Indian J. Pure Appl. Math., **43**(3): 197-210, June 2012

TRAPPING WAVES BY A SUBMERGED CYLINDER

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(*Received* 6 *April* 2011; *after final revision* 13 *March* 2012; *accepted* 13 *March* 2012)

For large values of the wavenumber k in the direction of cylinder, there is only one trapped wave. We construct asymptotics of these trapped modes and their frequencies as $k \to \infty$ in the case of one submerged cylinder into a plane water layer by means of reducing the initial problem to three integral equations on the boundaries and then solving them using a method suggested by Zhevandrov and Merzon (*Amer. Math. Soc. Translations* (2) **208**, 235-284 (2003)).

Key words : Boundary value problem, asymptotics, eigenfunctions, eigenvalues, water waves.

1. Introduction

It is well-known that submerged horizontal cylinders can serve as waveguides for water waves. The first result in this direction was obtained by Ursell [10] for a cylinder of circular cross-section. Later it was discovered that horizontal “bumps” on the bottom can also trap waves (see [2, 5]). In [2], Bonnet-Ben Dhia and Joly proved that for large values of the wavenumber k in the direction of the ridge, there is only one trapped mode. In [7] we have constructed explicitly this trapped mode for large values of k in the case of a ridge and also indicated the formula for the frequency in the case of one submerged cylinder. In [8] we obtained this trapped mode for large values of k in the case of one or two submerged cylinders.

In the present paper we construct the trapped mode for large values of the frequency in the case of submerged cylinder into a plane water layer. The problem of the ridge of small height was treated in [11], where a close analogy of the problem of water waves and small perturbations of the one-dimensional Schrödinger equation is established. The latter problem was studied by a number of authors (we mention, for example, [6, 9, 3], and, in the context of water waves, [4]). In our case, a technique similar to that of [11] yields the desired result. We note that in contrast to [11] the asymptotics turns out to be exponential, i.e., the distance of the trapped wave frequency to the cut-off frequency is exponentially small in k. This fact seemingly could have rendered the problem quite complicated from the point of view of asymptotic expansions, but, since in fact we construct an exact convergent expansion, no additional difficulties arise.

2. Mathematical Formulation and Main Results

The geometry of the problem is as follows: we assume that $\Gamma_C = \{x = x(t), y =$

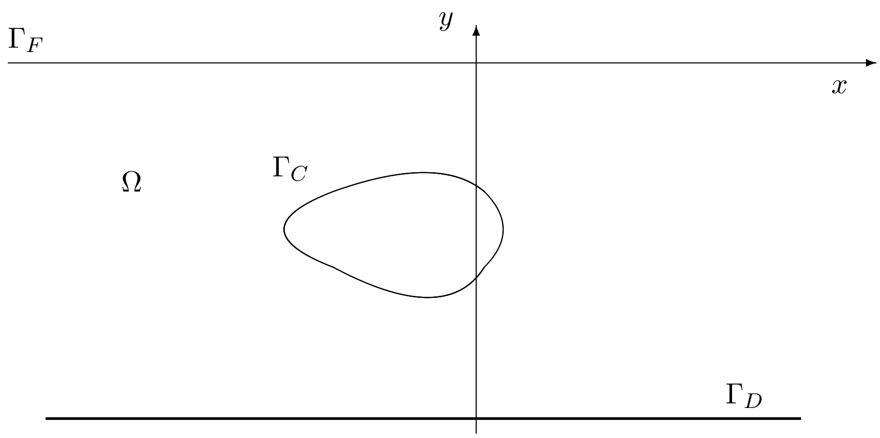

Figure 1:

$y(t), t \in [-\pi, \pi]\}$ with smooth $x(t)$ and $y(t)$,

$$x'^2 + y'^2 \neq 0, \tag{1}$$

and $\max y(t) = y(0)$, $y''(0) < 0$, $x'(0) > 0$; where y is the vertical coordinate, x is the horizontal coordinate orthogonal to y, Γ_C describes curve bounding her cross-section, similarly, $\Gamma_D = \{(x, -h_0) : x \in \mathbb{R}\}$ and Γ_D describes the bottom of the ocean. We assume that Γ_C and Γ_D do not intersect,i.e., at least $y(t) + h_0 \geq d > 0$. $\Gamma_F = \{(x, 0) : x \in \mathbb{R}\}$ is the free surface. The water layer Ω is the domain exterior to Γ_C and Γ_D and lying below Γ_F (see Fig. 1).

