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WAVE INTERACTIONS FOR A NONLINEAR DEGENERATE WAVE EQUATIONS[1]

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This paper is concerned with the interactions of the elementary waves for the nonlinear degenerate wave equations. By analyzing the expressions of the elementary waves and the relative locations of the left state U_l and the right state U_r in the phase plane (u, v) we deal with the interactions of the elementary waves, especially the overtaking of shock wave and rarefaction wave from the same family.

Key words : Degenerate wave equations, wave interaction, Riemann problem, degenerate shock, rarefaction wave, shock wave.

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1. Introduction

We are concerned with the nonlinear degenerate wave equations

$$\begin{cases} v_t - u_x = 0, \\ u_t - (|v|^{\gamma-1}v)_x = 0, \end{cases} \tag{1.1}$$

where v and u are the strain and the velocity, respectively, $|v|^{\gamma-1}v$ is the stress function, $\gamma > 1$ is a constant.

The stress function $\sigma(v) := |v|^{\gamma-1}v$ satisfies

$$\sigma'(v) = \gamma|v|^{\gamma-1} > 0, \quad v \neq 0, \quad \sigma'(0) = 0, \tag{1.2}$$

and

$$\sigma''(v) = \gamma(\gamma-1)|v|^{\gamma-1}v^{-1} = \begin{cases} < 0, & v < 0, \\ > 0, & v > 0. \end{cases} \tag{1.3}$$

So the stress function is not convex or concave and the shock condition may be degenerate. From the characteristic theory, we know that all the characteristics parallel with each other in the degenerate region $v = 0$, i.e., degenerate shock appears.

System (1.1) arises in the theory of elasticity by ignoring the effect of viscosity which attract many people's attention. Sun and Sheng [9] discussed the Riemann problem for this system with the characteristic method [1-3, 8-10]. By introducing the degenerate shock under the generalized shock condition [9], they obtained constructively the Riemann solutions for (1.1). Lu [4] studied the related second-order partial differential equation

$$v_{tt} = c(|v|^{\gamma-1}v)_{xx}, \qquad \gamma > 1, \tag{1.4}$$

and obtained the existence of the solutions to (1.4) by studying the compactness of entropy-entropy flux pairs to the viscosity form of (1.1). For one-dimensional Euler equations, discussions on the interactions of the elementary waves we refer the readers to [1, 2, 7] and for an isentropic magnetogasdynamic model, we refer to [6].

In this paper, our main purpose is to discuss all possible interactions of the elementary waves obtained in solving the Riemann problem for (1.1). By analyzing the explicit expressions of the shock waves and the rarefaction waves and investigating the relative locations of the left state U_l and the right state U_r in the (u, v) phase plane, we discuss the interactions of the elementary waves especially the interactions of shock waves and rarefaction waves from the same family.

This paper is arranged as follows. In Section 2, we restate the Riemann problem for system (1.1). In Section 3, by studying the relative locations of U_l and U_r in the phase plane we discuss all possible interactions of the elementary waves obtained from the corresponding Riemann problem.

2. PRELIMINARIES

For the convenience of the readers, we sketch the relevant results of the Riemann problem for (1.1). Consider the Riemann problem for (1.1) with the initial data

$$(v, u)(x, 0) = \begin{cases} (v_l, u_l), & x < 0, \\ (v_r, u_r), & x > 0. \end{cases} \tag{2.1}$$

System (1.1) has two real eigenvalues

$$\lambda_1 = -\sqrt{\gamma |v|^{\gamma-1}} \leq 0 \leq \sqrt{\gamma |v|^{\gamma-1}} = \lambda_2, \tag{2.2}$$

and the corresponding right-eigenvectors $\vec{r}_j = (1, \mp\sqrt{\gamma |v|^{\gamma-1}}),\ (j = 1, 2)$. It follows that the characteristic regions $v < 0$ and $v > 0$ are genuinely nonlinear.

For a given left state $U_l(v_l, u_l)$, let us consider all the states $U(v, u)$ that can be connected on the right to $U_l(v_l, u_l)$ by a 1-(2-)rarefaction wave curve denoted by R_1 (R_2).

Lemma 2.1 [9] — The 1-(2-)rarefaction wave curves are given respectively as follows

$$R_1: \quad u = u_l + \int_{v_l}^{v} \sqrt{\gamma |s|^{\gamma-1}} ds, \quad 0 < v < v_l \text{ or } v_l < v < 0, \tag{2.3}$$

$$R_2: \quad u = u_l - \int_{v_l}^{v} \sqrt{\gamma |s|^{\gamma-1}} ds, \quad v > v_l > 0 \text{ or } v < v_l < 0. \tag{2.4}$$

Furthermore,

$$\frac{du}{dv} > 0, \quad on \;\; R_1,$$

and

$$\frac{du}{dv} < 0, \quad on \;\; R_2.$$

We now turn to study the discontinuity line $x = x(t)$ which is a shock satisfying the Rankine-Hugoniot conditions and the generalized shock condition. Denote 1-(2-)shock wave curve by $S_1(S_2)$, which means a curve consisting of all the possible states $U(v, u)$ that can be connected on the right to the given left state $U_l(v_l, u_l)$.

The shock wave solutions satisfy the Rankine-Hugoniot conditions

$$\begin{cases} -s[v] - [u] = 0, \\ -s[u] - [|v|^{\gamma-1}v] = 0, \end{cases} \tag{2.5}$$

and the generalized shock condition

$$\frac{1}{s}h(v) \equiv \frac{1}{s}(-s^2(v - v_\pm) + \sigma(v) - \sigma(v_\pm)) \begin{cases} < 0, & v_r < v < v_l, \\ > 0, & v_l < v < v_r, \end{cases} \tag{2.6}$$

where $s = \frac{d}{dt}x(t)$, $[G] = G_l - G_r$.

