Indian J. Pure Appl. Math., **41**(3): 443-473, June 2010

(p, λ)-KOSZUL ALGEBRAS AND MODULES

Lü Jia-Feng* and Zhao Zhi-Bing**

**Department of Mathematics, Zhejiang Normal University, Jinhua, Zhejiang* 321004,
People's Republic of China
e-mails: jiafenglv@zjnu.edu.cn, jiafenglv@126.com
***School of Mathematical Science, Anhui University, Hefei, Anhui* 230039,
Peoples' Republic of China
e-mail: zbzhao066930@163.com

(*Received* 2 *April* 2009; *after final revision* 20 *November* 2009;
accepted 29 *January* 2010)

In this paper, the notions of (p, λ)-*Koszul algebra* and (p, λ)-*Koszul module* are introduced. Some criteria theorems for a positively graded algebra A to be (p, λ)-Koszul are given. The notion of *weakly* (p, λ)-*Koszul module* is defined as well and let $\mathcal{WK}^p_\lambda(A)$ denote the category of weakly (p, λ)-Koszul modules. We show that $M \in \mathcal{WK}^p_\lambda(A)$ if and only if it can be approximated by (p, λ)-Koszul submodules, which is equivalent to that $\mathbf{G}(M)$ is a (p, λ)-Koszul module, where $\mathbf{G}(M)$ denotes the associated graded module of M. As applications, the relationships of the minimal graded projective resolutions of M, $\mathbf{G}(M)$ and (p, λ)-Koszul submodules are established. In particular, for a module $M \in \mathcal{WK}^p_\lambda(A)$ we prove that $\bigoplus_{i \geq 0} \operatorname{Ext}^i_A(M, A_0) \in gr_0(E(A))$, we also get as a consequence that the *finitistic dimension conjecture* is valid in $\mathcal{WK}^p_\lambda(A)$ under certain conditions.

Key words : (p, λ)-Koszul algebras; Yoneda-Ext algebras; weakly (p, λ)-Koszul modules.

1. Introduction

Piecewise-Koszul algebras, a class of homogeneous algebras unifying the notions of *Koszul algebras* [13] and *D-Koszul algebras* [1], were first introduced and studied by A_∞-language (see [10]). In this paper, we introduce another class of homogeneous algebras, namely (p, λ)-*Koszul algebra*, which is a generalization of piecewise-Koszul algebra and agrees with both D-Koszul and piecewise-Koszul algebras in special cases. It turned out that this class of algebras has various good homological properties similar to Koszul and piecewise-Koszul algebras. For example, using Yoneda-Ext algebra to characterize Koszul, D-Koszul and piecewise-Koszul algebras is an effective method. Suppose that the positively graded algebra $A = \bigoplus_{i\geq 0} A_i$ is generated in degrees zero and one with A_0 semisimple, $E(A) = \bigoplus_{i\geq 0} \mathrm{Ext}^i_A(A_0, A_0)$ is the Yoneda-Ext algebra of A, which is bigraded by the (*ext, shift*)-degree: i.e., its $(i,j)^{th}$ component is $\mathrm{Ext}^i_A(A_0, A_0)_{-j}$. The following statements can be found in ([6], [8], [4] and [10]) respectively:

- A is a Koszul algebra if and only if $E(A)$ is minimally generated in the ext-degrees $0, 1$ (and $\mathrm{Ext}^1_A(A_0, A_0) = \mathrm{Ext}^1_A(A_0, A_0)_{-1}$);
- A is a D-Koszul algebra if and only if $E(A)$ is minimally generated in the ext-degrees $0, 1, 2$, and $\mathrm{Ext}^2_A(A_0, A_0) = \mathrm{Ext}^2_A(A_0, A_0)_{-D}$;
- A is a piecewise-Koszul algebra if and only if $E(A)$ is minimally generated in the ext-degrees $0, 1, p$, and $\mathrm{Ext}^p_A(A_0, A_0) = \mathrm{Ext}^p_A(A_0, A_0)_{-d}$.

For our new object, (p, λ)-Koszul algebras, we have the following result.

Theorem (Theorem 3.3) — *Let A be a graded algebra. Then A is a (p, λ)-Koszul algebra ($d > p \geq 2$) if and only if $E(A)$ is minimally generated in the ext-degrees* $1,\ p,\ 2p,\ \cdots,\ |\lambda|p$, *and* $\mathrm{Ext}^i_A(A_0, A_0) = \mathrm{Ext}^i_A(A_0, A_0)_{-\delta^p_\lambda(i)}$ *for $i = p,\ 2p,\ \cdots,\ |\lambda|p$, where δ^p_λ, λ and $|\lambda|$ are defined in the next section.*

It is well known that whether the Yoneda-Ext algebra of a graded algebra is finitely generated or not is too complicated to be answered. As an attempt to discuss this thesis, Green and Marcos introduced the notion of δ-*Koszul algebra* in [5] in 2005. In particular, they finished the paper with three questions:

- For which functions $\delta : \mathbb{N} \to \mathbb{N}$ is there a δ-resolution determined algebra?

- For which functions $\delta : \mathbb{N} \to \mathbb{N}$ is there a δ-Koszul algebra?
- Is there a bound $N_0 \in \mathbb{N}$, such that if $A = A_0 \oplus A_1 \oplus A_2 \oplus \cdots$ is a δ-Koszul algebra, then the Yoneda-Ext algebra $E(A) = \bigoplus_{n \geq 0} \mathrm{Ext}_A^n(A_0, A_0)$ is generated by $\mathrm{Ext}_A^0(A_0, A_0)$, $\mathrm{Ext}_A^1(A_0, A_0)$, $\cdots$, $\mathrm{Ext}_A^{N_0}(A_0, A_0)$?

Note that (p, λ)-Koszul algebras are a special class of δ-Koszul algebras. As an application of the above theorem, we can give an answer to the third question of Green and Marcos provided that for any natural number N, we can construct a (p, λ)-Koszul algebra with $p > N$ or $|\lambda|p > N$. Indeed, we give an answer in Section 3.

We also introduce (p, λ)-Koszul modules which are a special kind of pure graded modules. Inspired by [12] where the authors discussed the Koszulness for finitely generated graded modules over a Koszul algebra, one can ask a natural question: How to study the (p, λ)-Koszulness for an arbitrary finitely generated graded module over a (p, λ)-Koszul algebra? In order to do this, we define the so-called *weakly* (p, λ)*-Koszul module* and show that such modules can be approximated by (p, λ)-Koszul submodules. In particular, we prove the following results:

Theorem (Theorems 4.7 and 5.3, Corollary 4. 8 and Remark 2.5 (5)) *Let A be a (p, λ)-Koszul algebra, M an arbitrary finitely generated graded A-module, and $\{S_{d_1}, S_{d_2}, \cdots, S_{d_t}\}$ a set of minimal homogeneous generating spaces of M. Let $U_1 = \langle S_{d_1} \rangle$, $U_2 = \langle S_{d_1}, S_{d_2} \rangle$, $\cdots$, $U_t = \langle S_{d_1}, S_{d_2}, \cdots, S_{d_t} \rangle$. Setting $K_i := U_i/U_{i-1}$, $(1 \leq i \leq t)$. Then the following statements are equivalent:*

1. *M is a weakly (p, λ)-Koszul module;*
2. *All U_i are weakly (p, λ)-Koszul modules, where $1 \leq i \leq t$;*
3. *All K_i are (p, λ)-Koszul-like modules, where $1 \leq i \leq t$;*
4. *$\bigoplus_{n \geq 0} \mathrm{Ext}_A^n(K_i, A_0)$ are generated in degree 0 for all $1 i \leq t$;*
5. *$\mathbf{G}(M)$ is a (p, λ)-Koszul module as a graded A-module, where $\mathbf{G}(M)$ is the associated graded module of M.*

We also give some applications of the above theorem. Firstly, motivated by the above theorem, one may wonder the relationships of the minimal graded projective resolutions among M, K_i and $\mathbf{G}(M)$, where M is a weakly (p, λ)-Koszul module. The following is our result:

Theorem (Theorems 6.6 and 6.8) — *Let M be a weakly (p, λ)-Koszul module. Then we have the following statements.*