Looking for the velocity potential in the form $\exp\{i(\omega t - kz)\}\Phi(x, y)$, ω is the frequency, we come to the problem

$$\Phi_y = \lambda\Phi, \qquad y = 0, \tag{2}$$

$$\Phi_{xx} + \Phi_{yy} - k^2\Phi = 0 \qquad \text{in} \qquad \Omega, \tag{3}$$

$$\partial\Phi/\partial\vec{n}_C = 0 \qquad \text{on} \qquad \Gamma_C, \tag{4}$$

$$\Phi_y = 0 \qquad \text{on} \qquad \Gamma_D, \tag{5}$$

for the function Φ; here $\lambda = \omega^2/g$, g is the aceleration of gravity. Solutions of this problem from the Sobolev space $H_1(\Omega)$ are called trapped waves and exist only for certain values of λ (the eigenparameter) for k fixed. It is known that essential spectrum of (2)–(5) coincides with the interval $[k \tanh kh_0, \infty)$. There exists only one eigenvalue λ below the essential spectrum for large values of k. Our goal is to construct an asymptotics of this frequency. Our main result is as follows.

***Theorem* 1** — *The unique eigenvalue $\lambda(k)$ of (2)–(5) has the form*

$$\lambda = k \tanh kh_0 - \beta^2, \tag{6}$$

where

$$\beta = k \left(\sqrt{\frac{2\pi}{|y''(0)|}} e^{2y(0)k} x'(0)(1 + O(k^{-1})) \right). \tag{7}$$

In the next section we construct the corresponding eigenfunction.

3. Reduction to Integral Equations and their Solution

We reduce (2)–(5) to three integral equations on Γ_F, Γ_C and Γ_D for the functions $\varphi = \Phi|_{y=0}$, $\theta = \Phi|_{\Gamma_C}$ and $\alpha = \Phi|_{\Gamma_D}$. To this end, we consider the region $\Omega \setminus B_\rho(\xi,\eta)$, where $B_\rho(\xi,\eta) = \{(x,y) : \sqrt{(x-\xi)^2 + \eta^2} < \rho\}$ is the disk of radius ρ with center in (ξ,η), and we apply the Green formula to $\Phi(\xi,\eta)$ and $((-1/2\pi)K_0(kr))$ in $\Omega \setminus B_\rho$, where $r = \sqrt{(x-\xi)^2 + \eta^2}$ and K_0 is the Macdonald function (so that $(-1/2\pi)K_0(kr)$ is the fundamental solution of the operator $\Delta - k^2$).

Making $\rho \to 0$, we obtain

$$
\begin{aligned}
\Phi(\xi,\eta) \quad = \quad & \frac{\lambda}{2\pi}\int\limits_{-\infty}^{\infty} K_0(k\sqrt{(x-\xi)^2+\eta^2})\varphi(x)dx \\
& +\frac{k\eta}{2\pi}\int\limits_{-\infty}^{\infty} \frac{K_0'(k\sqrt{(x-\xi)^2+\eta^2})}{\sqrt{(x-\xi)^2+\eta^2}}\varphi(x)dx \\
& -\frac{k}{2\pi}\int\limits_{-\pi}^{\pi} \frac{K_0'(k\sqrt{(x(t)-\xi)^2+(y(t)-\eta)^2})}{\sqrt{(x(t)-\xi)^2+(y(t)-\eta)^2}} \\
& \times[y'(t)(x(t)-\xi)-x'(t)(y(t)-\eta)]\theta(t)dt \\
& -\frac{k}{2\pi}\int\limits_{-\infty}^{\infty} \frac{K_0'(k\sqrt{(x-\xi)^2+(h_0+\eta)^2})}{\sqrt{(x-\xi)^2+(h_0+\eta)^2}} \\
& \times(h_0+\eta)\alpha(x)dx, \\
& (\xi,\eta)\in\Omega. \qquad (8)
\end{aligned}
$$