When $v_l v_r \leq 0$, a kind of discontinuity is needed which can be defined as a degenerate shock wave for the case $s = \lambda_i(U_l)$ or $s = -\lambda_i(U_r)$ of (2.6) ([5]).

Definition 2.1 — A discontinuity $x = 0$ is called a degenerate shock of the system (1.1), if $v_l = v_r = 0, [u] \neq 0$ on its two sides. It is denoted by S_d.

Lemma 2.2 [9] — The 1-(2-)shock wave curves satisfy the entropy condition

$$\lambda_i(U_r) < s_i < \lambda_i(U_l), \quad i = 1, 2, \tag{2.7}$$

and given respectively by

$$S_1: \quad u = u_l + g(v_l, v), \tag{2.8}$$

$$S_2: \quad u = u_l - g(v_l, v), \tag{2.9}$$

where for 1-shock

$$g(v_l, v) = \begin{cases} \sqrt{[|v|^{\gamma-1}v][v]}, & v > v_l \geq 0, \\ \sqrt{[|v|^{\gamma-1}v][v]}, & v < v_l \leq 0, \end{cases} \tag{2.10}$$

and for 2-shock

$$g(v_l, v) = \begin{cases} \sqrt{[|v|^{\gamma-1}v][v]}, & v_l > v \geq 0, \\ \sqrt{[|v|^{\gamma-1}v][v]}, & v_l < v \leq 0. \end{cases} \tag{2.11}$$

In addition,

$$\frac{du}{dv} = \begin{cases} \frac{\gamma v^{\gamma-1}[v]+[v^\gamma]}{2\sqrt{[v^\gamma][v]}} > 0, & v > v_l \geq 0, \\ \frac{\gamma|v|^{\gamma-1}[|v|]+[|v|^\gamma]}{2\sqrt{[|v|^\gamma][|v|]}} > 0, & v < v_l \leq 0, \end{cases} \quad on \ S_1,$$

$$\frac{du}{dv} = \begin{cases} \dfrac{\gamma v^{\gamma-1}[v]+[v^\gamma]}{2\sqrt{[v^\gamma][v]}} < 0, & v_l > v \geq 0, \\ \frac{\gamma|v|^{\gamma-1}[|v|]+[|v|^\gamma]}{2\sqrt{[|v|^\gamma][|v|]}} < 0, & v_l < v \leq 0, \end{cases} \quad on \ S_2.$$

According to the relative location relations between U_l and U_r, we know that there are five cases for the solution of the Riemann problem. For simplicity, we just give the Riemann solution of the following cases: $v_l > 0, v_r \geq 0$, $v_l > 0, v_r < 0$, and $v_l = 0$. The other cases, which can be studied similarly, are omitted.

Case 1 : $\quad v_l > 0, v_r \geq 0.$

We can put all of these curves R_1, R_2, S_1 and S_2 together in the (v, u) plane (Fig. 2.1.). Here we remark that the curves $R_1(R_2)$ and $S_1(S_2)$ have second-order contact at U_l.

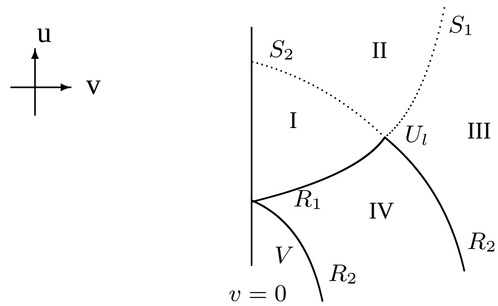

Fig. 2.1.

The solutions of the Riemann problem (1.1) and (2.1) for this case are depicted in Fig. 2.2.

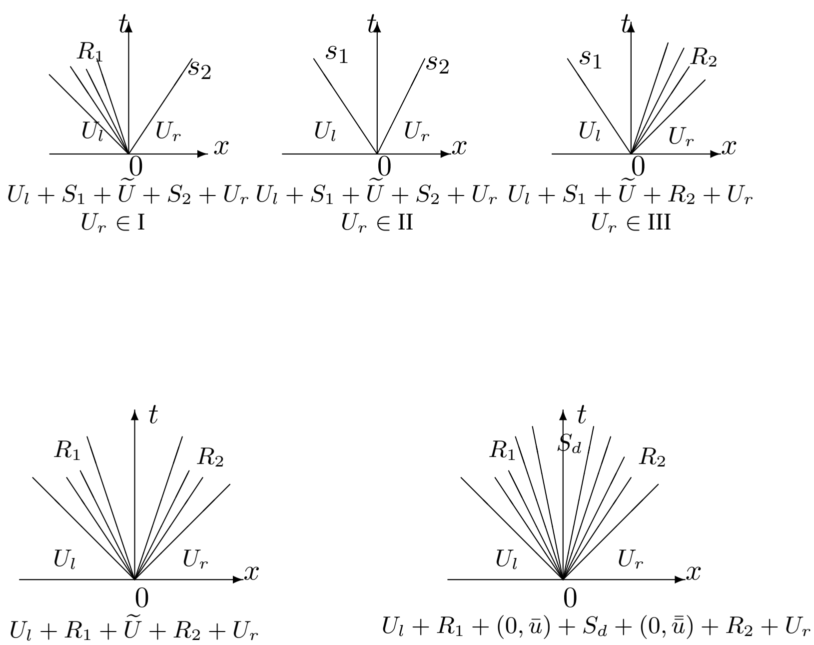

Fig. 2.2. $v_l > 0$

In Fig. 2.2, "+" means followed by, $\bar{u} = u_l + \int_{v_l}^{0} \sqrt{\gamma y^{\gamma-1}}dy$, $\bar{\bar{u}} = u_r + \int_0^{v_r} \sqrt{\gamma y^{\gamma-1}}dy$ and $\tilde{U}$ is determined by

$$\begin{cases} \tilde{u} - u_l = \int_{v_l}^{\tilde{v}} \sqrt{\gamma y^{\gamma-1}}dy, \\ u_r - \tilde{u} = \sqrt{(|\tilde{v}|^{\gamma-1}\tilde{v} - |v_r|^{\gamma-1}v_r)(\tilde{v} - v_r)}. \end{cases} \tag{2.12}$$

Case 2 : $\quad v_l > 0, v_r < 0.$

For this case, we still refer to Fig. 2.1. There exists a R_1 connecting U_l with $(0, \bar{u})$ in the region $v > 0$ and a R_2 connecting U_r with $(0, \bar{\bar{u}})$ in the region $v < 0$. We link $(0, \bar{u})$ to $(0, \bar{\bar{u}})$ by a degenerate shock S_d and obtain that the Riemann solution is $R_1 + S_d + R_2$.