1. *Let $0 = U_0 \subset U_1 \subset U_2 \subset \cdots \subset U_{t-1} \subset U_t = M$ be the natural submodule filtration of M. Let $\mathcal{P}_* \to M \to 0$ and $\mathcal{P}^i_* \to K_i \to 0$ be the minimal graded projective resolutions of M and K_i's, respectively. Then for all $n \geq 0$, we have*

$$\mathcal{P}_n \cong \bigoplus_{i=1}^{t} \mathcal{P}^i_n.$$

2. *Let $\mathcal{P}_* \longrightarrow M \longrightarrow 0$ and $\mathcal{Q}_* \longrightarrow \mathbf{G}(M) \longrightarrow 0$ be the minimal graded projective resolutions of M and the associated graded module $\mathbf{G}(M)$, respectively. Then for all $i \geq 0$, we have $Q_i \cong \mathbf{G}(P_i)[\delta^p_\lambda(i)]$.*

Secondly, we discuss the generation property of the Koszul dual of a (p, λ)-Koszul module and the finitistic dimension conjecture in $\mathcal{WK}^p_\lambda(A)$. We know that M is a (p, λ)-Koszul module if and only if $\bigoplus_{n \geq 0} \mathrm{Ext}^n_A(M, A_0)$ is generated in degree 0 as a graded $\bigoplus_{n \geq 0} \mathrm{Ext}^n_A(A_0, A_0)$-module, where A is a (p, λ)-Koszul algebra. But for weakly (p, λ)-Koszul modules, we only have the necessity of this result. From [7], we have the result: "Let A be a finite dimensional graded algebra and $\mathcal{K}^\delta(A)$ the category of δ-Koszul modules. Then the finitistic dimension conjecture is true in $\mathcal{K}^\delta(A)$." As an easy corollary, we prove that the finitistic dimension conjecture is true in $\mathcal{WK}^p_\lambda(A)$ under certain conditions. More precisely, we prove the following results:

Theorem (Theorems 6.9 and 6.11) — *The following statements are true:*

1. *Let $M \in \mathcal{WK}^p_\lambda(A)$. Then $\bigoplus_{n \geq 0} \mathrm{Ext}^n_A(M, A_0)$ is generated in degree 0 as a graded $\bigoplus_{n \geq 0} \mathrm{Ext}^n_A(A_0, A_0)$-module.*

2. *Let A be a finite dimensional (p, λ)-Koszul algebra. Then the finitistic dimension conjecture is true in $\mathcal{WK}^p_\lambda(A)$.*

The following is an outline of the paper. Section 2 gives some notations, definitions, examples and basic properties. In Section 3, we will discuss the properties of the Yoneda-Ext algebras of (p, λ)-Koszul algebras. Further, we give an answer to the third question of Green and Marcos. In Section 4, we will prove that weakly (p, λ)-Koszul modules can be approximated by (p, λ)-Koszul submodules. In Section 5, we will discuss the (p, λ)-Koszulness of the associated graded module of a weakly (p, λ)-Koszul module. In section 6, we will give some applications of the results obtained in Sections 4 and 5.

In this paper, $\mathbb{Z}$ denotes the set of integers, $\mathbb{N} = \{0,\ 1,\ 2,\ \cdots,\ n,\ \cdots\}$ the set of natural numbers and $\mathbb{N}^* = \{1,\ 2,\ \cdots,\ n,\ \cdots\}$ the set of positive integers and $\mathbb{F}$ will denote an arbitrary ground field.

2. Preliminaries

In this paper, all the graded $\mathbb{F}$-algebras $A = \bigoplus_{i \geq 0} A_i$ are assumed with the following properties: (i) $A_0 = \mathbb{F} \times \mathbb{F} \times \cdots \times \mathbb{F}$, a finite product of $\mathbb{F}$; (ii) A is generated in degrees 0 and 1; that is, $A_i \cdot A_j = A_{i+j}$ for all $0 \leq i, j < \infty$ and (iii) $\dim_{\mathbb{F}} A_i < \infty$ for all $i \geq 0$.

The graded Jacobson radical of A will be denoted by J it is obvious that $J = \bigoplus_{i \geq 1} A_i$.

Endowed with the Yoneda product, $E(A) := \bigoplus_{i \geq 0} \mathrm{Ext}_A^i(A_0, A_0)$ is an associative bigraded algebra and we call it the *Yoneda-Ext algebra* of A. Let M be a finitely generated graded A-module. Setting $\mathcal{E}(M) := \bigoplus_{i \geq 0} \mathrm{Ext}_A^i(M, \mathbb{F})$, which is a bigraded $E(A)$-module, we will call it the *Koszul dual* of M.

Throughout, let $Gr(A)$ denote the category of graded A-modules, and $gr(A)$, its full subcategory of finitely generated modules. The morphisms in these categories are the A-module maps of degree zero. We denote $Gr_s(A)$ and $gr_s(A)$ the full subcategory of $Gr(A)$ and $gr(A)$ whose objects are generated in degree s respectively. An object in $Gr_s(A)$ or $gr_s(A)$ is called a graded *pure* A-module.

Definition 2.1 — Let A be a graded algebra and $E(A)$ its Yoneda-Ext algebra. We call $\mathrm{Ext}_A^{\geq 2}(A_0, A_0)$ the *higher ext-degree* parts of $E(A)$. Let $h(A)$ denote the number of higher ext-degree parts in the generating space of $E(A)$. That is, if V is the minimal generating graded space of $E(A)$, then

$$h(A) = \sharp\{d \geq 2 \mid \mathrm{Ext}_A^d(A_0, A_0) \cap V \neq 0\}.$$

Let A be a graded algebra. From [8], [9] and [10], we get that if $h(A) > 0$, then there exists at least one nontrivial higher multiplication on the Yoneda-Ext algebra $E(A)$. In order to understand $h(A)$ well, let us see some easy examples.

Example 2.2 —

1. Let A be a Koszul algebra (see [6]). Then $h(A) = 0$ since the Yoneda-Ext algebra of A is minimally generated in ext-degrees 0 and 1.

2. Let A be a D-Koszul algebra with $D > 2$ (see [4]). Then $h(A) = 1$ since the Yoneda-Ext algebra of A is minimally generated in ext-degrees 0, 1 and 2.

3. Let A be a piecewise-Koszul algebra with $p > d \geq 2$ (see [10]). Then $h(A) = 1$ since the Yoneda-Ext algebra of A is minimally generated in ext-degrees 0, 1 and p.

Now we will give the definition of our new object: (p, λ)-Koszul algebra.

In this work, $\lambda : \mathbb{N}^* \to \mathbb{N}^*$ will always denote a periodic function which will obey the following properties:

Let $|\lambda|$ denote the smallest positive period of λ then $\lambda(1) \geq 2$ and λ is strictly increasing in the interval $[1, |\lambda|]$.

Introduce a function

$$\delta_\lambda^p : \mathbb{N} \to \mathbb{N}$$

with the following properties:

(i) $\delta_\lambda^p(0) = 0$, $\delta_\lambda^p(p) = d$, where $d = \lambda(1) + p - 1$ and $p \geq 2$ are fixed integers;

(ii) $\delta_\lambda^p(pn + i) - \delta_\lambda^p(pn + i - 1) = 1$ for all $1 \leq i \leq p - 1$;

(iii) $\delta_\lambda^p(pn) - \delta_\lambda^p(pn - 1) = \lambda(n)$ for all $n \geq 1$.

To understand the function δ_λ^p clearly, we give a concrete example.

Example 2.3 — Let $p = 3$, $d = 4$, $|\lambda| = 2$, $\lambda(1) = 2$ and $\lambda(2) = 5$. Then $\{\delta_\lambda^p(i) | i \in \mathbb{N}\} = \{0, 1, 2, 4, 5, 6, 11, 12, 13, 15, 16, 17, 22, 23, 24, 26, \cdots\}$.

Definition 2.4 — Let A be a graded algebra and $M \in gr(A)$. We call M a (p,λ)-*Koszul module* if it has a minimal graded projective resolution of the form

$$\mathbf{Q}: \quad \cdots \to Q_n \xrightarrow{f_n} \cdots \to Q_1 \xrightarrow{f_1} Q_0 \xrightarrow{f_0} M[-s] \to 0,$$

in which $s \in \mathbb{Z}$ and each Q_n is generated in degree $\delta_\lambda^p(n)$. Let $\mathcal{K}_\lambda^p(A)$ denote the category of (p,λ)-Koszul modules. A graded algebra $A = \bigoplus_{i\geq 0} A_i$ is called a (p,λ)-*Koszul algebra* if the trivial A-module $A_0 \in \mathcal{K}_\lambda^p(A)$, and the number of elements of the image of λ is called the *jump-degree* of A, denoted by $\|A\|$. Obviously $\|A\| = \sharp\{\lambda(i) | i \in [1, |\lambda|]\}$.

