(r, f)$ is of at most finite linear measure.

Now from (1.4), we can find a real number $\epsilon > 0$ such that

$$\sum_{j=1}^{q}(\alpha_{j+1} - \beta_j + 2\epsilon) + 2\epsilon < \frac{4}{\sigma + 2\varepsilon} \arcsin\sqrt{\frac{\delta(\infty, f')}{2}}, \quad \alpha_{q+1} = \alpha_1 + 2\pi, \tag{3.6}$$

and

$$\lambda(f') > \sigma + 2\epsilon > \mu(f'). \tag{3.7}$$

Applying Lemma 5 to f' gives the existence of the Pólya peak $\{r_n\}$ of order $\sigma + 2\epsilon$ of f' outside the set E. Then, noting that $\sigma + 2\epsilon > \omega_j \geq 1/2$, by applying Lemma 6 to the Pólya peak $\{r_n\}$, for sufficiently large n we have

$$\text{meas} D_\Lambda\left(r_n, \infty\right) > \frac{4}{\sigma + 2\varepsilon} \arcsin\sqrt{\frac{\delta(\infty, f')}{2}} - \epsilon. \tag{3.8}$$

Without loss of generality, we can assume that (3.8) holds for all $n \in \mathbb{N}$. Set

$$K_n := \text{meas}\left(D_\Lambda(r_n, \infty) \cap \bigcup_{j=1}^{q}(\alpha_j + \epsilon, \beta_j - \epsilon)\right).$$

It then follows from (3.6) and (3.8) that

$$\begin{aligned} K_n &\geq \text{meas}(D_\Lambda(r_n, \infty)) - \text{meas}\left([-\pi, \pi) \setminus \bigcup_{j=1}^{q}(\alpha_j + \epsilon, \beta_j - \epsilon)\right) \\ &= \text{meas}(D_\Lambda(r_n, \infty)) - \text{meas}\left(\bigcup_{j=1}^{q}(\beta_j - \epsilon, \alpha_{j+1} + \epsilon)\right) \\ &= \text{meas}(D_\Lambda(r_n, \infty)) - \sum_{j=1}^{q}(\alpha_{j+1} - \beta_j + 2\epsilon) > \epsilon > 0. \end{aligned} \tag{3.9}$$

By (3.9), it is easy to see that there exists a j_0 such that, for infinitely many $n \in \mathbb{N}$, we have

$$\text{meas}\left(D_\Lambda(r_n,\infty)\cap(\alpha_{j_0}+\epsilon,\beta_{j_0}-\epsilon)\right)\geq\frac{K_n}{q}>\frac{\epsilon}{q}. \tag{3.10}$$

Without loss of generality, we can assume that (3.10) holds for all $n \in \mathbb{N}$. Set $E_n = D_\Lambda(r_n,\infty)\cap(\alpha_{j_0}+\epsilon,\beta_{j_0}-\epsilon)$ and $\Lambda(r)=[\log r]^{-1}$.

From the definition of $D_\Lambda(r_n,\infty)$, we deduce that

$$\begin{aligned}\int_{\alpha_{j_0}+\epsilon}^{\beta_{j_0}-\epsilon}\log^+\left|f'\left(r_ne^{i\theta}\right)\right|d\theta &\geq \int_{E_n}\log^+\left|f'\left(r_ne^{i\theta}\right)\right|d\theta\\ &\geq \frac{\epsilon}{q}\frac{T\left(r_n,f'\right)}{\log r_n}.\end{aligned} \tag{3.11}$$

On the other hand, by the definition of $B_{\alpha,\beta}(r,*)$ and (3.5), it follows that

$$\begin{aligned}\int_{\alpha_{j_0}+\epsilon}^{\beta_{j_0}-\epsilon}\log^+\left|f'\left(r_ne^{i\theta}\right)\right|d\theta &\leq \frac{\pi r_n^{\omega_{j_0}}}{2\omega_{j_0}\sin(\epsilon\omega_{j_0})}B_{\alpha_{j_0},\beta_{j_0}}\left(r_n,f'\right)\\ &\leq K_{j_0,\varepsilon}\left(r_n^{\rho+\epsilon}+r_n^{\omega_{j_0}}\log\left(r_nT(r_n,f)\right)\right),\end{aligned} \tag{3.12}$$

where $r_n \notin E$, $\omega_{j_0}=\pi/(\beta_{j_0}-\alpha_{j_0})$, and $K_{j_0,\varepsilon}$ is a positive number depending on only j_0 and ε. Combining (3.11) with (3.12) gives

$$T\left(r_n,f'\right)\leq\frac{qK_{j_0,\varepsilon}\log r_n}{\epsilon}\left(r_n^{\rho+\epsilon}+r_n^{\omega_{j_0}}\log\left(r_nT(r_n,f)\right)\right), \tag{3.13}$$

implying together with Lemma 4 and (iii) in Lemma 5 that

$$\begin{aligned}\log T\left(r_n,f'\right)\leq 3\log\log r_n+\max\{\rho+\epsilon,\omega_{j_0}\}\\ \log r_n+\log\log T(r_n,f')+O(1).\end{aligned} \tag{3.14}$$

Thus, from (ii) in Lemma 5 for $\sigma+2\epsilon$, we have

$$\sigma+2\epsilon\leq\limsup_{n\to\infty}\frac{\log T(r_n,f')}{\log r_n}\leq\max\{\rho+\epsilon,\omega_{j_0}\}\leq\sigma+\epsilon,$$

which is impossible.

Case $2: \lambda(f) = \mu(f)$. Then $\sigma = \mu(f) = \lambda(f) = \lambda(f') = \mu(f')$. By the same argument as in Case 1 with all the $\sigma + 2\epsilon$ replaced by $\sigma = \mu(f)$, we can derive

$$\mu(f) = \sigma \leq \max\{\rho + \epsilon, \omega_{j_0}\} < \lambda(f),$$

which is also impossible.