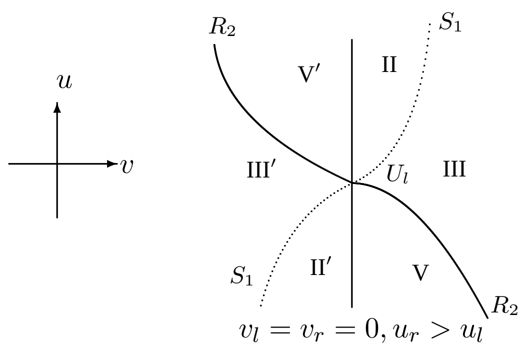

Fig. 2.3.

When $v_l = v_r = 0$, there are two subcases $u_r > u_l$ and $u_r < u_l$. Just consider the former subcase (Fig. 2.3.). We can use a degenerate shock S_d to connect U_l and U_r, while we find that it is unstable under the perturbation to U_l and U_r. Thus, we choose $S_1(v < 0)$ on the left to connect U_l with $\hat{U}$ and $S_2(\hat{v} < v < 0)$ on the right to connect $\hat{U}$ with U_r, where $\hat{U} = (-(\frac{u_l - u_r}{2})^{-\frac{2}{\gamma+1}}, \frac{u_l + u_r}{2})$. Thus, the Riemann solution is $S_1 + S_2$.

When $v_r \neq 0$, the Riemann solution is given by

$$S_1 + S_2, \text{ if } U_r \in \text{II}', \qquad S_1 + R_2, \text{ if } U_r \in \text{III}', \qquad S_d + R_2, \text{ if } U_r \in \text{V}',$$

$$S_1 + S_2, \text{ if } U_r \in \text{II}, \qquad S_1 + R_2, \text{ if } U_r \in \text{III}, \qquad S_d + R_2, \text{ if } U_r \in \text{V}.$$

3. Interactions of the Elementary Waves

In this section, we investigate all possible interactions of the elementary waves obtained from the Riemann problem (1.1) and (2.1). By analyzing the specific expressions of the shock wave curves and the rarefaction wave curves, we study the relative locations of U_l and U_r in the phase plane and discuss the interactions of elementary waves especially the overtaking of the shocks and the rarefaction waves from the same family.

Consider the equations (1.1) with the following initial data

$$(v,u)(x,t_0) = \begin{cases} (v_l, u_l), & -\infty < x \le x_1, \\ (v_*, u_*), & x_1 < x \le x_2, \\ (v_r, u_r), & x_2 < x < \infty, \end{cases} \tag{3.1}$$

for arbitrary $x_1, x_2 \in R$ and we should choose appropriate U_* and U_r with regard to U_l.

Since the interactions of the elementary waves containing the degenerate shock are easy to investigate, we just need to consider the following cases: the collision of waves belonging to different families(S_2S_1, R_2R_1, R_2S_1, S_2R_1) and the overtaking of waves belonging to the same family(S_2S_2, S_1S_1, S_2R_2, R_1S_1, S_1R_1, R_2S_2, R_2R_2, R_1R_1). By a direct computation we have the following results which are useful for our later discussion.

Lemma 3.1 — The function $g(v_l, v)$ satisfies

$$g_1'(v_l, v) = \frac{\gamma v^{\gamma-1}(v - v_l) + (v^\gamma - v_l^\gamma)}{2\sqrt{(v^\gamma - v_l^\gamma)(v - v_l)}} \begin{cases} > 0, & v > v_l \ge 0, \\ < 0, & v_l > v \ge 0, \end{cases} \tag{3.2}$$

$$g_2'(v_l, v) = \frac{\gamma v^{\gamma-1}(v_l - v) + (v^\gamma - v_l^\gamma)}{2\sqrt{(v_l^\gamma - v^\gamma)(v - v_l)}} \begin{cases} < 0, & v > v_l \ge 0, \\ > 0, & v_l > v \ge 0. \end{cases} \tag{3.3}$$

Based on the relations of v_l, v_* and v_r, there are two possibilities $v_l > 0, v_* > 0, v_r > 0$ and $v_l < 0, v_* < 0, v_r < 0$. We chiefly study the former case, the discussions of the other cases are omitted which can be studied similarly.

3.1. The collision of the elementary waves from the same family.

(i) The collision of two shocks: $S_2 S_1$.

The state U_l is connected to the state U_* by a 2-shock, and the state U_* is connected to the state U_r by a 1-shock (Fig. 3.1.). In other words, for a given state U_l, we choose U_* and U_r such that $u_* = u_l + g(v_l, v_*)$, $v_l > v_*$ and $u_r = u_* + g(v_*, v_r)$, $v_r > v_*$. we easily know that the two shocks collide with each other at a finite time and a new Riemann problem is formed. In order to solve the new Riemann problem, we should determine in what region U_r lies in with respect to U_l. Our next claim is that U_r lies in the region II for any given U_l and the new Riemann problem is resolved by a 1-shock wave and a 2-shock wave. It suffices to show that

$$g(v_*, v) - g(v_l, v) + g(v_l, v_*) > 0, \;\; v > v_*, \; v_l > v_*, \tag{3.4}$$

i.e.,

$$\begin{aligned} \sqrt{(v^\gamma - v_*^\gamma)(v - v_*)} \;&-\; \sqrt{(v^\gamma - v_l^\gamma)(v - v_l)} \\ &+\; \sqrt{(v_*^\gamma - v_l^\gamma)(v_* - v_l)} > 0, \;\; v > v_*, \; v_l > v_*. \end{aligned} \tag{3.5}$$