Remark 2.5 :

1. Let A be a (p,λ)-Koszul algebra with $d > p \geq 2$. Then $\|A\| = |\lambda|$.

2. In particular, if $|\lambda| = 2$ and λ is defined by $\lambda(1) = d - p + 1$ and $\lambda(2) = s + d - p + 1$ for integers $d > p \geq 2$ and $s > 0$. In this case, the function δ_λ^p can be written explicitly,

$$\delta_\lambda^p(n) = \begin{cases} \dfrac{nd}{p} + \dfrac{ns}{2p}, & \text{if } n \equiv 0 \ (\text{mod} 2p), \\ \dfrac{(n-1)d}{p} + \dfrac{(n-1)s}{2p} + 1, & \text{if } n \equiv 1 \ (\text{mod} 2p), \\ \dfrac{(n-2)d}{p} + \dfrac{(n-2)s}{2p} + 2, & \text{if } n \equiv 2 \ (\text{mod} 2p), \\ \cdots & \\ \dfrac{(n-p+1)d}{p} + \dfrac{(n-p+1)s}{2p} + p - 1, & \text{if } n \equiv p-1 \ (\text{mod} 2p), \\ \dfrac{nd}{p} + \dfrac{(n-p)s}{2p}, & \text{if } n \equiv p \ (\text{mod} 2p), \\ \dfrac{(n-1)d}{p} + \dfrac{(n-p-1)s}{2p} + 1, & \text{if } n \equiv p+1 \ (\text{mod} 2p), \\ \cdots & \\ \dfrac{(n-p+1)d}{p} + \dfrac{(n-2p+1)s}{2p} + p - 1, & \text{if } n \equiv 2p-1 \ (\text{mod} 2p). \end{cases}$$

3. Let A be a graded algebra and A^{op} its opposite algebra. Then A is a (p,λ)-Koszul algebra if and only if A^{op} is a (p,λ)-Koszul algebra.

4. A is a (p,λ)-Koszul algebra if and only if for each $n \geq 0$, $\mathrm{Ext}_A^n(A_0, A_0)$ is concentrated in the bidegree $(n,\ \delta_\lambda^p(n))$, which is also equivalent to that, for each $n \geq 0$, $\mathrm{Ext}_A^n(A_0, A_0[-m]) = 0$ unless $m = \delta_\lambda^p(n)$.

5. Let A be a (p, λ)-Koszul algebra and $M \in gr_s(A)$ and $s \in \mathbb{N}$. By Proposition 3.5 in [4], $M \in \mathcal{K}^p_\lambda(A)$ if and only if $\mathcal{E}(M)$ is generated in degree 0 as a graded $E(A)$-module.

6. The category $\mathcal{K}^p_\lambda(A)$ is closed under extensions and preserves cokernels of monomorphisms, where A is a (p, λ)-Koszul algebra. That is, for a short exact sequence $0 \to K \to M \to N \to 0$ in $gr(A)$, then
   (a) If K and N are in $\mathcal{K}^p_\lambda(A)$, then $M \in \mathcal{K}^p_\lambda(A)$,
   (b) If K and M are in $\mathcal{K}^p_\lambda(A)$, then $M/K \cong N \in \mathcal{K}^p_\lambda(A)$.

7. Let A be a (p, λ)-Koszul algebra with $|\lambda| = T$ and $M \in \mathcal{K}^p_\lambda(A)$. Then
   (a) All the $(kpT)^{th}$ syzygies $\Omega^{kpT}(M) \in \mathcal{K}^p_\lambda(A)$;
   (b) All the $(kpT-1)^{th}$ syzygies $\Omega^{kpT-1}(JM) \in \mathcal{K}^p_\lambda(A)$.

Now we will give some examples of (p, λ)-Koszul objects.

Example 2.6 — Setting

$$A = \frac{\mathbb{F}\langle x, y, z, w\rangle}{(yx, z^2y, wz)}.$$

Then under a routine computation, the trivial module ${}_A\mathbb{F}$ has the following minimal resolution

$$0 \to A(-5) \to A(-4)^2 \to A(-2)^2 \oplus A(-3) \to A(-1)^4 \to A \to \mathbb{F} \to 0.$$

Denoting $M := \dfrac{A \oplus A}{\big((x,0),(0,y)\big)}$, which has the following minimal graded free resolution

$$(*) : 0 \to A(-5) \stackrel{M_4}{\to} A(-4)^2 \stackrel{M_3}{\to} A(-2) \oplus A(-3) \stackrel{M_2}{\to} A(-1)^2 \stackrel{M_1}{\to} A^2 \stackrel{\varepsilon}{\to} M \to 0,$$

where $M_4 = \begin{pmatrix} w & 0 \end{pmatrix}$, $M_3 = \begin{pmatrix} z^2 & 0 \\ 0 & w \end{pmatrix}$, $M_2 = \begin{pmatrix} y & 0 \\ 0 & z^2 \end{pmatrix}$ and $M_1 = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$. By some computations, $(*)$ can be decomposed into the following two exact sequences

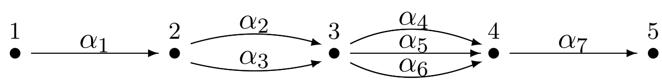
$$0 \to A(-5) \to A(-4) \to A(-2) \to A(-1) \to A \to A/(x) \to 0,$$

and

$$0 \to A(-4) \to A(-3) \to A(-1) \to A \to A/(y) \to 0.$$

Therefore, $A/(x)$ is a (p, λ)-Koszul module with respect to $p = 3,\ |\lambda| = 1$ and $\lambda(1) = 2$.

Example 2.7 — Let Γ be the quiver:



Now setting

$$A = \frac{\mathbb{F}\Gamma}{\langle \alpha_1\alpha_2 - \alpha_1\alpha_3, \alpha_4\alpha_7 - \alpha_5\alpha_7, \alpha_5\alpha_7 - \alpha_6\alpha_7, \alpha_2\alpha_4, \alpha_3\alpha_6 \rangle}.$$

It is not difficult to check that the trivial A-module $\mathbb{F}^{\otimes 5}$ has the following minimal graded projective resolution $0 \to P_3 \to P_2 \to P_1 \to P_0 \to \mathbb{F}^{\otimes 5} \to 0$ such that $P_3,\ P_2,\ P_1$ and P_0 are generated in degree $4,\ 2,\ 1$ and 0, respectively. Therefore, A is a (p, λ)-Koszul algebra with respect to $p = 3,\ |\lambda| = 1$ and $\lambda(1) = 2$. This example also shows that (p, λ)-Koszul algebras are different from almost Koszul algebras, which was introduced by Brenner, Butler and King in [3].

Example 2.8 — Let $N_0 \geq 3$ be a fixed integer and Γ be the following quiver:

$$\bullet^1 \leftrightarrows^{\alpha_1}_{\beta_1} \bullet^2 \leftrightarrows^{\alpha_2}_{\beta_2} \bullet^3 \leftrightarrows^{\alpha_3}_{\beta_3} \cdots \leftrightarrows \bullet^{N_0-1} \leftrightarrows^{\alpha_{N_0-1}}_{\beta_{N_0-1}} \bullet^{N_0}.$$

Now let

$$A = \frac{\Bbbk\Gamma}{\langle \alpha_i\beta_i - \beta_{i+1}\alpha_{i+1},\ \alpha_{i+1}\alpha_i,\ \beta_i\beta_{i+1} : i = 1,\ 2, \cdots, N_0 - 2 \rangle}.$$

Then A is a (p, λ)-Koszul algebra with respect to $p = N_0,\ |\lambda| = 1$ and $\lambda(1) = 2$. We will discuss this algebra in Remark 3.4 in detail.

We can use the minimal graded projective resolution to characterize the (p, λ)-Koszul objects:

Proposition 2.9 — Let A be a (p, λ)-Koszul algebra with $|\lambda| = T$, $M \in gr_0(A)$ and

$$\mathbf{P}: \quad \cdots \to P_n \xrightarrow{f_n} \cdots \to P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \to 0$$

be a minimal graded projective resolution of M. Then $M \in \mathcal{K}^p_\lambda(A)$ if and only if

1. For $i \equiv pn+j \pmod{pT}$, ($n \in [0,\ T-1]$, $j = 0,\ 1,\ \cdots,\ p-2$), we have $\ker f_i \subseteq JP_i$ and $J\ker f_i = J^2P_i \cap \ker f_i$;
2. For $i \equiv pn-1 \pmod{pT}$, ($n \in [1,\ T]$), we have $\ker f_i \subseteq J^{\lambda(n)}P_i$ and $J\ker f_i = J^{\lambda(n)+1}P_i \cap \ker f_i$.

PROOF : The proof is similar to that of ([6], Proposition 3.1) and we omit the details.

