

RANDERS CHANGE OF m^{th} ROOT METRIC

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The present paper deals with a Randers metric that has been derived after a particular β -change in the m th root metric. Various geometers such as [7], [9], [10] etc. have studied the m th root metric and its transformations. We have obtained some tensors and theorems holding the relation between the Finsler space equipped with the m th root metric and the one obtained after its Randers change.

Key words : Finsler space, Randers space, m th root metric and β -change.

1. INTRODUCTION

A Finsler space F^n is said to be equipped with (α, β) metric, if the metric function $L(x, y)$ is positively homogeneous of degree one in α and β . Matsumoto [4] in 1971 introduced a particular transformation of Finsler metric $\alpha(x, y)$ defined as:

$$L(x, y) = \alpha(x, y) + \beta(x, y), \quad (1.1)$$

where $\alpha(x, y)$ is the fundamental function of Finsler space F^n and $\beta(x, y) = b_i(x) y^i$ is a differential one form. Such a change of Finsler space is termed as β -change.

Randers [8] then introduced the Randers space with metric $L = \alpha + \beta$, where $\alpha = (a_{ij}(x)y^i y^j)^{1/2}$ is a Riemannian metric and β is as defined above. Later in 1978 Numata [6] introduced another metric $L = \mu + \beta$, where $\mu = (a_{ij}(y)y^i y^j)^{1/2}$ is a Minkowski metric and $\beta(x, y) = b_i(x) y^i$. This metric is similar to the Randers one but with different geometrical properties. In papers [1], [2], [4] etc. the corresponding β -change of various Finsler metrics have been studied from different viewpoints to obtain several theorems.

In 1979 Shimada [9] introduced the m th root metric on the differentiable manifold M^n defined as:

$$L = \sqrt[m]{a_{i_1 \dots i_m} y^{i_1} \dots y^{i_m}}, \quad (1.2)$$

where the coefficients $a_{i_1 \dots i_m}$ are the components of symmetric covariant tensor field of order $(0, m)$ being the functions of positional co-ordinates only.

Since then various geometers such as [7], [10] etc. have explored the theory of m th root metric and studied its transformations.

In the present paper we have considered a transformation of the m th root metric such that it transforms to a similar metric as the Randers one defined in (1.1) in a way that the Riemannian metric α is replaced with the m th root metric λ defined in (1.2). Hence this paper deals with a Finsler space F^n equipped with the fundamental function $L(x, y)$ defined as:

$$L(x, y) = \lambda(x, y) + \beta(x, y), \quad (1.3)$$

where $\lambda = (a_{i_1 \dots i_m}(x)y^{i_1} \dots y^{i_m})^{1/m}$ is the metric function of Finsler space \overline{F}^n and $\beta(x, y) = b_i(x) y^i$ is a differential one form. In this paper we shall study the relation between various tensors of Finsler space F^n with that of \overline{F}^n .

2. PRELIMINARIES

Let M^n be an n -dimensional smooth manifold and $F^n = (M^n, L)$ be an n -dimensional Finsler space equipped with the fundamental function $L(x, y)$ defined in (1.3). As we are concerned here with the transformation of m th root metric, we shall therefore restrict ourselves for $m > 2$ throughout the paper.

The normalized supporting element l_i , angular metric tensor h_{ij} , metric tensor g_{ij} and Cartan's C -tensor C_{ijk} [3] are defined respectively as:

$$l_i = \frac{\partial L}{\partial y^i}, \quad h_{ij} = L \frac{\partial^2 L}{\partial y^i \partial y^j}, \quad g_{ij} = \frac{1}{2} \frac{\partial^2 L^2}{\partial y^i \partial y^j}$$

and

$$C_{ijk} = \frac{1}{4} \frac{\partial^3 L^2}{\partial y^i \partial y^j \partial y^k}. \quad (2.1)$$

Dealing with the m th root metric we shall first introduce the tensors $a_i(x, y)$, $a_{ij}(x, y)$ and $a_{ijk}(x, y)$ as defined in [7] :

$$\begin{aligned} (a) \quad & \alpha^{m-1} a_i = a_{ii_2 \dots i_m} y^{i_2} \dots y^{i_m}, \\ (b) \quad & \alpha^{m-2} a_{ij} = a_{ij i_3 \dots i_m} y^{i_3} \dots y^{i_m} \\ (c) \quad & \alpha^{m-3} a_{ijk} = a_{ijk i_4 \dots i_m} y^{i_4} \dots y^{i_m}. \end{aligned} \quad (2.2)$$

The differentiation of $a_{i_1 \dots i_r}$ with respect to y^j is given by [10]:

$$\dot{\partial}_j a_{i_1 \dots i_r} = \frac{(m-r)}{\lambda} (a_{i_1 \dots i_r, j} - a_{i_1 \dots i_r} a_j) \quad (2.3)$$

where $r = 1 \dots n$.