By a direct computation, we obtain that the left hand side of (3.5) equals to

$$[\sqrt{(v_l^\gamma - v_*^\gamma)(v - v_*)} + \sqrt{(v_*^\gamma - v^\gamma)(v_* - v_l)}]^2$$

which is strictly positive. Therefore (3.4) holds which means that the shock wave curve $S_1(U_*)$ lies above the shock wave curve $S_1(U_l)$ and the curve $S_1(U_*)$ lies entirely in the region II. It follows that $S_2 S_1 \to S_1 S_2$ which indicates that these two shocks cross with each other immediately after their collision.

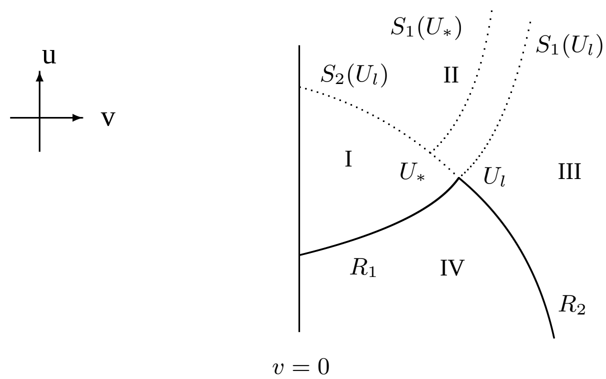

Fig. 3.1. Collision S_2S_1.

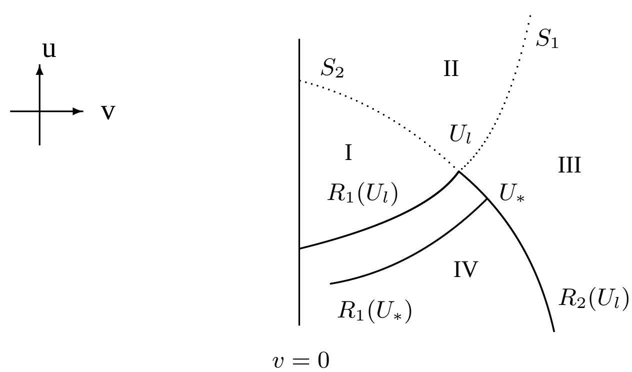

Fig. 3.2. Collision R_2R_1.

(ii) The collision of two rarefaction waves: R_2R_1.

Here U_l is connected to U_* by a 2-rarefaction wave, and U_* is connected to U_r by a 1-rarefaction wave (Fig. 3.2.). That is, for a given state U_l, we choose U_* and U_r such that $u_* = u_l - \int_{v_l}^{v_*} \sqrt{\gamma |s|^{\gamma-1}} ds, \quad v_* > v_l$, and $u_r = u_* + \int_{v_l}^{v_*} \sqrt{\gamma |s|^{\gamma-1}} ds, \quad v_r < v_*$. It is obvious that the two rarefaction waves collide with each other at a finite time. Since $v_* > v_l$ and

$$\int_{v_l}^{v} \sqrt{\gamma |s|^{\gamma-1}} ds + \int_{v_l}^{v_*} \sqrt{\gamma |s|^{\gamma-1}} ds - \int_{v}^{v_l} \sqrt{\gamma |s|^{\gamma-1}} ds > 0, \tag{3.6}$$

we know that the curve $R_1(U_*)$ lies below the curve $R_1(U_l)$. It follows that the state U_r lies in the region IV and $R_2R_1 \to R_1R_2$.

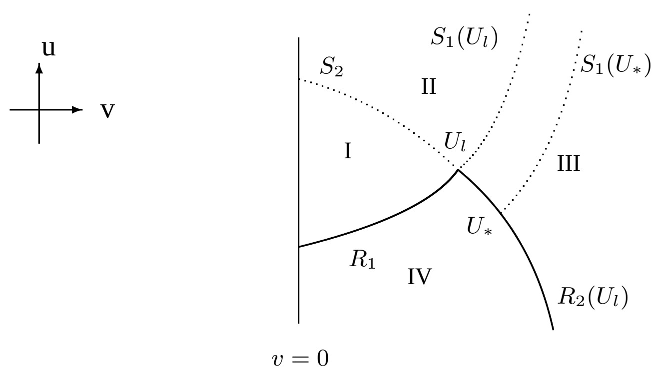

Fig. 3.3. Collision R_2S_1.

(iii) The collision of a rarefaction waves and a shock: R_2S_1.

Consider $U_* \in R_2(U_l)$ and $U_r \in S_1(U_*)$, i.e., we choose U_* and U_r such that $u_* = u_l - \int_{v_l}^{v_*} \sqrt{\gamma|s|^{\gamma-1}}ds,\ \ v_* > v_l$, and $u_r = u_* + g(v_*, v_r),\ \ v > v_*$ for a given state U_l (Fig. 3.3.). We claim that the curve $S_1(U_*)$ lies below the curve $S_1(U_l)$, hence for any given U_l the state U_r lies in the region III. In fact, we have

$$\int_{v_l}^{v_*} \sqrt{\gamma|s|^{\gamma-1}}ds + g(v_l, v) - g(v_*, v) > 0,\ \ v_l < v_* < v, \tag{3.7}$$

since $v_l < v_*$ and the function $g(v_l, v)$ is decreasing with respect to the first variable v_l as a result of Lemma 3.1. Therefore we obtain that $R_2S_1 \to S_1R_2$.

(iv) The collision of a shock and a rarefaction waves: S_2R_1.