3. On the Yoneda-Ext Algebras

In this section, we will study the Yoneda-Ext algebra of a (p,λ)-Koszul algebra and the Koszul dual of a weakly (p,λ)-Koszul module in detail.

Lemma 3.1 — [4] Let A be a graded $\mathbb{F}$ algebra. Suppose that $\mathbf{P}$ is the minimal graded projective resolution of the trivial A-module A_0 and P_n is finitely generated with generators in degree $f(n)$, where $f : \mathbb{N} \to \mathbb{N}$ is a function. Assume that $f(i+j) = f(i) + f(j)$. Then the Yoneda map $\mathrm{Ext}^i_A(A_0,A_0) \otimes_{\mathbb{F}} \mathrm{Ext}^j_A(A_0,A_0) \to \mathrm{Ext}^{i+j}_A(A_0,A_0)$ is surjective. Moreover, we have $\mathrm{Ext}^{i+j}_A(A_0,A_0) = \mathrm{Ext}^i_A(A_0,A_0) \cdot \mathrm{Ext}^j_A(A_0,A_0) = \mathrm{Ext}^j_A(A_0,A_0) \cdot \mathrm{Ext}^i_A(A_0,A_0)$.

Lemma 3.2 — Let δ^p_λ and λ be as before. Then $\delta^p_\lambda(n+pk|\lambda|) = \delta^p_\lambda(n) + \delta^p_\lambda(pk|\lambda|)$ for all $k,\ n \geq 0$.

PROOF : It suffices to prove $\delta^p_\lambda(n+|\lambda|p) = \delta^p_\lambda(n) + \delta^p_\lambda(|\lambda|p)$. By property (ii) of the definition of δ^p_λ, we have $\delta^p_\lambda(pn+i) - \delta^p_\lambda(pn+i-1) = 1$, ($\forall 1 \leq i \leq p-1$) and by property (iii) of the definition of δ^p_λ, we have $\delta^p_\lambda(pn) - \delta^p_\lambda(pn-1) = \lambda(n)$, ($\forall n \geq 1$). Thus we have $\delta^p_\lambda(pn) - \delta^p_\lambda(pn-p) = \lambda(n)+p-1$. Now consider the case of $n = kp$, where $k \in \mathbb{N}$. To prove the statement, we have to check the equation $\delta^p_\lambda(kp+|\lambda|p) = \delta^p_\lambda(kp) + \delta^p_\lambda(|\lambda|p)$. But $\delta^p_\lambda(kp+|\lambda|p) - \delta^p_\lambda(kp) = \sum_{i=1}^{|\lambda|} \lambda(k+i) + |\lambda|(p-1)$ is obvious. So to finish the proof of the case of $n = kp$, we only need to show $\delta^p_\lambda(|\lambda|p) = \sum_{i=1}^{|\lambda|} \lambda(k+i) + |\lambda|(p-1)$, which is implied by $\delta^p_\lambda(|\lambda|p) = \sum_{i=1}^{|\lambda|} \lambda(i) + |\lambda|(p-1)$ and $\sum_{i=1}^{|\lambda|} \lambda(i) = \sum_{i=1}^{|\lambda|} \lambda(k+i)$ for a fixed integer k.

For $n = kp+i$, where $1 \leq i \leq p-1$. Note that $\delta^p_\lambda(|\lambda|p+kp+i) = \delta^p_\lambda((|\lambda|+k)p+i) = \delta^p_\lambda(|\lambda|p+kp) + \delta^p_\lambda(i) = \delta^p_\lambda(|\lambda|p) + \delta^p_\lambda(kp) + \delta^p_\lambda(i) = \delta^p_\lambda(|\lambda|p) + \delta^p_\lambda(kp+i)$.

Therefore, we are done. □

***Theorem* 3.3** — *Let A be a graded algebra and $E(A)$ its Yoneda-Ext algebra. Then A is a (p, λ)-Koszul algebra ($d > p \geq 2$) if and only if*

1. *$E(A)$ is minimally generated in ext-degrees $1,\ p,\ 2p,\ \cdots,\ |\lambda|p$; and*

2. *$\mathrm{Ext}_A^{kp}(A_0, A_0) = \mathrm{Ext}_A^{kp}(A_0, A_0)_{-\delta_\lambda^p(kp)}$, where $k = 1,\ 2,\ \cdots,\ |\lambda|$.*

PROOF : ($\Rightarrow$) First, we will prove the "minimal property." That is, we have to prove that $\mathrm{Ext}_A^{pi}(A_0, A_0)$ $(i = 1,\ 2,\ \cdots,\ |\lambda|)$ can not be generated by the lower ext-degrees. For the simplicity, we will show $\mathrm{Ext}_A^p(A_0, A_0)$ can not be generated in the lower ext-degrees. In fact,

$$\begin{aligned}
\mathrm{Ext}_A^p(A_0, A_0) &= \sum_{m+n=p} \mathrm{Ext}_A^m(A_0, A_0) \cdot \mathrm{Ext}_A^n(A_0, A_0) \\
&= \sum_{m,\ n>0;\ m+n=p} \mathrm{Ext}_A^m(A_0, A_0)_{-m} \cdot \mathrm{Ext}_A^n(A_0, A_0)_{-n} \\
&\subseteq \mathrm{Ext}_A^p(A_0, A_0)_{-p}.
\end{aligned}$$

But

$$\mathrm{Ext}_A^p(A_0, A_0) = \mathrm{Ext}_A^p(A_0, A_0)_{-d}$$

and $d > p$, which implies that $\sum_{m,\ n>0;\ m+n=p} \mathrm{Ext}_A^m(A_0, A_0) \cdot \mathrm{Ext}_A^n(A_0, A_0) = 0$.

Suppose that A is a (p, λ)-Koszul algebra with $|\lambda| := T$, assertion (2) is obvious. For $1 \leq n \leq p-1$, it is just the Koszul case and $\mathrm{Ext}_A^n(A_0, A_0) = (\mathrm{Ext}_A^1(A_0, A_0))^n$ for all $1 \leq n \leq p-1$.

Note that for all $2 \leq n \leq T$, $\delta_\lambda^p(pn-1) = \delta_\lambda^p(pn-p) + \delta_\lambda^p(p-1)$, by Lemma 3.1, we have $\mathrm{Ext}_A^{pn-1}(A_0, A_0) = \mathrm{Ext}_A^{pn-p}(A_0, A_0) \cdot \mathrm{Ext}_A^{p-1}(A_0, A_0)$ for all $2 \leq n \leq T$. Suppose the statement is true for less than n, where $n \geq Tp$. Now consider the case of n. By Lemma 3.2, $\delta_\lambda^p(n) = \delta_\lambda^p(n-Tp) + \delta_\lambda^p(Tp)$, by Lemma 3.1, we have $\mathrm{Ext}_A^n(A_0, A_0) = \mathrm{Ext}_A^{Tp}(A_0, A_0) \cdot \mathrm{Ext}_A^{n-Tp}(A_0, A_0)$ and we are done.

($\Leftarrow$) By the hypothesis, it is easy to see that we have $\mathrm{Ext}_A^n(A_0, A_0) = \mathrm{Ext}_A^n(A_0, A_0)_{-\delta_\lambda^p(n)}$ for all $0 \leq n \leq Tp$. Now let $n \geq Tp$, write $n = kTp + i$, where $i \in [1,\ Tp-1]$.

$$\begin{aligned}
\mathrm{Ext}_A^n(A_0, A_0) &= \mathrm{Ext}_A^{kTp+i}(A_0, A_0) \\
&= \sum_{m+n=kTp+i, m,\ n \geq 0} \mathrm{Ext}_A^m(A_0, A_0) \cdot \mathrm{Ext}_A^n(A_0, A_0) \\
&\subseteq \mathrm{Ext}_A^{kTp+i}(A_0, A_0)_{-\delta_\lambda^p(kTp)-\delta_\lambda^p(i)} \\
&= \mathrm{Ext}_A^{kTp+i}(A_0, A_0)_{-\delta_\lambda^p(kTp+i)} \\
&= \mathrm{Ext}_A^n(A_0, A_0)_{-\delta_\lambda^p(n)}.
\end{aligned}$$

Therefore, we finish the proof. □

Remark 3.4 : We claim that by Theorem 3.3, Example 2.8 gives an answer to the third question raised by Green and Marcos in [5]. Indeed, (p, λ)-Koszul algebras are special "δ-Koszul algebras". If we can construct a (p, λ)-Koszul algebra with $|\lambda|p > N$ for any enough large $N \in \mathbb{N}$, then by Theorem 3.3, the (p, d, λ)-Koszul algebras will give a negative answer to the question (see also [11]). Now we will compute out the minimal graded projective resolution of the trivial A-module A_0 as follows.