This completes the proof of Theorem 2. □

3.2 *Proof of Theorem 1*

By Theorem 2, it suffices to prove that the lower order $\mu(f)$ of f is finite. As in the proof of Theorem 2, we have, for each $j \in \{1, 2, \ldots, q\}$,

$$B_{\theta_j,\theta_{j+1}}(r, f') \leq K_{j,\varepsilon} r^{\rho+\epsilon-\omega_j} + O\left(\log rT(r, f)\right), \quad \omega_j = \frac{\pi}{\theta_{j+1} - \theta_j}, \quad \forall r \notin E, \tag{3.15}$$

where the exceptional set E associated with $R(r, f)$ is of at most finite linear measure.

For F in Lemma 7 and E in (3.15), $\overline{\log \text{dens}}(F \cup E) = 0$ and hence for $M(2)$ in Lemma 8 when $K = 2$, $\underline{\log \text{dens}}\,(M(2) \setminus (F \cup E)) \geq d(2) > 0$. Applying Lemma 7 to f' gives the existence of a sequence $\{r_n\}$ of positive numbers such that $r_n \to \infty$ $(n \to \infty)$, $r_n \in M(2) \setminus (F \cup E)$, and

$$\text{meas} E(r_n, f') > \frac{1}{T^{\epsilon}(r_n, f')[\log r_n]^{1+\epsilon}}. \tag{3.16}$$

Set

$$\epsilon_n = \frac{1}{2q+1} \frac{1}{T^{\epsilon}(r_n, f')[\log r_n]^{1+\epsilon}}. \tag{3.17}$$

Then, from (3.17), it follows that

$$\begin{aligned}
&\text{meas}\left(E(r_n, f') \cap \bigcup_{j=1}^{q} (\theta_j + \epsilon_n, \theta_{j+1} - \epsilon_n)\right) \\
\geq\; &\text{meas}(E(r_n, f') - \text{meas}\left(\bigcup_{j=1}^{q} (\theta_j - \epsilon_n, \theta_j + \epsilon_n)\right) \\
\geq\; &(2q+1)\epsilon_n - 2q\epsilon_n = \epsilon_n > 0.
\end{aligned} \tag{3.18}$$

Hence, there exists a $j \in \{1, 2, \dots, q\}$ such that, for infinitely many $n \in \mathbb{N}$, we have

$$\text{meas}\left(E(r_n, f') \cap (\theta_j + \epsilon_n, \theta_{j+1} - \epsilon_n)\right) > \frac{\epsilon_n}{q}. \tag{3.19}$$

Without loss of generality, we can assume that this holds for all $n \in \mathbb{N}$. Let $E_n = E(r_n, f') \cap (\theta_j + \epsilon_n, \theta_{j+1} - \epsilon_n)$. Thus, by the definition of $E(r, f)$ and (3.18), it follows that

$$\begin{aligned} \int_{\theta_j+\epsilon}^{\theta_{j+1}-\epsilon} \log^+ \left| f'\left(r_n e^{i\theta}\right) \right| d\theta &\geq \int_{E_n} \log^+ \left| f'\left(r_n e^{i\theta}\right) \right| d\theta \\ &\geq \text{meas}(E_n) \frac{\delta(\infty, f')}{4} T\left(r_n, f'\right) \\ &\geq \frac{\epsilon_n \delta(\infty, f')}{4q} T\left(r_n, f'\right). \end{aligned} \tag{3.20}$$

On the other hand, by the definition of $B_{\alpha,\beta}(r, *)$ and (3.15), we have

$$\begin{aligned} \int_{\theta_j+\epsilon}^{\theta_{j+1}-\epsilon} \log^+ \left| f'\left(r_n e^{i\theta}\right) \right| d\theta &\leq \tfrac{\pi r^{\omega_j}}{2\omega_j \sin(\epsilon_n \omega_j)} B_{\theta_j, \theta_{j+1}}\left(r_n, f'\right) \\ &\leq \tfrac{\pi^2 K_{j,\varepsilon}}{4\omega_j^2 \epsilon_n} \left(r_n^{\rho+\epsilon} + r_n^{\omega_j} \log\left(r_n T(r_n, f)\right)\right), \end{aligned} \tag{3.21}$$

where $r_n \notin E \cup F$, $\omega_j = \pi/(\theta_{j+1} - \theta_j)$, and $K_{j,\varepsilon}$ is a positive number depending on only j and ε. Combining (3.20) with (3.21) now yields

$$\epsilon_n^2 T\left(r_n, f'\right) \leq \frac{q\pi^2 K_{j,\varepsilon}}{\omega_j^2 \delta(\infty, f')} \left(r_n^{\rho+\epsilon} + r_n^{\omega_j} \log\left(r_n T(r_n, f)\right)\right), \tag{3.22}$$

so that, together with (3.17) and (2.5), we have

$$\begin{aligned} T^{1-2\epsilon}(r_n, f') \quad \leq \quad & \frac{q\pi^2 (2q+1)^2 [\log r_n]^{2+2\epsilon} K_{j,\varepsilon}}{\omega_j^2 \delta(\infty, f')} \left(r_n^{\rho+\epsilon}\right) \\ & + r_n^{\omega_j} \left(\log r_n + \log T(r_n, f') + \log(6e)\right). \end{aligned} \tag{3.23}$$

Thus $\mu(f) = \mu(f') \leq \max\{\rho + \epsilon, \omega_j\}/(1 - 2\epsilon) < \infty$ and so Theorem 1 follows from Theorem 2.

This completes the proof of Theorem 1. □

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