Here $U_* \in S_2(U_l)$ and $U_r \in R_1(U_*)$, i.e., we choose U_* and U_r such that $u_* = u_l + g(v_l, v_*),\ \ v_* < v_l$, and $u_r = u_* + \int_{v_*}^{v_r} \sqrt{\gamma|s|^{\gamma-1}}ds,\ \ v_r < v_*$ for a given state U_l (Fig. 3.4.). Next we prove that the curve $R_1(U_*)$ lies above the curve $R_1(U_l)$ and the state U_r lies in the region I. It follows that $S_2R_1 \to R_1S_2$. It is sufficient to prove that

$$\int_{v}^{v_l} \sqrt{\gamma |s|^{\gamma-1}} ds - \int_{v}^{v_*} \sqrt{\gamma |s|^{\gamma-1}} ds + g(v_l, v_*) > 0, \quad v < v_* < v_l,$$

i.e.,

$$\int_{v_*}^{v_l} \sqrt{\gamma |s|^{\gamma-1}} ds + g(v_l, v_*) > 0.$$

The above inequality obviously holds since $v_* < v_l$.

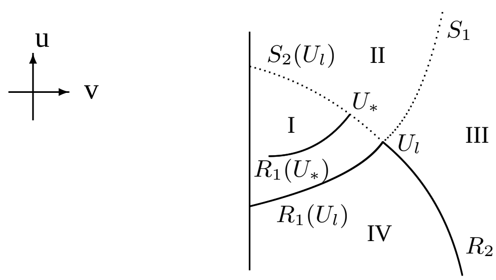

Fig. 3.4. Collision S_2R_1.

3.2. The overtaking of the elementary waves from the same family.

(i) 2-shock overtakes another 2-shock: S_2S_2.

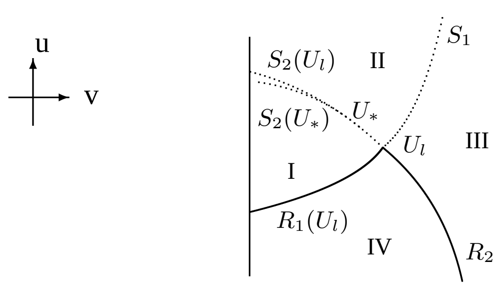

Fig. 3.5. S_2 overtakes S_2.

From (2.7), we know that

$$s_2(U_*, U_r) < \lambda_2(U_*) < s_2(U_l, U_*),$$

which indicates that the 2-shock connecting U_l to U_* must overtake the 2-shock connecting U_* to U_r at a finite time and a new Riemann problem is formed (Fig. 3.5.). In order to clarify the structure of the solution for this new Riemann problem, we should determine the region in which U_r lies for a given state U_l. Our claim is that the curve $S_2(U_*)$ lies below the curve $S_2(U_l)$ and the state U_r lies in the region I which means that $S_2S_2 \to R_1S_2$. In order to prove the curve $S_2(U_*)$ lies entirely in the region I we need only consider the following inequality

$$g(v_l, v) - g(v_*, v) - g(v_l, v_*) > 0, \;\; v_l > v_* > v, \tag{3.8}$$

i.e.,

$$\sqrt{(v^\gamma - v_l^\gamma)(v - v_l)} - \sqrt{(v_*^\gamma - v_l^\gamma)(v_* - v_l)} - \sqrt{(v^\gamma - v_*^\gamma)(v - v_*)} > 0.$$

The above inequality holds since it's left hand equals to

$$[\sqrt{(v_*^\gamma - v_l^\gamma)(v - v_l)} + \sqrt{(v_l^\gamma - v^\gamma)(v_l - v_*)}]^2,$$

which is strictly positive.

(ii) 2-shock overtakes another 2-shock: S_1S_1.

Similarly, we can prove that U_r lies in the region III and $S_1S_1 \to S_1R_2$.

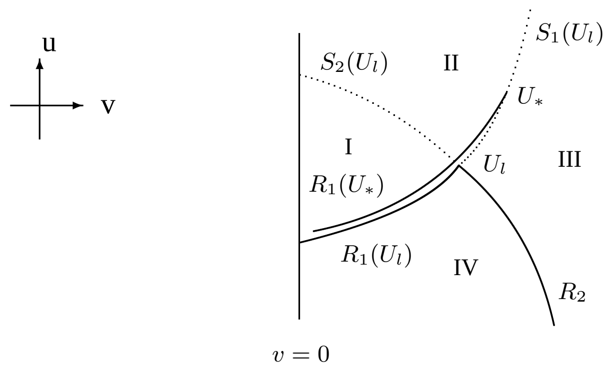

Fig. 3.6. R_1 overtakes S_1.

(iii) 1-rarefaction wave overtakes 1-shock: S_1R_1.

Here $U_* \in S_1(U_l)$ and $U_r \in R_1(U_*)$. That is, we choose U_* and U_r such that $u_* = u_l + g(v_l, v_*),\ \ v_l < v_*$, and $u_r = u_* + \int_{v_*}^{v_r} \sqrt{\gamma |s|^{\gamma-1}} ds,\ \ v_r < v_*$ for given U_l (Fig. 3.6.).

First we show that the curve $R_1(U_*)$ lies above the curve $S_1(U_l)$ for $v_l \le v < v_*$. It is sufficient to prove that

$$g(v_l, v_*) - g(v_l, v) + \int_{v_*}^{v} \sqrt{\gamma |s|^{\gamma-1}} ds > 0,\ \ v_l \le v < v_*. \tag{3.9}$$

Define the left side of (3.14) as a function $H_1(v)$ for $v_l \le v < v_*$, then $H_(v_*) = 0$. After a direct computation, we obtain $H_1'(v) = \frac{-[\sqrt{\gamma v^{\gamma-1}(v-v_l)} - \sqrt{v^\gamma - v_l^\gamma}]^2}{2\sqrt{(v^\gamma - v_l^\gamma)(v-v_l)}} < 0$, which shows that $H_1(v) > H_1(v_*) = 0$. This is our claim, i.e., $R_1(U_*)$ lies above $S_1(U_l)$.