Let P_i denote the simple A-module related to the vertex i.

If $N_0 = 3$, then $\mathbb{F}^{\otimes 3}$ has the following minimal graded projective resolution

$\cdots \to (A \oplus P_2)[6] \to (A \oplus P_2)[5] \to A[4] \to (A \oplus P_2)[2] \to (A \oplus P_2)[1] \to A \to \mathbb{F}^{\otimes 3} \to 0.$

If $N_0 = 4$, then $\mathbb{F}^{\otimes 4}$ has the following minimal graded projective resolution

$\cdots \to (A \oplus P_2 \oplus P_3)[7] \to (A \oplus P_2 \oplus P_3)[6] \to A[5] \to (A \oplus P_2 \oplus P_3)[3] \to (A \oplus P_2 \oplus P_3)[2] \to (A \oplus P_2 \oplus P_3)[1] \to A \to \mathbb{F}^{\otimes 4} \to 0.$

By an induction, we get that the minimal graded projective resolution of the trivial A-module $\Bbbk^{\otimes N_0}$ has the following general form:

$\cdots \to (A \oplus (P_2 \oplus \cdots \oplus P_{N_0-1}))[N_0+2] \to A[N_0+1] \to A[N_0-1] \to \cdots \to (A \oplus (P_2 \oplus \cdots \oplus P_{N_0-1}) \oplus (P_3 \oplus \cdots \oplus P_{N_0-2}) \oplus \cdots \oplus P_{\frac{N_0-1}{2}} \oplus P_{\frac{N_0+1}{2}} \oplus P_{\frac{N_0+3}{2}}) \left[\frac{N_0+1}{2}\right] \to (A \oplus (P_2 \oplus \cdots \oplus P_{N_0-1}) \oplus (P_3 \oplus \cdots \oplus P_{N_0-2}) \oplus \cdots \oplus\ P_{\frac{N_0+1}{2}}\Big)$

$\left[\frac{N_0-1}{2}\right] \to \cdots \to (A \oplus (P_2 \oplus \cdots \oplus P_{N_0-1}) \oplus (P_3 \oplus \cdots \oplus P_{N_0-2}))[2] \to (A \oplus (P_2 \oplus \cdots \oplus P_{N_0-1}))[1] \to A \to \mathbb{F}^{\otimes N_0} \to 0$ for N_0 being odd;

$\cdots \to (A \oplus (P_2 \oplus \cdots \oplus P_{N_0-1}))[N_0+2] \to A[N_0+1] \to A[N_0-1] \to \cdots \to (A \oplus (P_2 \oplus \cdots \oplus P_{N_0-1}) \oplus (P_3 \oplus \cdots \oplus P_{N_0-2}) \oplus \cdots \oplus P_{\frac{N_0}{2}} \oplus P_{\frac{N_0}{2}+1}) \left[\frac{N_0}{2}\right] \to (A \oplus (P_2 \oplus \cdots \oplus P_{N_0-1}) \oplus (P_3 \oplus \cdots \oplus P_{N_0-2}) \oplus \cdots \oplus P_{\frac{N_0}{2}} \oplus P_{\frac{N_0}{2}+1}) \left[\frac{N_0}{2} - 1\right] \to \cdots \to (A \oplus (P_2 \oplus \cdots \oplus P_{N_0-1}) \oplus (P_3 \oplus \cdots \oplus P_{N_0-2}))[2] \to (A \oplus (P_2 \oplus \cdots \oplus P_{N_0-1}))[1] \to A \to \mathbb{F}^{\otimes N_0} \to 0$ for N_0 being even.

It is obvious that most terms of the above resolutions are made of many brackets, in order to avoid some misunderstandings, we stipulate the following: Given a concrete $N \in \mathbb{N}$, whether the bracket appears or not is completely determined by the subscripts of the first object and the last object in the bracket. If the subscript of the first object is smaller than that of the last object, then such bracket appears. Otherwise, the bracket does not appear.

Now it is easy to see that the algebra constructed above is a (p, λ)-Koszul algebra with respect to $p = N_0$, $|\lambda| = 1$ and $\lambda(1) = 2$. Note that the number p equals to the number of the vertexes of the quiver Γ. Thus for any given $N \in \mathbb{N}$, we can construct a (p, λ)-Koszul algebra with $|\lambda|p = p > N$.

Corollary 3.5 — Let A be a (p, λ)-Koszul algebra and $E(A)$ its Yoneda-Ext algebra. Then $\|A\| = h(A)$.

Definition 3.6 — Let $A = T(A_1)/(R_A)$ be a quadratic algebra, where $T(A_1)$ is the tensor algebra of A_0-A_0-bimodule ${}_{A_0}A_{1A_0}$. Define

$$R_A^{\perp} := \{f \in A_1^* \otimes_{A_0} A_1^* | f(R_A) = 0\}, \quad A^! := T(A_1^*)/(R_A^{\perp}),$$

where $(R_A^{\perp})$ is the ideal of $T(A_1^*)$ generated by $R^{\perp}$. We call $A^!$ the *dual algebra* of A. Let M be a graded left A-module generated in degree 0. Writing $M = (A \otimes_{A_0} M_0)/(R_M)$ with $R_M \subset M_1$. The *dual module* of the M is the right $A^!$-module $M^! := (M_0^* \otimes A^!)/(R_M^{\perp})$ with $R_M^{\perp} \subseteq M_0^* \otimes_{A_0} A_1^*$ being the orthogonal complement of R_M. Here $(\ )^* := \mathrm{Hom}_{A_0}(\ , A_0)$.

If M is a right A-module, then $M^!$ can be defined similarly.

Lemma 3.7 ([14]) — Let A be a graded algebra and $M = \bigoplus_{i \geq 0} M_i$ be a finitely generated module over A. Then we have the following isomorphisms:

$$\bigoplus_{i\geq 0} \mathrm{Ext}_A^i(A_0, A_0)_{-i} \cong ({}_qA)^!, \qquad \bigoplus_{i\geq 0} \mathrm{Ext}_A^i(M, \mathbb{F})_{-i} \cong ({}_qM)^!_{{}_qA},$$

where ${}_qA$ and ${}_qM$ are the quadratic part of A and M, respectively.

Proposition 3.8 — Let A be a (p, λ)-Koszul algebra ($d > p > 2$) and $E(A)$ its Yoneda-Ext algebra.

1. Let B be the subalgebra of $E(A)$ generated by $\mathrm{Ext}_A^1(A_0, A_0)$ then
   (a) B is finite dimensional,
   (b) as a graded algebra $B \cong A^!$.
2. If $M \in gr_0(A)$ is a (p, λ)-Koszul module, then $\bigoplus_{i=0}^{p-1} \mathrm{Ext}_A^i(M, \mathbb{F}) \cong (M)^!_B$.

PROOF : For (1), note that A is a (p, λ)-Koszul algebra with $d > p > 2$, then $\mathrm{Ext}_A^i(A_0, A_0) = \mathrm{Ext}_A^i(A_0, A_0)_{-i}$ for all $0 \leq i < p$. It is easy to see that B is of finite dimensional since A is a locally finite algebra and $B = \bigoplus_{i=0}^{p-1} \mathrm{Ext}_A^i(A_0, A_0)_{-i}$. Note that (p, λ)-Koszul algebras with $d > p > 2$ are quadratic algebras, thus $A =_q A$. Now by Lemma 3.7 and we are done. (2) is immediate from the fact that (p, λ)-Koszul modules are quadratic modules and Lemma 3.7. □

4. Weakly (p, λ)-Koszul Modules

In this section, we will study the "(p, λ)-Koszulness" of an arbitrary finitely generated graded module over a (p, λ)-Koszul algebra.

In this section, A denotes a (p, λ)-Koszul algebra with $|\lambda| = T \geq 1$.

Motivated by Proposition 2.9, we give the next definition:

Definition 4.1 — Let $M \in gr(A)$. We call M a *weakly* (p, λ)*-Koszul module* if there exists a minimal graded projective resolution of M $\cdots \to P_i \xrightarrow{f_i} \cdots \longrightarrow P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \to 0$, such that for all $i, k \geq 0$, we have the following conditions

1. For $i \equiv pn + j \pmod{pT}$, ($n \in [0,\ T-1]$, $j = 0,\ 1,\ \cdots,\ p-2$), we have $\ker f_i \subseteq JP_i$ and $J^k \ker f_i = J^{k+1}P_i \cap \ker f_i$;

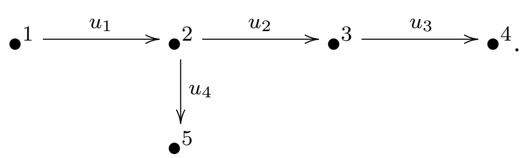
2. For $i \equiv pn - 1 \pmod{pT}$, $(n \in [1,\ T])$, we have $\ker f_i \subseteq J^{\lambda(n)} P_i$ and $J^k \ker f_i = J^{\lambda(n)+k} P_i \cap \ker f_i$.