3. FUNDAMENTAL TENSORS

We consider an n -dimensional Finsler space F^n with metric $L(x, y)$ defined in (1.3). The differentiation of (1.3) with respect to y^i yields the normalized supporting element l_i as:

$$l_i = a_i + b_i, \quad (3.1)$$

where l_i and a_i are defined in (2.1) and (2.2) respectively.

In view of (2.2) and (2.3) differentiation of (3.1) with respect to y^j yields:

$$h_{ij} = (m-1) \frac{L}{\lambda} (a_{ij} - a_i a_j). \quad (3.2)$$

The above equation can be re-written as:

$$\frac{h_{ij}}{L} = \frac{\lambda_{ij}}{\lambda}, \quad (3.3)$$

where λ_{ij} is the angular metric tensor of the Finsler space \overline{F}^n equipped with metric λ .

On account of (2.1), (2.2) and (2.3) the fundamental tensor $g_{ij}(x, y)$ of Finsler space F^n is given as:

$$g_{ij} = (m-1)\tau a_{ij} + (a_i b_j + b_i a_j) + b_i b_j + (1 - \overline{m-1}\tau) a_i a_j, \quad (3.4)$$

where $\tau = L/\lambda$.

We may now rewrite the above expression as:

$$g_{ij} = p_0 a_{ij} + p_1 (a_i b_j + b_i a_j) + p_2 b_i b_j + p_3 a_i a_j, \quad (3.5)$$

where $p_0 = (m-1)\tau$,

$$p_1 = 1,$$

$$p_2 = 1$$

and

$$p_3 = 1 - (m-1)\tau.$$

The tensor $a_{ij}(x, y)$ is called the basic tensor and is supposed to have a non-vanishing determinant.

In order to obtain the contravariant metric tensor $g^{ij}(x, y)$ we here assume the covariant entity H_{ij} defined as:

$$H_{ij} = p_0 a_{ij} + c_i c_j, \quad (3.6)$$

where $c_i = \pi b_i$ and p_0 is defined in (3.5).

The contravariant tensor $H^{ij}(x, y)$ is now defined as:

$$H^{ij} = \frac{1}{p_0} \left[a^{ij} - \frac{c^i c^j}{p_0 + c^2} \right], \quad (3.7)$$

where $c^i = g^{ij} c_j$ and c_i is already defined in (3.6).

In view of (3.5) the unknown quantity π is obtained using the following relations:

$$\pi_{-1}^2 = p_3,$$

$$(ii) \quad \pi_0 = \frac{p_2}{\pi_{-1}}$$

and

$$(iii) \quad \pi^2 = p_1 - \pi_0^2.$$

Substituting the above result in equation (3.7) we get:

$$H^{ij} = \frac{1}{(m-1)\tau} \left[a_{ij} + \frac{b_i b_j}{1 - (m-1)\tau - b^2} \right], \quad (3.8)$$

where $b^i = a^{ij} b_j$,

$$b^2 = g^{ij} b_i b_j = b_i b^i \quad (3.9)$$

and τ being defined in (3.4).

The metric tensor g_{ij} can now be written in the following form:

$$g_{ij} = H_{ij} + d_i d_j, \quad (3.10)$$

$$\text{where } d_i = \pi_0 b_i + \pi_{-1} a_i = \frac{1}{\sqrt{1 - (m-1)\tau}} [b_i + (1 - \overline{m-1}\tau) a_i].$$

Then $g^{ij}(x, y)$ is defined as:

$$g^{ij}(x, y) = H^{ij} - \frac{d^i d^j}{1 + d^2}, \quad (3.11)$$

$$\text{where } d^i = H^{ij} d_j = \frac{\sqrt{1 - \overline{m-1}\tau}}{(m-1)\tau} \left[\frac{(1+q)}{1 - \overline{m-1}\tau - b^2} b^i + a^i \right],$$

$$q = \frac{\beta}{\lambda} = b_i a^i = a_i b^i,$$

$$a^i = a^{ij} a_j$$

and $d^2 = H^{ij} d_i d_j$

$$\begin{aligned} &= \frac{1}{(m-1)(1 - \overline{m-1}\tau - b^2)\tau} [(1 - \overline{m-1}\tau - b^2)(q + 1 - \overline{m-1}\tau) \\ &\quad + (1+q)\{q(1 - \overline{m-1}\tau) + b^2\}]. \end{aligned}$$