Second we show the curve $R_1(U_*)$ lies above the curve $R_1(U_l)$ for $v \le v_l < v_*$, i.e.,

$$g(v_l, v_*) - \int_{v_l}^{v} \sqrt{\gamma |s|^{\gamma-1}} ds + \int_{v_*}^{v} \sqrt{\gamma |s|^{\gamma-1}} ds > 0,\ \ v \le v_l < v_*. \tag{3.10}$$

Notice that the left hand of (3.15) is $H_1(v_l)$ which has been showed to be positive.

Last we show that $R_1(U_*)$ and $S_2(U_l)$ intersect with each other uniquely at a point (v_3, u_3), where $v_3 < v_l < v_*$. Now we define a function $H_2(v) = g(v_l, v_*) - g(v_l, v) + \int_{v_*}^{v} \sqrt{\gamma|s|^{\gamma-1}}ds$ for $v < v_l$. Since $H_2(v_l) > 0$ and we can choose a $\bar{v}$ such that $H_2(\bar{v}) < 0$ as v is close to zero. Thus, there exists a unique v_1, $v_1 < v_l < v_*$ such that $H_2(v_1) = 0$, i.e., $R_1(U_*)$ intersects with $S_2(U_l)$ uniquely.

Depending on the value of v_1, we distinguish the following three cases:

(a) When $v_r < v_1$, we obtain that U_r lies in the region I and $S_1R_1 \to R_1S_2$;

(b) When $v_r = v_1$, we know U_r lies on the curve $S_2(U_l)$ and $S_1R_1 \to S_2$;

(c) When $v_r > v_1$, we obtain that U_r lies in the region II and $S_1R_1 \to S_1S_2$.

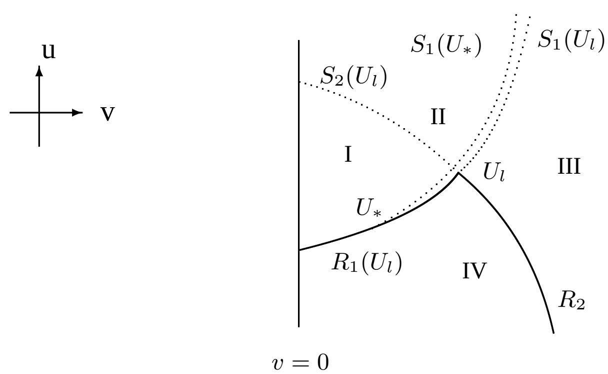

Fig. 3.7. S_1 overtakes R_1.

(iv) 1-shock overtakes 1-rarefaction wave: R_1S_1.

Consider $U_* \in R_1(U_l)$ and $U_r \in S_1(U_*)$. We choose U_* and U_r so as to $u_* = u_l + \int_{v_l}^{v_*} \sqrt{\gamma|s|^{\gamma-1}}ds, \;\; v_l > v_*$, and $u_r = u_* + g(v_*, v_r), \;\; v_r > v_*$ for a given U_l (Fig. 3.7.).

First we show that the curve $S_1(U_*)$ lies above the curve $R_1(U_l)$ for $v_* < v \leq v_l$. We just need to show that

$$g(v_*, v) + \int_{v_l}^{v_*} \sqrt{\gamma |s|^{\gamma-1}} ds - \int_{v_l}^{v} \sqrt{\gamma |s|^{\gamma-1}} ds > 0, \;\; v_* < v \leq v_l. \qquad (3.11)$$

Define a function $G_1(v) = g(v_*, v) + \int_{v_l}^{v_*} \sqrt{\gamma |s|^{\gamma-1}} ds - \int_{v_l}^{v} \sqrt{\gamma |s|^{\gamma-1}} ds$ for $v_* < v \leq v_l$. Since $G_1'(v) = \frac{[\sqrt{\gamma v^{\gamma-1}(v-v_*)} - \sqrt{v^\gamma - v_*^\gamma}]^2}{2\sqrt{(v^\gamma - v_*^\gamma)(v-v_*)}} > 0$ which indicates thereby that $G_1(v) > G_1(v_*) = 0$. That is to say (3.9) holds.

Second we show the curve $S_1(U_*)$ lies above the curve $S_1(U_l)$ for $v > v_l$. It is enough to show that

$$g(v_*, v) - g(v_l, v) + \int_{v_l}^{v_*} \sqrt{\gamma |s|^{\gamma-1}} ds > 0, \;\; v > v_l > v_*. \qquad (3.12)$$

To see this, we first of all show

$$g(v_*, v) - g(v_l, v) - g(v_*, v_l) > 0, \;\; v > v_l > v_*,$$

i.e.,

$$\sqrt{(v^\gamma - v_*^\gamma)(v - v_*)} - \sqrt{(v^\gamma - v_l^\gamma)(v - v_l)} - \sqrt{(v_l^\gamma - v_*^\gamma)(v_l - v_*)} > 0. \quad (3.13)$$

It is easy to check that (3.11) holds since the left side of the above inequality equals to $[\sqrt{(v^\gamma - v_l^\gamma)(v - v_*)} + \sqrt{(v^\gamma - v_*^\gamma)(v - v_l)}]^2$ which is obviously strictly positive.

It follows that

$$\begin{aligned} g(v_*, v) - g(v_l, v) + \textstyle\int_{v_l}^{v_*} \sqrt{\gamma |s|^{\gamma-1}} ds &> g(v_*, v_l) + \textstyle\int_{v_l}^{v_*} \sqrt{\gamma |s|^{\gamma-1}} ds \\ &= H_2(v_l) \\ &> 0, \end{aligned}$$

which shows that (3.10) holds.