Let $\mathcal{WK}^p_\lambda(A)$ denote the category of weakly (p, λ)-Koszul modules.

Example 4.2 — The following are some easy examples of weakly (p, λ)-Koszul modules.

1. (p, λ)-Koszul modules are special weakly (p, λ)-Koszul modules.
2. Let A be a (p, λ)-Koszul algebra and $M^s := A_0[1] \oplus \cdots \oplus A_0[s]$, $s \in \mathbb{N}^*$. Then M^s is a weakly (p, λ)-Koszul module.

Example 4.3 — Let Γ be the following quiver:



Setting $A = \mathbb{F}/(u_1 u_2 u_3)$. Then under a routine computation, A is a (p, λ)-Koszul algebra with $p = 2, |\lambda| = 1$ and $\lambda(1) = 3$. Let $e_1, \cdots, e_5$ be the idempotents of A corresponding to the vertexes. Let $V = \mathbb{F}v_0 \oplus \mathbb{F}v_1$ be a graded vector space with basis v_0 and v_1. Assume that the degree of v_0 is 0 and that of v_1 is 1. Define a left A_0-module action on V as follows: $e_4 \cdot v_0 = v_0$ and $e_i \cdot v_0 = 0$ for $i \neq 4$; $e_5 \cdot v_1 = v_1$ and $e_i \cdot v_1 = 0$ for $i \neq 5$. Let

$$M = \frac{A \otimes_{A_0} V}{\langle u_2 \otimes_{A_0} u_3 \otimes_{A_0} v_0 - u_4 \otimes_{A_0} v_1 \rangle}.$$

Now it is not hard to check that M is a weakly (p, λ)-Koszul module with respect to $p = 2, |\lambda| = 1$ and $\lambda(1) = 3$.

Proposition 4.4 — Let $M \in gr_s(A)$ and $s \in \mathbb{N}$. The following statements are equivalent:

1. $M \in \mathcal{K}^p_\lambda(A)$;
2. M has the following resolution $\mathbf{P}$: $\cdots \to P_n \xrightarrow{f_n} \cdots \to P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \to 0$ such that
   (a) For $i \equiv pn+j \pmod{pT}$, $(n \in [0,\ T-1],\ j = 0,\ 1,\ \cdots,\ p-2)$, we have $\ker f_i \subseteq JP_i$ and $J \ker f_i = J^2 P_i \cap \ker f_i$;

(b) For $i \equiv pn-1 \pmod{pT}$, $(n \in [1,\ T])$, we have $\ker f_i \subseteq J^{\lambda(n)}P_i$ and $J\ker f_i = J^{\lambda(n)+1}P_i \cap \ker f_i$.

3. $M \in \mathcal{WK}^p_\lambda(A)$.

Lemma 4.5 — Let $0 \to K \to M \to N \to 0$ be an exact sequence in $gr(A)$. The following are true:

1. If $K, M \in \mathcal{WK}^p_\lambda(A)$ with $J^kK = K \cap J^kM$ for all $k \geq 0$, then $N \in \mathcal{WK}^p_\lambda(A)$.
2. If $K, N \in \mathcal{WK}^p_\lambda(A)$ with $JK = K \cap JM$, then $M \in \mathcal{WK}^p_\lambda(A)$.

Proof : We will only show (1) since (2) can be shown similarly.

Similar to the Koszul case ([12]), we have $J^k\Omega^1(K) = \Omega^1(K) \cap J^k\Omega^1(M)$ and $J^k\Omega^1(N) = \Omega^1(N) \cap J^{k+1}P_0^N$. Repeating the above steps and by induction, we have $J^k\Omega^i(K) = \Omega^i(K) \cap J^k\Omega^i(M)$ and $J^k\Omega^i(N) = \Omega^i(N) \cap J^{k+1}P_{i-1}^N$ for $1 \leq i \leq p$. Similarly, we have the following commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & A/J^k \otimes_A \Omega^{p+1}(K) & \longrightarrow & A/J^k \otimes_A \Omega^{p+1}(M) & \longrightarrow & A/J^k \otimes_A \Omega^{p+1}(N) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A/J^k \otimes_A J^{\lambda(1)}P_p^K & \longrightarrow & A/J^k \otimes_A J^{\lambda(1)}P_p^M & \longrightarrow & A/J^k \otimes_A J^{\lambda(1)}P_p^N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A/J^k \otimes_A J^{\lambda(1)}\Omega^p(K) & \longrightarrow & A/J^k \otimes_A J^{\lambda(1)}\Omega^p(M) & \longrightarrow & A/J^k \otimes_A J^{\lambda(1)}\Omega^p(N) & \longrightarrow & 0, \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

which implies that $J^k\Omega^{p+1}(N) = \Omega^{p+1}(N) \cap J^{k+\lambda(1)}P_p^N$. By induction, we have that $N \in \mathcal{WK}^p_\lambda(A)$.

Lemma 4.6 — Let $M = \bigoplus_{i\geq 0} M_i$ be a weakly (p, λ)-Koszul module with $M_0 \neq 0$. Setting $K_M = \langle M_0 \rangle$. Then

1. K_M is a (p, λ)-Koszul module;

2. $K_M \cap J^k M = J^k K_M$ for each $k \geq 0$;
3. M/K_M is a weakly (p,λ)-Koszul module.

PROOF : For the first assertion of the theorem, we need to show that K_M admits a minimal graded projective resolution $\cdots \to Q_n \to \cdots \to Q_2 \to Q_1 \to Q_0 \to K_M \to 0$ such that each Q_i is generated in degree $\delta^p_\lambda(i)$. Similar to the proof of ([12], Theorem 2.4), we have the following commutative diagram with exact rows,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega^p(K_M) & \longrightarrow & P^{p-1}_{p-1} & \longrightarrow & \Omega^{p-1}(K_M) & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow{\scriptstyle i} & & \big\downarrow{\scriptstyle j} & & \\ 0 & \longrightarrow & \Omega^p(M) & \longrightarrow & P_{p-1} & \longrightarrow & \Omega^{p-1}(M) & \longrightarrow & 0 \end{array}$$

where i and j are natural embeddings.

Next we need to show $\langle \Omega^p(M)_{\delta^p_\lambda(p)} \rangle = \Omega^p(K_M)$. We get $\Omega^p(M) \subseteq J^{\lambda(1)}P_{p-1}$ since M is weakly (p,λ)-Koszul. Thus we have the following exact sequence:

$$0 \to \Omega^p(M) \to J^{\lambda(1)}P_{p-1} \to J^{\lambda(1)}\Omega^{p-1}(M) \to 0.$$

Since $\Omega^p(M) \cap J^k(J^{\lambda(1)}P_{p-1}) = \Omega^p(M) \cap J^{k+\lambda(1)}P_{p-1} = J^k\Omega^p(M)$, we have

$$\langle \Omega^p(M)\langle \omega^p(M)^{p(\delta)}_{\delta\lambda} \rangle = \Omega^p(M) \cap \langle (J^{\lambda(1)}P_{p-1})_{\delta^p_\lambda(p)} \rangle = \Omega^p(M) \cap \langle (J^{\lambda(1)}P^{p-1}_{p-1})_{\delta^p_\lambda(p)} \rangle.$$

We have $\Omega^p(M) \cap \langle (J^{\lambda(1)}P^{p-1}_{p-1})_{\delta^p_\lambda(p)} \rangle = \Omega^p(M) \cap P^{p-1}_{p-1} = \Omega^p(K_M)$ since P^{p-1}_{p-1} is generated in degree $p-1$. Hence $\Omega^p(K_M) = \langle \Omega^p(M)_{\delta^p_\lambda(p)} \rangle$, which is generated in degree $\delta^p_\lambda(p)$. Hence Q_p is generated in degree $\delta^p_\lambda(p)$. Then repeating the above argument, we get that Q_i is generated in degree $\delta^p_\lambda(i)$ for all $i \geq 0$.