Passing in equation (8) to the limit when $\eta \to 0^-$; $\xi \to x(t)$, $\eta \to y(t)$; and $\eta \to -h_0$, we obtain the following integral equations

$$
\begin{aligned}
\pi\varphi(\xi) \quad = \quad & \lambda\int\limits_{-\infty}^{\infty} K_0(k|x-\xi|)\varphi(x)dx \\
& -k\int\limits_{-\pi}^{\pi} \frac{K_0'(k\sqrt{(x(t)-\xi)^2+y(t)^2})}{\sqrt{(x(t)-\xi)^2+y(t)^2}} \\
& \times[y'(t)(x(t)-\xi)-x'(t)y(t)]\theta(t)dt \\
& -k\int\limits_{-\infty}^{\infty} \frac{K_0'(k\sqrt{(x-\xi)^2+h_0^2})}{\sqrt{(x-\xi)^2+h_0^2}} \times h_0\alpha(x)dx, \qquad (9)
\end{aligned}
$$

$$\begin{aligned}
\pi\theta(t) &= \lambda \int_{-\infty}^{\infty} K_0(k\sqrt{(x-x(t))^2+y(t)^2})\varphi(x)dx \\
&+ky(t)\int_{-\infty}^{\infty} \frac{K_0'(k\sqrt{(x-x(t))^2+y(t)^2})}{\sqrt{(x-x(t))^2+y(t)^2}}\varphi(x)dx \\
&-k\int_{-\pi}^{\pi} \frac{K_0'(k\sqrt{(x(t_1)-x(t))^2+(y(t_1)-y(t))^2})}{\sqrt{(x(t_1)-x(t))^2+(y(t_1)-y(t))^2}} \\
&\times[y'(t_1)(x(t_1)-x(t))-x'(t_1)(y(t_1)-y(t))]\theta(t_1)dt_1 \\
&-k\int_{-\infty}^{\infty} \frac{K_0'(k\sqrt{(x-x(t))^2+(h_0+y(t))^2})}{\sqrt{(x-x(t))^2+(h_0+y(t))^2}} \\
&\times(h_0+y(t))\alpha(x)dx, \qquad (10)
\end{aligned}$$

$$\begin{aligned}
\pi\alpha(\iota) &= \lambda \int_{-\infty}^{\infty} K_0(k\sqrt{(x-\iota)^2+h_0^2})\varphi(x)dx \\
&-kh_0\int_{-\infty}^{\infty} \frac{K_0'(k\sqrt{(x-\iota)^2+h_0^2})}{\sqrt{(x-\iota)^2+h_0^2}}\varphi(x)dx \\
&-k\int_{-\pi}^{\pi} \frac{K_0'(k\sqrt{(x(t_1)-\iota)^2+(y(t_1)+h_0)^2})}{\sqrt{(x(t_1)-\iota)^2+(y(t_1)+h_0)^2}} \\
&\times[y'(t_1)(x(t_1)-\iota)-x'(t_1)(y(t_1)+h_0)]\theta(t_1)dt_1 \qquad (11)
\end{aligned}$$

In order to apply the technique of [11] to (9) it is necessary to pass to the Fourier transform $\tilde{\varphi}$ of the function φ,

$$\mathcal{F}_{\xi\to p}[\varphi(\xi)](p) \equiv \tilde{\varphi}(p) = \int_{-\infty}^{\infty} e^{ip\xi}\varphi(\xi)d\xi.$$

We use the formulas (see [1])

$$
\begin{aligned}
K_0'(z) &= -K_1(z), \quad \mathcal{F}_{\xi\to p}[K_0(k|\xi|)](p) = \frac{\pi}{\sqrt{k^2+p^2}}, \\
& \mathcal{F}_{\xi\to p}\left[\frac{K_1(k\sqrt{\xi^2+h_0^2})}{\sqrt{\xi^2+h_0^2}}\right](p) = \frac{\pi}{kh_0}e^{-h_0\sqrt{k^2+p^2}}, \\
& \mathcal{F}_{\xi\to p}\left[K_0(k\sqrt{\xi^2+h_0^2})\right](p) = \frac{\pi}{\sqrt{k^2+p^2}}e^{-h_0\sqrt{k^2+p^2}}. \quad (12)
\end{aligned}
$$