Last we show that $S_1(U_*)$ and $S_2(U_l)$ intersect at some point (v_1, u_1) uniquely, where $v_* < v_2 < v_l$. For this it remains to show that the equation $g(v_l, v) - g(v_l, v_*) + \int_{v_*}^{v} \sqrt{\gamma |s|^{\gamma-1}} ds = 0$ has a unique root. Define a function

$$G_2(v) = g(v_l, v) - g(v_*, v) + \int_{v_l}^{v_*} \sqrt{\gamma |s|^{\gamma-1}} ds, \;\; v_* < v < v_1.$$

Since $G_2(v_l) = -G_1(v_l) < 0$ and $G_2(v_*) = g(v_l, v_*) + \int_{v_l}^{v_*} \sqrt{\gamma |s|^{\gamma-1}} ds > 0$, we know that there exists a unique point $(v_2, u_2), \; v_* < v_2 < v_l$ such that $G_2(v_2) = 0$. This is the desired conclusion.

Similarly, there are three cases as follows:

(a) When $v_r < v_2$, U_r lies in the region I and $R_1 S_1 \to R_1 S_2$;

(b) When $v_r = v_2$, U_r lies on the curve $S_2(U_l)$ and $R_1 S_1 \to S_2$;

(c) When $v_r > v_2$, U_r lies in the region II and $R_1 S_1 \to S_1 S_2$.

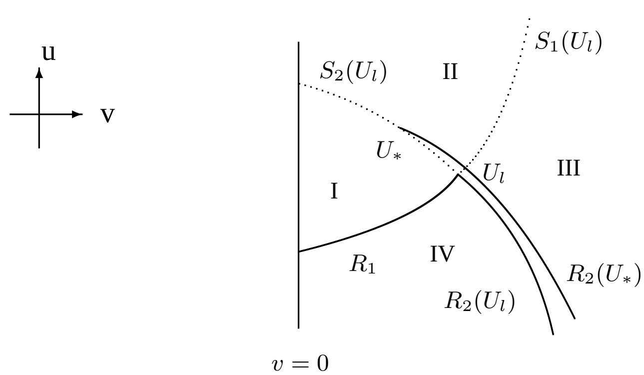

Fig. 3.8. R_2 overtakes S_2.

(v) 2-rarefaction wave overtakes 2-shock: $S_2 R_2$.

Consider $U_* \in S_2(U_l)$ and $U_r \in R_2(U_*)$, i.e., for a given state U_l we choose U_* and U_r such that $u_* = u_l + g(v_l, v_*), \;\; v_l > v_*$, and $u_r = u_* - \int_{v_*}^{v_r} \sqrt{\gamma |s|^{\gamma-1}} ds, \;\; v_r > v_*$ (Fig. 3.8.).

First we show that the curve $S_2(U_l)$ lies below the curve $R_2(U_*)$ for $v_* < v \leq v_l$.

It suffices to show that

$$g(v_l, v_*) - g(v_l, v) - \int_{v_*}^{v} \sqrt{\gamma |s|^{\gamma-1}} ds > 0, \ \ v_* < v \leq v_l. \tag{3.14}$$

Define the left side of (3.12) as a function $F_1(v)$ for $v_* < v \leq v_l$, then $F_1(v_*) = 0$. By a direct computation, we get $F_1'(v) = \frac{[\sqrt{\gamma v^{\gamma-1}(v_l - v)} - \sqrt{v_l^\gamma - v^\gamma}]^2}{2\sqrt{(v^\gamma - v_l^\gamma)(v - v_l)}} > 0$, indicating thereby that $F_1(v) > F_1(v_*) = 0$, i.e., $S_2(U_l)$ lies below the curve $R_2(U_*)$ for $v_* < v \leq v_l$.

Second we show the curve $R_2(U_*)$ lies above the curve $R_2(U_l)$ for $v_* < v_l \leq v$, i.e.,

$$g(v_l, v_*) - \int_{v_*}^{v} \sqrt{\gamma |s|^{\gamma-1}} ds + \int_{v_l}^{v} \sqrt{\gamma |s|^{\gamma-1}} ds > 0, \ \ v_* < v_l \leq v. \tag{3.15}$$

Notice that the left side of (3.13) is just $F_1(v_l)$ which has been proved to be positive.

Last we show that $R_2(U_*)$ intersects with $S_1(U_l)$ at a point uniquely, say (v_1, u_1) for $v_* < v_l < v_1$. It is sufficient to show that the equation $g(v_l, v) - g(v_l, v_*) + \int_{v_*}^{v} \sqrt{\gamma |s|^{\gamma-1}} ds = 0$ has a unique root. In fact, we define a function $F_2(v) = g(v_l, v) - g(v_l, v_*) + \int_{v_*}^{v} \sqrt{\gamma |s|^{\gamma-1}} ds$. Since $F_2(v_l) < 0$ and we can choose a $\hat{v}$ such that $F_2(v) > 0$ for all $v > \hat{v}$, then $R_2(U_*)$ and $S_1(U_l)$ intersect with each other at a unique point (v_3, u_3) for $v_* < v_3 < v_1$ by virtue of monotonicity of $R_2(U_*)$, $S_1(U_l)$ and intermediate value theorem.

There are the following three cases:

(a) When $v_r < v_3$, we see U_r lies in the region II and $S_2R_2 \to S_1S_2$;

(b) When $v_r = v_3$, we know U_r lies on the curve $S_1(U_l)$ and $S_2R_2 \to S_1$;

(c) When $v_r > v_3$, we obtain that U_r lies in the region III and $S_2R_2 \to S_1R_2$.

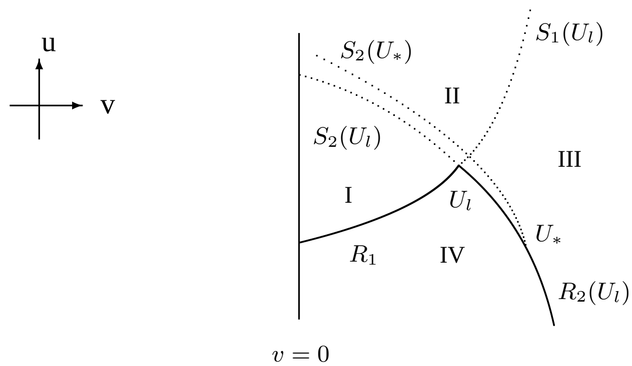

Fig. 3.9. S_2 overtakes R_2.