The second assertion is trivial and the third assertion is immediate from the exact sequence $0 \to K_M \to M \to M/K_M \to 0$ and Lemma 4.5 (1). $\square$

Let M be an arbitrary finitely generated graded A-module, $\{S_{d_1}, S_{d_2}, \cdots, S_{d_t}\}$ a set of minimal homogeneous generating spaces of M. We may assume

$S_{d_i} \subseteq M_{d_i}$ for $1 \leq i \leq t$ and $d_1 < d_2 < \cdots < d_t$. Then M admits a natural filtration of graded submodules

$$\mathcal{F}M: \quad 0 = U_0 \subset U_1 \subset U_2 \subset \cdots \subset U_t = M,$$

where $U_1 = \langle S_{d_1} \rangle$, $U_2 = \langle S_{d_1}, S_{d_2} \rangle$, $\cdots$, $U_t = \langle S_{d_1}, S_{d_2}, \cdots, S_{d_t} \rangle$.

***Theorem* 4.7** — *Let $M \in gr(A)$. Using the above notations. Then $M \in \mathcal{WK}^p_\lambda(A)$ if and only if, for all $1 \leq i \leq t$, $U_i/U_{i-1} \in \mathcal{K}^p_\lambda(A)$, where $U_0 = 0$.*

PROOF : Assume that M is a weakly (p, λ)-Koszul module. If $p = 0$, then the theorem is trivial. Now suppose $p \geq 1$. Applying Lemma 4.6, we get that U_1 is a (p, λ)-Koszul module and $U_1 \cap J^k M = J^k U_1$ for all $k \geq 0$. Let $W = M/U_1$. By Lemma 4.5, W is a weakly (p, λ)-Koszul module. Clearly, W is support in degrees not less than d_2. Consider the exact sequence $0 \to K_W \to W \to W/K_W \to 0$, where $K_W = \langle W_{d_2} \rangle = U_2/U_1$. Applying Lemma 4.6 again, we get that K_W is a (p, λ)-Koszul module and $K_W \cap J^k M = J^k K_W$ for all $k \geq 0$. Repeating the above argument, we get that all U_i/U_{i-1} are (p, λ)-Koszul modules for $1 \leq i \leq t$.

Conversely, consider the exact sequence $0 \to U_0 \to U_1 \to U_1/U_0 \to 0$. Evidently, $JU_0 = U_0 \cap JU_1$. Since U_0 and U_1/U_0 are (p, λ)-Koszul modules, we get that U_1 is a weakly (p, λ)-Koszul module by Lemma 4.5 (2). By Corollary 6.5 and Lemma 4.5 several times, we get that M is a weakly (p, λ)-Koszul module. □

Corollary 4.8 — Using the notations of Theorem 4.7. Then M is a weakly (p, λ)-Koszul module if and only if all U_i $(1 \leq i \leq t)$ are weakly (p, λ)-Koszul modules.

PROOF : $(\Rightarrow)$ Clearly, U_i possesses a following natural filtration $0 \subset U_1 \subset U_2 \subset \cdots \subset U_i$ such that all U_j/U_{j-1} are (p, λ)-Koszul modules, where $1 \leq j \leq i$. By Theorem 4.7, all U_i $(1 \leq i \leq t)$ are weakly (p, λ)-Koszul modules. □

$(\Leftarrow)$ Note that $M = U_t$.

5. ON THE ASSOCIATED GRADED MODULE $\mathbf{G}(M)$

Let A be a graded $\mathbb{F}$-algebra and $M \in gr(A)$. We can get another graded module, denoted by $\mathbf{G}(M)$, called the *associated graded module of* M as follows:

$$\mathbf{G}(M) = M/JM \oplus JM/J^2M \oplus J^2M/J^3M \oplus \cdots ,$$

and $\mathbf{G}(A)$ can be defined similarly.

The following are the basic properties of the functor $\mathbf{G}$, we omit the proof since it is canonical.

Proposition 5.1 — Let $M \in gr(A)$. Then

1. $\mathbf{G}(A) \cong A$ as a graded $\mathbb{F}$-algebra;
2. $\mathbf{G}(M) \in gr_0(\mathbf{G}(A)) = gr_0(A)$;
3. If M is pure, then $\mathbf{G}(M)[i] \cong M$ as a graded A-module for some i;
4. $\mathbf{G}^2 = \mathbf{G}$;
5. $\mathbf{G}(J^nM) \cong J^n\mathbf{G}(M)$ for all $n \geq 0$;
6. $M \in gr(A)$ if and only if $\mathbf{G}(M) \in gr(A)$;
7. $\mathbf{G}$ preserves finite direct sums;
8. Let $P \in gr(A)$ be a graded projective module. Then $\mathbf{G}(P)$ is a pure graded projective $\mathbf{G}(A)$-module;
9. Let $f \in \mathrm{Hom}_A(M, N)$ an epimorphism. Then $\mathbf{G}(f)$ is also an epimorphism from $\mathbf{G}(M)$ to $\mathbf{G}(N)$).

Lemma 5.2 [12] — Let $M = M_0 \oplus M_1 \oplus M_2 \oplus \cdots \in gr(A)$ with $M_0 \neq 0$. Then we have a split exact sequence in $gr(\mathbf{G}(A)) = gr(A)$

$$0 \longrightarrow \mathbf{G}(\langle M_0 \rangle) \longrightarrow \mathbf{G}(M) \longrightarrow \mathbf{G}(M/\langle M_0 \rangle) \longrightarrow 0.$$

Theorem 5.3 — *Let A be a (p, λ)-Koszul algebra and $M \in gr(A)$. Then M is a weakly (p, λ)-Koszul module if and only if $\mathbf{G}(M)$ is a (p, λ)-Koszul module.*

Proof : Assume that M is generated by a minimal set of homogeneous elements lying in degrees $k_0 < k_1 < \cdots < k_t$ since $M \in gr(A)$. By Lemma 5.2, we get a split exact sequence,

$$0 \longrightarrow \mathbf{G}(\langle M_{k_0} \rangle) \longrightarrow \mathbf{G}(M) \longrightarrow \mathbf{G}(M/\langle M_{k_0} \rangle) \longrightarrow 0.$$

($\Rightarrow$) Recall that M is a weakly (p,λ)-Koszul module, to prove that $\mathbf{G}(M)$ is a (p,λ)-Koszul module, by induction on t. If $t=0$, M is a weakly (p,λ)-Koszul module generated in a single degree, clearly M is a (p,λ)-Koszul module and $M \cong \mathbf{G}(M)$ as a graded A-module. Hence $\mathbf{G}(M)$ is a (p,λ)-Koszul module. Now we assume that the statement holds for less than t. By Lemma 4.6, $\langle M_{k_0}\rangle$ is a (p,λ)-Koszul module, of course a weakly (p,λ)-Koszul module. Consider the exact sequence

$$0 \longrightarrow \langle M_{k_0}\rangle \longrightarrow M \longrightarrow M/\langle M_{k_0}\rangle \longrightarrow 0.$$

By Lemmas 4.5 and 4.6 (2), we get that $M/\langle M_{k_0}\rangle$ is a weakly (p,λ)-Koszul module. Observe that the number of generators of $M/\langle M_{k_0}\rangle$ is less than t, by the induction assumption, $\mathbf{G}(M/\langle M_{k_0}\rangle)$ ia a (p,λ)-Koszul module. Note that $\mathbf{G}(\langle M_{k_0}\rangle)$ is obvious a (p,λ)-Koszul module, we get that $\mathbf{G}(M)$ is a (p,λ)-Koszul module induced from the split exact sequence

$$0 \longrightarrow \mathbf{G}(\langle M_{k_0}\rangle) \longrightarrow \mathbf{G}(M) \longrightarrow \mathbf{G}(M/\langle M_{k_0}\rangle) \longrightarrow 0.$$

($\Leftarrow$) Suppose that $\mathbf{G}(M)$ is a (p,λ)-Koszul module, we also do it by induction. If $t=0$, then $\mathbf{G}(M) \cong M$ as a graded A-module. Clearly M is a weakly (p,λ)-Koszul module. Now we assume that the statement holds for less than t. We get that $\mathbf{G}(\langle M_{k_0}\rangle)$ and $\mathbf{G}(M/\langle M_{k_0}\rangle)$ are (p,λ)-Koszul modules induced from the split exact sequence

$$0 \longrightarrow \mathbf{G}(\langle M_{k_0}\rangle) \longrightarrow \mathbf{G}(M) \longrightarrow \mathbf{G}(M/\langle M_{k_0}\rangle) \longrightarrow 0.$$

By the induction assumption, $\langle M_{k_0}\rangle$ and $M/\langle M_{k_0}\rangle$ are weakly (p,λ)-Koszul modules. Consider the exact sequence

$$0 \longrightarrow \langle M_{k_0}\rangle \longrightarrow M \longrightarrow M/\langle M_{k_0}\rangle \longrightarrow 0$$

and by Lemma 4.5, we get that M is a weakly (p,λ)-Koszul module.