Passing to the Fourier transform $\tilde{\varphi}$ of the function φ and using (12), we come to the following system for $\tilde{\varphi}(p)$, $\theta(t)$ and $\alpha(t)$:

$$
\begin{aligned}
\left(1-\frac{\lambda}{\tau(p)}\right)\tilde{\varphi}(p) &= \int_{-\pi}^{\pi} e^{ipx(t)+y(t)\tau(p)}\left(x'(t)-\frac{ipy'(t)}{\tau(p)}\right)\theta(t)dt \\
&\quad + e^{-h_0\tau(p)}\tilde{\alpha}(p), \qquad (13)
\end{aligned}
$$

$$
\begin{aligned}
\theta(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ip'x(t)+y(t)\tau(p')}\left(1+\frac{\lambda}{\tau(p')}\right)\tilde{\varphi}(p')dp' \\
&\quad -\frac{k}{\pi}\int_{-\pi}^{\pi}\frac{K_0'(k\sqrt{\varrho(t_1,t)})}{\sqrt{\varrho(t_1,t)}}\sigma_1(t_1,t)\theta(t_1)dt_1 \\
&\quad +\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ipx(t)-(h_0+y(t))\tau(p)}\tilde{\alpha}(p)dp, \qquad (14)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\alpha}(p) &= e^{-h_0\tau(p)}\left(1+\frac{\lambda}{\tau(p)}\right)\tilde{\varphi}(p) \\
&\quad + \int_{-\pi}^{\pi} e^{-ipx(t_1)-(y(t_1)+h_0)\tau(p)}\left(\frac{ip}{\tau(p)}y'(t_1)+x'(t_1)\right) \\
&\quad \theta(t_1)dt_1, \qquad (15)
\end{aligned}
$$

where

$$\begin{aligned}\tau(p) &:= \sqrt{k^2+p^2},\\ \varrho(t_1,t) &:= (x(t_1)-x(t))^2+(y(t_1)-y(t))^2,\\ \sigma(t_1,t) &:= y'(t_1)(x(t_1)-x(t))-x'(t_1)(y(t_1)-y(t)).\end{aligned} \tag{16}$$

Rewrite system (13)–(15) as

$$\left(1-\frac{\lambda}{\tau(p)}\right)\tilde{\varphi}(p) = (\hat{M}_1\theta)(p)+(\hat{M}_2\tilde{\alpha})(p), \tag{17}$$

$$[(1-\hat{M}_5)\theta](t) = (\hat{M}_3\tilde{\varphi})(t)+(\hat{M}_4\tilde{\alpha})(t), \tag{18}$$

$$\tilde{\alpha}(p) = (\hat{M}_6\tilde{\varphi})(p)+(\hat{M}_7\theta)(p). \tag{19}$$

where

$$\begin{aligned}(\hat{M}_1\theta)(p) &= \int_{-\pi}^{\pi} M_1(p,t)\theta(t)dt,\\ (\hat{M}_2\tilde{\alpha})(p) &= e^{-h_0\tau}\tilde{\alpha}(p),\\ (\hat{M}_3\tilde{\varphi})(t) &= \int_{-\infty}^{\infty} M_3(p',t)\tilde{\varphi}(p')dp',\\ (\hat{M}_4\tilde{\alpha})(t) &= \int_{-\infty}^{\infty} M_4(p,t)\tilde{\alpha}(p)dp,\\ (\hat{M}_5\theta)(t) &= \int_{-\pi}^{\pi} M_5(t_1,t)\theta(t_1)dt_1,\\ (\hat{M}_6\tilde{\varphi})(p) &= e^{-h_0\tau(p)}\left(1+\frac{\lambda}{\tau(p)}\right)\tilde{\varphi}(p),\\ (\hat{M}_7\theta)(p) &= \int_{-\pi}^{\pi} M_7(p,t_1)\theta(t_1)dt_1,\end{aligned}$$