Hence $g^{ij}(x, y)$ is given as:

$$g^{ij} = \frac{1}{(m-1)\tau} \left[a^{ij} - \frac{1}{(1+q)} (a^i b^j + b^i a^j) + \frac{(b^2 + \overline{m-1}\tau - 1)}{(1+q)^2} a^i a^j \right]. \quad (3.12)$$

Theorem 3.1 — *The metric tensors $g_{ij}(x, y)$ and $g^{ij}(x, y)$ of the Finsler space F^n equipped with the Randers metric obtained by transforming the m th root metric are given as:*

$$g_{ij} = (m-1)\tau a_{ij} + (a_i b_j + b_i a_j) + b_i b_j + (1 - \overline{m-1}\tau) a_i a_j$$

and

$$g^{ij} = \frac{1}{(m-1)\tau} \left[a^{ij} - \frac{1}{(1+q)} (a^i b^j + b^i a^j) + \frac{(b^2 + \overline{m-1}\tau - 1)}{(1+q)^2} a^i a^j \right].$$

The differentiation of τ with respect to y^i is given as:

$$\frac{\partial \tau}{\partial y^i} = \frac{(1 - \tau)a_i + b_i}{\lambda}. \quad (3.13)$$

In view of (2.2), (2.3) and (3.12) we differentiate (3.4) with respect to y^k to obtain the following result:

$$\begin{aligned} 2\lambda C_{ijk} &= (m-1)[(m-2)\tau a_{ijk} + (1 - \overline{m-1}\tau)(a_i a_{jk} + a_j a_{ik} + a_k a_{ij}) \\ &\quad + (a_{ij} b_k + a_{jk} b_i + a_{ki} b_j) - (a_i a_j b_k + a_j a_k b_i + a_k a_i b_j) \\ &\quad + (\overline{2m-1}\tau - 3)a_i a_j a_k]. \end{aligned} \quad (3.14)$$

Theorem 3.2 — *The Cartan's C -tensor $C_{ijk}(x, y)$ under the Randers change of m th root metric takes the following form:*

$$\begin{aligned} 2\lambda C_{ijk} &= (m-1)[(m-2)\tau a_{ijk} + (1 - \overline{m-1}\tau)(a_i a_{jk} + a_j a_{ik} + a_k a_{ij}) \\ &\quad + (a_{ij} b_k + a_{jk} b_i + a_{ki} b_j) - (a_i a_j b_k + a_j a_k b_i + a_k a_i b_j) \\ &\quad + (\overline{2m-1}\tau - 3)a_i a_j a_k]. \end{aligned}$$

Now, C_{ijk} is re-written as:

$$C_{ijk} = \tau p_{ijk} + (\lambda_{ij} m_k + \lambda_{jk} m_i + \lambda_{ik} m_j) / 2\lambda, \quad (3.15)$$

where p_{ijk} is the (h)hv- torsion tensor of the Cartan's connection CT of the m th root Finsler metric λ given as:

$$\begin{aligned} p_{ijk} &= \frac{(m-1)(m-2)}{2\lambda} [a_{ijk} - (a_i a_{jk} + a_j a_{ki} + a_k a_{ij}) + 2a_i a_j a_k], \\ \lambda_{ij} &= (m-1)(a_{ij} - a_i a_j) \\ \text{and } m_i &= b_i - \frac{\beta}{\lambda} a_i. \end{aligned} \quad (3.16)$$

Theorem 3.3— *The (h)hv-torsion tensor $p_{ijk}(x, y)$ of the Cartan's connection CT of the Randers metric obtained by β transformation of the m th root metric is given by:*

$$p_{ijk} = \frac{(m-1)(m-2)}{2\lambda} [a_{ijk} - (a_i a_{jk} + a_j a_{ki} + a_k a_{ij}) + 2a_i a_j a_k].$$

4. THE V-CURVATURE TENSOR OF F^n

In view of (3.15) and definition of a^i we now have the following identities:

$$\begin{aligned} a^i a_i &= 1, & a_{ijk} a^i &= a_{jk}, & m_i a^i &= 0, & m_i b^i &= b^2 - q^2, \\ \lambda_{ij} a^i &= 0, & \lambda_{ij} b^i &= (m-1)m_i & \text{and} & p_{ijk} a^i &= 0. \end{aligned} \quad (4.1)$$

In order to find the v-curvature tensor of the Finsler space F^n it is necessary to obtain the (h)hv-torsion tensor $C_{jk}^i = g^{ir} C_{jrk}$.