(vi) 2-shock overtakes 2-rarefaction wave: R_2S_2.

Here $U_* \in R_2(U_l)$ and $U_r \in S_2(U_*)$, i.e., for a given state U_l we choose U_* and U_r so as to $u_* = u_l - \int_{v_l}^{v_*} \sqrt{\gamma|s|^{\gamma-1}}ds,\ \ v_l < v_*$, and $u_r = u_* + g(v_*, v_r),\ \ v_r < v_*$ (Fig. 3.9.).

First we show that the curve $S_2(U_*)$ lies above the curve $R_2(U_l)$ for $v_l \le v < v_*$. It remains to show that

$$g(v_*, v) + \int_{v_l}^{v} \sqrt{\gamma|s|^{\gamma-1}}ds - \int_{v_l}^{v_*} \sqrt{\gamma|s|^{\gamma-1}}ds > 0, \ \ v_l \le v < v_*. \qquad (3.16)$$

Define a function $K_1(v) = g(v_*, v) + \int_{v_l}^{v} \sqrt{\gamma|s|^{\gamma-1}}ds - \int_{v_l}^{v_*} \sqrt{\gamma|s|^{\gamma-1}}ds$, $v_l \le v < v_*$, then $K_1(v_*) = 0$. By a direct computation, we get $K_1'(v) = -\frac{[\sqrt{\gamma v^{\gamma-1}(v_*-v)} - \sqrt{v_*^{\gamma} - v^{\gamma}}]^2}{2\sqrt{(v^{\gamma} - v_*^{\gamma})(v - v_*)}} < 0$, and $K_1(v) > K_1(v_*) = 0$, i.e., $S_2(U_*)$ lies above $R_2(U_l)$.

Second we show the curve $S_2(U_*)$ lies above the curve $S_2(U_l)$ for $v \le v_l < v_*$, i.e.,

$$g(v_*, v) - g(v_l, v) - \int_{v_l}^{v_*} \sqrt{\gamma|s|^{\gamma-1}}ds > 0, \ \ v \le v_l < v_*. \qquad (3.17)$$

Next we show the following inequality

$$g(v_*, v) - g(v_l, v) - g(v_*, v_l) > 0, \ \ v < v_l < v_*, \tag{3.18}$$

holds, i.e.,

$$\sqrt{(v^\gamma - v_*^\gamma)(v - v_*)} - \sqrt{(v^\gamma - v_l^\gamma)(v - v_l)} - \sqrt{(v_l^\gamma - v_*^\gamma)(v_l - v_*)} > 0.$$

It is easily seen that (3.18) holds since the left side of the above inequality satisfies

$$[\sqrt{(v^\gamma - v_l^\gamma)(v - v_*)} + \sqrt{(v^\gamma - v_*^\gamma)(v - v_l)}]^2 > 0.$$

From

$$\begin{array}{rcl} g(v_*, v) - g(v_l, v) - \int_{v_l}^{v_*} \sqrt{\gamma |s|^{\gamma-1}} ds & > & g(v_*, v_l) - \int_{v_l}^{v_*} \sqrt{\gamma |s|^{\gamma-1}} ds \\ & = & K_1(v_l) \\ & > & 0, \end{array}$$

we obtain that (3.17) holds.

Last we show that $S_2(U_*)$ intersects with $S_1(U_l)$ at a unique point (v_4, u_4) for $v_l < v_4 < v_*$, i.e., the equation $g(v_l, v) - g(v_*, v) + \int_{v_l}^{v_*} \sqrt{\gamma |s|^{\gamma-1}} ds = 0$ has a unique root. To show this, we define a function $K_2(v) = g(v_l, v) - g(v_*, v) + \int_{v_l}^{v_*} \sqrt{\gamma |s|^{\gamma-1}} ds$. Since $K_2(v_l) = -K_1(v_l) < 0$ and we can choose a $\tilde{v}$ such that $F_2(v) > 0$ for all $v > \tilde{v}$, then there exists a unique point (v_4, u_4), $\ v_l < v_4 < v_*$ such that $K_2(v_4) = 0$. That is, $R_2(U_*)$ and $S_1(U_l)$ intersect with each other at (v_4, u_4) uniquely.

Again similarly, there are the following three cases depending on the value of v_4:

(a) When $v_r < v_4$, we obtain that U_r lies in the region II and $R_2 S_2 \to S_1 S_2$;

(b) When $v_r = v_4$, we know U_r lies on the curve $S_1(U_l)$ and $R_2 S_2 \to S_1$;

(c) When $v_r > v_4$, we obtain that U_r lies in the region III and $R_2 S_2 \to S_1 R_2$.

(vii) 2-rarefaction wave overtakes another 2-rarefaction wave: $R_2 R_2$.

Since the propagating speed of the wave front in $R_2(U_l, U_*)$ equals to the propagating speed of the wave back in $R_2(U_*, U_R)$, we know that they can never overtake each other and we may regard $R_2(U_l, U_*)$ and $R_2(U_*, U_R)$ together as a forward rarefaction wave: $R_2R_2 \to R_2$.

(viii) 1-rarefaction wave overtakes another 1-rarefaction wave: R_1R_1.

Similarly as above, we may regard $R_1(U_l, U_*)$ and $R_1(U_*, U_R)$ together as a backward rarefaction wave: $R_1R_1 \to R_1$.

So far, we have finished the discussions for all kinds of wave interactions. It is important to study the interactions of elementary waves for system (1.1) not only because of their significance in practical applications in elasticity system such as comparison with the numerical and experimental results, but also because of their basic role as building blocks for the theory of elasticity.

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