with

$$
\begin{aligned}
M_1(p,t) &= e^{ipx(t)+y(t)\tau(p)}\left(x'(t) - \frac{ipy'(t)}{\tau(p)}\right), \qquad (20)\\
M_2(p) &= e^{-h_0\tau(p)},\\
M_3(p',t) &= \frac{1}{2\pi}e^{-ip'x(t)+y(t)\tau(p')}\left(1+\frac{\lambda}{\tau(p')}\right),\\
M_4(p,t) &= \frac{1}{2\pi}e^{-ipx(t)-(h_0+y(t))\tau(p)},\\
M_5(t_1,t) &= -\frac{k}{\pi}\frac{K_0'(k\sqrt{\varrho(t_1,t)})}{\sqrt{\varrho(t_1,t)}}\sigma(t_1,t),\\
M_6(p) &= \frac{1}{2\pi}e^{-h_0\tau(p)}\left(1+\frac{\lambda}{\tau(p)}\right),\\
M_7(p,t_1) &= e^{-ipx(t_1)-(y(t_1)+h_0)\tau(p)}\left(\frac{ip}{\tau(p)}y'(t_1)+x'(t_1)\right),
\end{aligned}
$$

Consider equations (18), (19). It is not hard to see, using the asymptotics of $K_1(z)$ for small and large z, that the operator $\hat{M}_5$ in (18) is bounded by const $k^{-1/2}$. In fact, the following lemma holds.

Lemma 1 — We have

$$\left|\int_{-\pi}^{\pi} M_5(t_1,t)\theta(t_1)\,dt_1\right| \le Ck^{-1/2}\|\theta\|,$$

where C is a constant and $\|\theta\| = \sup_{t\in[-\pi,\pi]}|\theta(t)|$.

PROOF : For a given $\delta > 0$, we divide the interval of integration in two domains, $k|t_1 - t| < \delta$ and $k|t_1 - t| > \delta$. In the first domain we use the asymptotics $K_0'(z) \sim \frac{1}{z}$, and in the second, the asymptotics $K_0'(z) \sim \sqrt{\frac{\pi}{2z}}e^{-z}$. For $k|t - t_1| < \delta$, we have by (20)

$$|M_5(t_1,t)| \le C_1\frac{|\sigma(t_1,t)|}{\varrho(t_1,t)}.$$

The numerator here is $O((t_1 - t)^2)$, and

$$\sqrt{\varrho(t_1,t)} \ge c|t_1 - t|, \qquad c > 0, \tag{21}$$

by (1). Hence $M_5(t_1,t)$ is bounded in this domain. For $k|t_1-t|>\delta$ we have, again by (1),

$$|M_5(t_1,t)| \le C_2 k^{1/2} e^{-k\sqrt{\varrho(t_1,t)}} \frac{|\sigma(t_1,t)|}{(\varrho(t_1,t))^{3/4}}.$$

The last factor is bounded by virtue of (21) and by the same inequality we obtain

$$|M_5(t_1,t)| \le C_3 k^{1/2} e^{-ck|t_1-t|}. \tag{22}$$

Since $e^{-k|t_1-t|} \ge e^{-\delta}$ = const for $|t_1-t|<\delta/k$, we see that (22) holds for all t_1, t. Now

$$\left| \int_{-\pi}^{\pi} M_5(t_1,t)\theta(t_1)\,dt_1 \right| \le \text{const} \int_{-\pi}^{\pi} k^{1/2} e^{-ck|t_1-t|} dt_1 \|\theta\| \le Ck^{-1/2}\|\theta\|$$

as claimed.

We see that $\|\hat{M}_5\| \le \text{const}k^{-1/2}$. Hence we can invert the operator $(1-\hat{M}_5)$ using the Neumann series. Moreover, $\hat{M}_{4,7}$ are exponentially small since $h_0 + y(t) \ge d > 0$. Solving (18), (19) for θ and $\tilde{\alpha}$, we obtain

$$\begin{aligned}\theta(t) =& \{[1-(1-\hat{M}_5)^{-1}\hat{M}_4\hat{M}_7]^{-1} \\ & \times (1-\hat{M}_5)^{-1}[\hat{M}_3+\hat{M}_4\hat{M}_6]\tilde{\varphi}\}(t),\end{aligned} \tag{23}$$

$$\begin{aligned}\tilde{\alpha}(p) =& \{[1-\hat{M}_7(1-\hat{M}_5)^{-1}\hat{M}_4]^{-1} \\ & \times [\hat{M}_6+\hat{M}_7(1-\hat{M}_5)^{-1}\hat{M}_3]\tilde{\varphi}\}(p),\end{aligned} \tag{24}$$