Hence transvection of (3.13) with g^{ir} yields:

$$\begin{aligned} C_{jk}^i &= \frac{1}{(m-1)} p_{jk}^i + \frac{1}{2L(m-1)} (\lambda_j^i m_k + \lambda_k^i m_j + \lambda_{jk} m^i) \\ &- \frac{a^i}{L(1+q)} \left[m_j m_k + \frac{(b^2 - q^2)}{2(m-1)} \lambda_{jk} \right] - \frac{1}{(m-1)(1+q)} a^i p_{jrk} b^r, \end{aligned} \quad (4.2)$$

where

$$\begin{aligned} p_{jk}^i &= p_{jrk} a^{ir} = \frac{(m-1)(m-2)}{2\lambda} [a_{jk}^i - (\delta_j^i a_k + \delta_k^i a_j + a^i a_{jk}) + 2a^i a_j a_k], \\ \lambda_j^i &= \lambda_{jr} a^{ir} = (m-1)(\delta_j^i - a^i a_j), \\ m^i &= m_r a^{ir} = b^i - q a^i \\ \text{and } a_{jk}^i &= a^{ir} a_{jrk}. \end{aligned} \quad (4.3)$$

From (4.1) and (4.3) we now have another set of identities as following:

$$\begin{aligned}
 p_{ijr}\lambda_h^r &= p_{ij}^r\lambda_{rh} = (m-1)p_{ijh}, & p_{ijr}m^r &= p_{ijr}b^r, & m_r\lambda_i^r &= (m-1)m_i, \\
 m_im^i &= b^2 - q^2, & \lambda_{ir}\lambda_j^r &= (m-1)\lambda_{ij} \\
 \text{and} & & \lambda_{ir}m^r &= (m-1)m_i.
 \end{aligned} \tag{4.4}$$

In view of (3.13), (4.1), (4.2) and (4.3) we have:

$$\begin{aligned}
 C_{ijr}C_{hk}^r &= \frac{\tau}{(m-1)}p_{ijr}p_{hk}^r + \frac{1}{2\lambda(m-1)}(p_{ijr}\lambda_{hk} + p_{hkr}\lambda_{ij})b^r \\
 &+ \frac{1}{4\lambda L}(2m_hm_k\lambda_{ij} + m_im_k\lambda_{hj} + m_im_h\lambda_{jk} + 2m_im_j\lambda_{hk} \\
 &+ m_jm_k\lambda_{ih} + m_jm_h\lambda_{ik}) \\
 &+ \frac{1}{2\lambda}(p_{ijh}m_k + p_{ijk}m_h + p_{jkh}m_i + p_{ihk}m_j) \\
 &+ \frac{1}{4\lambda L}(b^2 - q^2)\lambda_{ij}\lambda_{hk}.
 \end{aligned} \tag{4.5}$$

The v-curvature tensor S_{hijk} is given by [3]:

$$S_{hijk} = C_{ijr}C_{hk}^r - C_{ikr}C_{hj}^r. \tag{4.6}$$

Now, interchanging indices j and k in (4.4) we get:

$$\begin{aligned}
 C_{ikr}C_{hj}^r &= \frac{\tau}{(m-1)}p_{ikr}p_{hj}^r + \frac{1}{2\lambda(m-1)}(p_{ikr}\lambda_{hj} + p_{hjr}\lambda_{ik})b^r \\
 &+ \frac{1}{4\lambda L}(2m_hm_j\lambda_{ik} + m_im_j\lambda_{hk} + m_im_h\lambda_{jk} + 2m_im_k\lambda_{hj} \\
 &+ m_jm_k\lambda_{ih} + m_km_h\lambda_{ij}) \\
 &+ \frac{1}{2\lambda}(p_{ikh}m_j + p_{ijk}m_h + p_{jkh}m_i + p_{ihj}m_k) \\
 &+ \frac{1}{4\lambda L}(b^2 - q^2)\lambda_{ik}\lambda_{hj}.
 \end{aligned} \tag{4.7}$$

In view of (4.6) subtracting (4.7) from (4.5) we get:

$$S_{hijk} = Q_{(jk)} \left\{ \frac{\tau}{(m-1)} p_{ijr} p_{hk}^r + \lambda_{hk} m_{ij} + \lambda_{ij} m_{hk} \right\}, \quad (4.8)$$

where

$$m_{ij} = \frac{1}{2\lambda(m-1)} \left\{ p_{ijr} b^r + \frac{(m-1)}{2L} m_i m_j + \frac{1}{4L} (b^2 - q^2) \lambda_{ij} \right\}$$

and the symbol $Q_{(jk)}$ denotes the interchange of indices j and k followed by subtraction.

Theorem 4.1 — *The v-curvature tensor $S_{hijk}(x, y)$ under the Randers change of m th root metric takes the following form:*

$$S_{hijk} = Q_{(jk)} \left\{ \frac{\tau}{(m-1)} p_{ijr} p_{hk}^r + \lambda_{hk} m_{ij} + \lambda_{ij} m_{hk} \right\}.$$

The reciprocal $\bar{g}^{ij}(x, y)$ of metric tensor $\bar{g}_{ij}(x, y)$ of the Finsler space \bar{F}^n equipped with m th root metric is given by [7]:

$$\bar{g}^{ij} = \frac{1}{(m-1)} \{ a^{ij} + (m-2) a^i a^j \}. \quad (4.9)$$

In order to relate the v-curvature tensor of F^n with the m th root Finsler space of \bar{F}^n we may now find the (h)hv-torsion tensor of \bar{F}^n given as:

$$\bar{p}_{jk}^i = \bar{g}^{ir} p_{jrk} = \frac{1}{(m-1)} p_{jk}^i, \quad (4.10)$$

where \bar{g}^{ij} is defined in (4.9).

Thus, the v-curvature tensor \bar{S}_{hijk} of m th root Finsler space \bar{F}^n is given as:

$$\bar{S}_{hijk} = p_{ijr} \bar{p}_{hk}^r - p_{ikr} \bar{p}_{hj}^r = \frac{1}{(m-1)} (p_{ijr} p_{hk}^r - p_{ikr} p_{hj}^r). \quad (4.11)$$

In view of (4.11) S_{hijk} may now be re-written as:

$$S_{hijk} = \tau \bar{S}_{hijk} + Q_{(jk)}(\lambda_{ij}m_{hk} + \lambda_{hk}m_{ij}), \quad (4.12)$$

where λ_{ij} , m_{ij} and \bar{S}_{hijk} are defined in (3.3), (4.7) and (4.10) respectively.

From the definition of $S4$ -like Finsler space we know that the v -curvature tensor of the four dimensional Finsler space may be written as [3]:

$$L^2 S_{hijk} = Q_{(jk)}(h_{hj}K_{ik} + h_{ik}K_{hj}), \quad (4.13)$$

where K_{ij} is a symmetric tensor field of type $(0, 2)$ such that $K_{ij}y^j = 0$.

Hence, if the v -curvature tensor of the Finsler space F^n vanishes, then the Finsler space \bar{F}^n is $S4$ -like.

Theorem 4.2 — *The Finsler space \bar{F}^n equipped with the m th root metric is $S4$ -like, if the v -curvature tensor $S_{hijk}(x, y)$ of the Finsler space F^n obtained by the Randers change of m th root metric vanishes.*

If the v -curvature tensor of the Finsler space \bar{F}^n vanishes then (4.12) reduces to:

$$S_{hijk} = \lambda_{ij}m_{hk} + \lambda_{hk}m_{ij} - \lambda_{hj}m_{ik} - \lambda_{ik}m_{hj}. \quad (4.14)$$

Contraction of (4.14) with g^{hj} yields the Ricci tensor $S_{ik} = g^{hj}S_{hijk}$ given as:

$$S_{ik} = -\frac{1}{(m-1)\tau} [m\lambda_{ik} + (m-1)(n-3)m_{ik}], \quad (4.15)$$

where $m = m_{ij}a^{ij}$.

Using the definition of m_{ij} the above expression may be re-written as:

$$S_{ik} + H_1\lambda_{ik} + H_2p_{ikt}b^r = H_3m_im_k, \quad (4.16)$$

where

$$H_1 = \frac{m}{(m-1)\tau} + \frac{(n-3)}{8(m-1)L^2}(b^2 - q^2),$$

$$H_2 = \frac{(n-3)}{2L(m-1)}$$

and

$$H_3 = -\frac{(n-3)}{4L^2}.$$

Theorem 4.3 — *If the v-curvature tensor \bar{S}_{hijk} of the Finsler space \bar{F}^n equipped with the m th root metric vanishes then there exist scalars H_1 and H_2 in the Finsler space F^n ($n \geq 4$) obtained under the Randers change of m th root metric, such that the matrix $\|S_{ik} + H_1 \lambda_{ik} + H_2 p_{ikr} b^r\|$ is of rank less than two.*

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