where $(1-\hat{M}_j)^{-1} = \sum_{n=0}^{\infty} \hat{M}_j^n$. Substituting (23), (24) in (13) we finally come to

$$\left(1-\frac{\lambda}{\tau(p)}\right)\tilde{\varphi}(p) = [\hat{M}_{11}\tilde{\varphi}](p), \tag{25}$$

where

$$\hat{M}_{11} = \hat{M}_9\hat{M}_3 + \hat{M}_{10}\hat{M}_6,$$

$$\begin{aligned}\hat{M}_9 =&\hat{M}_1[1-(1-\hat{M}_5)^{-1}\hat{M}_4\hat{M}_7]^{-1}(1-\hat{M}_5)^{-1}\\ &+\hat{M}_2[1-\hat{M}_7(1-\hat{M}_5)^{-1}\hat{M}_4]^{-1}\hat{M}_7(1-\hat{M}_5)^{-1},\\ \hat{M}_{10} =&\hat{M}_1[1-(1-\hat{M}_5)^{-1}\hat{M}_4\hat{M}_7]^{-1}(1-\hat{M}_5)^{-1}\hat{M}_4\\ &+\hat{M}_2[1-\hat{M}_7(1-\hat{M}_5)^{-1}\hat{M}_4]^{-1}.\end{aligned}$$

We know that λ is given by (6), where β is exponentially small in k, i.e., the distance of the trapped wave frequency to the cut-off frequency is exponentially small in k, (see [2]). Hence the first factor in the left-hand side of (25),

$$L(p) := 1 - \frac{\lambda}{\tau(p)} = 1 - \frac{k \tanh kh_0 - \beta^2}{\sqrt{k^2+p^2}}, \tag{26}$$

is exponentially small in k for $p = 0$. In fact, the roots of $L(p) = 0$ which tend to zero as $k \to \infty$, as it is not hard to see, are simple and given by

$$p = p_\pm = \pm\frac{i\sqrt{2}\beta}{\sqrt{\varepsilon}} + O(\varepsilon^{1/2}\beta^3), \qquad \varepsilon = \frac{1}{k}. \tag{27}$$

Since $L(p) \sim Const(p^2+\beta^2)$ for small p, we see that integral equation (25) is similar to the integral equation of Section 2 of heuristic considerations of [11] and our arguments are still valid if we change $\tilde{\varphi}$ in the form $\tilde{\varphi}(p) = A(p)/L(p)$. As we shall see (see formula (29) below), that $A(p)$ is analytic and using the fact that $M_3(p,t)$, is analytic in a strip containing the real axis, then we can change the contour of integration in the integral

$$\int_{-\infty}^{\infty} M_3(p,t)\frac{A(p)}{L(p)}dp,$$

to the one shown in Fig. 2.

We have, by the residue theorem,

$$\int_{-\infty}^{\infty} M_3(p,t)\frac{A(p)}{L(p)}dp = \int_{\gamma} M_3(p,t)\frac{A(p)}{L(p)}dp + 2\pi i\frac{M_3(p_+,t)A(p_+)}{\frac{d}{dp}L(p)|_{p=p_+}}. \tag{28}$$

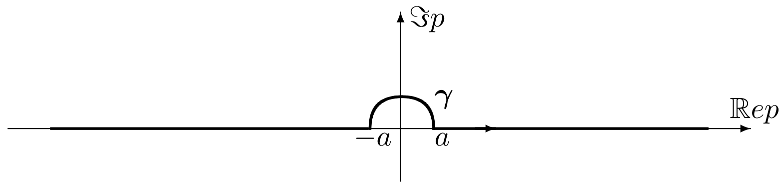

Figure 2:

Thus (25) transforms into

$$A(p) = \hat{M}_{11}^{\gamma} A(p) + g(p)A(p_+), \tag{29}$$

where

$$\hat{M}_{11}^{\gamma} = \hat{M}_9 \hat{M}_3^{\gamma} + \hat{M}_{10}\hat{M}_6,$$

$$[\hat{M}_3^{\gamma} A](t) = \int\limits_{\gamma} M_3(p', t)\frac{A(p')}{L(p')}dp' \qquad f_3(t) = 2\pi i \frac{M_3(p_+, t)}{\frac{d}{dp}L(p)|_{p=p_+}},$$

$$g(p) = (\hat{M}_9 f_3(t) + \hat{M}_{10}\hat{M}_6)(p).$$

Observe now that the operator $\hat{M}_{11}^{\gamma}$ is small in ε, $\varepsilon = 1/k$ since $|L(p)| \geq const\ k^{-2}$ along γ and $M_2(p)$ is exponentially small. Indeed, on the arc we have up to $O(k^{-\infty})$

$$|L(p)| = \left| 1 - \frac{1}{\sqrt{1 + p^2/k^2}} \right| = \frac{a^2}{2k^2} + O(k^{-4}), \tag{30}$$

and on the part of the contour which lies on the real axis the minimum of $|L(p)|$ is attained at the points $p = \pm a$, hence, the above estimate still holds.

Rewriting (29) as

$$[(1 - \hat{M}_{11}^{\gamma})A](p) = g(p)A(p_+), \tag{31}$$

we see that $(1 - \hat{M}_{11}^{\gamma})$ is invertible and

$$A(p) = [(1 - \hat{M}_{11}^{\gamma})^{-1} g](p)A(p_+). \tag{32}$$

Putting $p = p_+$ in the last equality and dividing by $A(p_+)$, we obtain an equation for β:

$$1 = [(1 - \hat{M}_{11})^{-1} g](p)|_{p=p_+}. \tag{33}$$

A standard application of the Laplace method of asymptotic evaluation of integrals to the leading term in (33) yields formula (7). In fact, from the leading term in (33),

$$\beta \sim \frac{\sqrt{2}\pi}{\varepsilon^{3/2}} \left(\int_{-\pi}^{\pi} M_1(p_+, t) M_3(p_+, t) dt \right), \tag{34}$$

with $M_j(p_+, t_1)$, $j = 1, 3$, defined in (20). We have

$$\beta \sim \frac{\sqrt{2}}{\varepsilon^{3/2}} \left(\int_{-\pi}^{\pi} e^{2ky(t)} x'(t) dt \right).$$

Applying the Laplace method to the last integrals, we obtain (7).

ACKNOWLEDGMENT

The authors express their deep gratitude to CONACYT-México, Programa de Mejoramiento del Profesorado (PROMEP)-México, CIC-UMSNH, and Universidad de Cartagena for financial support.

REFERENCES

1. M. Abramowitz and I. A. Stegun, *Handbook of mathematical functions*, Dover Publ., Inc., New York (1970).

2. A. S. Bonnet-Ben Dhia and P. Joly, Mathematical analysis of guided water waves, *SIAM J. Appl. Math.*, **53** (1993), 1507–1550.

3. R. D. Gadyl′shin, Local perturbations of the Schrödinger operator on the axis, *Theor. Math. Phys.* **132** (2002), 976–982.

4. D. S. Kuznetsov, A spectrum perturbation problem and its applications to waves above an underwater ridge, *Siberian Math. J.*, **42** (2001), 668–694.

5. N. Kuznetsov, V. Maz'ya and B. Vainberg, *Linear water waves*, Cambridge University Press, Cambridge (2002).

6. L. D. Landau and E. M. Lifshitz, *Quantum Mechanics*, Pergamon, London (1958).

7. A. M. Marín, R. D. Ortíz and P. Zhevandrov, High-frequency asymptotics of waves trapped by underwater ridges and submerged cylinders, *J. Comp. Appl. Math.*, **204** (2007), 356–362.

8. A. M. Marín, R. D. Ortíz and P. Zhevandrov, Waves Trapped by Submerged Obstacles at High Frequencies, *J. Appl. Math.*, Article ID 80205, **2007** (2007), 17 pages.

9. B. Simon, The bound state of weakly coupled Schrödinger operator in one and two dimensions, *Ann. Phys.*, (NY) **97** (1976), 279–288.

10. F. Ursell, Trapping modes in the theory of surface waves, *Proc. Cambridge Phil. Soc.*, **47** (1951), 347–358.

11. P. Zhevandrov and A. Merzon, Asymptotics of eigenfunctions in shallow potential wells and related problems, *Amer. Math. Soc. Translations*, (2) **208** (2003), 235–284.