6. Some Applications

6.1 *On the relationship of the minimal graded projective resolutions between M and these U_i/U_{i-1}'s.*

Lemma 6.1 — Let A be a graded algebra and

$$0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$$

be an exact sequence in $gr(A)$. Then $JK = K \cap JM$ if and only if we have the following commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega^1(K) & \longrightarrow & \Omega^1(M) & \longrightarrow & \Omega^1(N) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_0 & \longrightarrow & Q_0 & \longrightarrow & L_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0, \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where P_0, Q_0 and L_0 are graded projective covers.

PROOF : ($\Rightarrow$) By the assumption, we obtain the exact sequence

$$0 \longrightarrow K/JK \longrightarrow M/JM \longrightarrow N/JN \longrightarrow 0.$$

Note that for a finitely generated graded module over a positively graded algebra, M/JM is the minimal generating space of M and we denote $M/JM := S_M = S_M^{d_1} \oplus S_M^{d_2} \oplus \cdots \oplus S_M^{d_p}$, where $S_M^{d_i}$ is the set of homogeneous elements of M of degree d_i and the "$\oplus$" is with respect to A_0-modules. Therefore, we get an exact sequence of A_0-modules $0 \longrightarrow S_K \longrightarrow S_M \longrightarrow S_N \longrightarrow 0$.

Now setting $P_0 := A \otimes_{A_0} S_K$, $Q_0 := A \otimes_{A_0} S_M$ and $L_0 := A \otimes_{A_0} S_N$. We have the following exact sequence $0 \longrightarrow P_0 \longrightarrow Q_0 \longrightarrow L_0 \longrightarrow 0$ since A_0 is semisimple. Therefore, we have the desired diagram.

($\Leftarrow$) Suppose that we have the above commutative diagram. Note that the projective cover of a module is unique up to isomorphisms. We may assume that $P_0 := A \otimes_{A_0} S_K$, $Q_0 := A \otimes_{A_0} S_M$ and $L_0 := A \otimes_{A_0} S_N$, where $S_K := K/JK$, $S_M := M/JM$ and $S_N := N/JN$. From the middle row of the diagram, we have the following exact sequence

$$0 \longrightarrow A \otimes_{A_0} S_K \longrightarrow A \otimes_{A_0} S_M \longrightarrow A \otimes_{A_0} S_N \longrightarrow 0,$$

which implies the following short exact sequence as A_0-modules

$$0 \longrightarrow S_K \longrightarrow S_M \longrightarrow S_N \longrightarrow 0.$$

That is, we have the exact sequence $0 \to K/JK \to M/JM \to N/JN \to 0$, which implies $JK = K \cap JM$.

Lemma 6.2 — Let $M = \bigoplus_{i\geq 0} M_i$ be a weakly (p, λ)-Koszul module with $M_0 \neq 0$.

1. Denoting $K := \langle M_0 \rangle$ and $N := M/K$. Then the "Minimal Horseshoe Lemma" (replacing all the projective resolutions by minimal projective resolutions) holds for the natural exact sequence

$$0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0.$$

2. Using the notions of Theorem 4.7. Then for integers $j \geq 1$, the "Minimal Horseshoe Lemma" holds for

$$0 \longrightarrow U_j \longrightarrow U_{j+1} \longrightarrow U_{j+1}/F_j \longrightarrow 0.$$

Proof : (1) By Lemma 4.6, we get $JK = K \cap JM$. By Lemma 6.1, we have the commutative diagram as Lemma 6.1 and the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Omega^1(K) & \longrightarrow & \Omega^1(M) & \longrightarrow & \Omega^1(N) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & JP_0 & \longrightarrow & JL_0 & \longrightarrow & JQ_0 & \longrightarrow & 0, \end{array}$$

where P_0, Q_0 and L_0 are graded projective covers. Of course, $L_0 = P_0 \oplus Q_0$ since the exact sequence $0 \longrightarrow P_0 \longrightarrow L_0 \longrightarrow Q_0 \longrightarrow 0$ and Q_0 is a graded projective module. By Lemma 4.5, we have M and N are weakly (p, λ)-Koszul modules. Applying the functor $A/J \otimes_A -$ to the above diagram, we get the following commutative diagram

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ & & A/J \otimes_A \Omega^1(K) & \xrightarrow{\beta} & A/J \otimes_A \Omega^1(M) & \longrightarrow & A/J \otimes_A \Omega^1(N) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A/J \otimes_A JP_0 & \longrightarrow & A/J \otimes_A JL_0 & \longrightarrow & A/J \otimes_A JQ_0 & \longrightarrow & 0. \end{array}$$

By Lemma 4.6, K is a (p,λ)-Koszul module, which implies $J\Omega^1(K) = \Omega^1(K) \cap J^2P_0$. Thus, α is a monomorphism, which implies that β is also a monomorphism. Hence we have $J\Omega^1(K) = \Omega^1(K) \cap J\Omega^1(M)$. Now replacing $0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$ by

$$0 \longrightarrow \Omega^1(K) \longrightarrow \Omega^1(M) \longrightarrow \Omega^1(N) \longrightarrow 0,$$

repeating the above argument, we are done.

(2) By (1), we have the following commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{P}^1_* & \longrightarrow & \mathcal{P}^2_* & \longrightarrow & \mathcal{Q}^2_* & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & U_1 & \longrightarrow & U_2 & \longrightarrow & U_2/U_1 & \longrightarrow & 0, \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

where $\mathcal{P}^1_*$, $\mathcal{P}^2_*$ and $\mathcal{Q}^2_*$ are the minimal graded projective resolutions. Clearly, for each i, the terms P^1_i, P^2_i and Q^2_i in the complexes $\mathcal{P}^1_*$, $\mathcal{P}^2_*$ and $\mathcal{Q}^2_*$ respectively satisfy $P^2_i = P^1_i \oplus Q^2_i$.

Similarly, we also have the following two commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{Q}^2_* & \longrightarrow & \mathcal{Q} & \longrightarrow & \mathcal{Q}^3_* & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & U_2/U_1 & \longrightarrow & U_3/U_1 & \longrightarrow & U_3/U_2 & \longrightarrow & 0, \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

where $\mathcal{Q}^2_*$, $\mathcal{Q}$ and $\mathcal{Q}^3_*$ are the minimal graded projective resolutions.

Now consider the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{P}_*^1 & \longrightarrow & \mathcal{P}_*^3 & \longrightarrow & \mathcal{Q} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & U_1 & \longrightarrow & U_3 & \longrightarrow & U_3/U_1 & \longrightarrow & 0, \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where $\mathcal{P}_*^1$, $\mathcal{P}_*^3$ and $\mathcal{Q}$ are the minimal graded projective resolutions. If we further denote the terms in complexes $\mathcal{P}_*^3$ and $\mathcal{Q}_*^3$ by P_i^3 and Q_i^3. Then it is clear that $P_i^3 = P_i^1 \oplus Q_i^2 \oplus Q_i^3$.

For exact sequence $0 \longrightarrow U_2 \longrightarrow U_3 \longrightarrow U_3/U_2 \longrightarrow 0$, by 'Horseshoe Lemma', we have the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{P}_*^2 & \longrightarrow & \mathcal{P}_* & \longrightarrow & \mathcal{Q}_*^3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & U_2 & \longrightarrow & U_3 & \longrightarrow & U_3/U_2 & \longrightarrow & 0, \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

with exact rows and columns, where $\mathcal{P}_*^2$ and $\mathcal{Q}_*^3$ are the minimal graded projective resolutions. For each term P_i in $\mathcal{P}_*$, it is clear that $P_i = P_i^2 \oplus Q_i^3 = P_i^1 \oplus Q_i^2 \oplus Q_i^3$, which shows that $\mathcal{P}_*$ is the minimal graded projective resolution of U_3. Then we can get the desired result by induction. □

Remark 6.3 : Some one may think that the Minimal Horseshoe Lemma holds automatically, but the following easy example suggests that it is not the case in general.

Example 6.4 — Let $M \in gr(A)$ with $Rad(M) \neq 0$, where $Rad(M)$ denotes the graded Jacobson radical of M. Setting $K = Rad(M)$, $N = M/Rad(M)$. Then we have the following commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc} & & \vdots & & \vdots & & \vdots & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & P_1 & \longrightarrow & Q_1 & \longrightarrow & L_1 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & P_0 & \longrightarrow & Q_0 & \longrightarrow & L_0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0, \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

where the both sides are minimal graded projective resolutions and the middle column is a graded projective resolution. Now we claim that $Q_0 \longrightarrow M \longrightarrow 0$ must not be a graded projective cover. If not, note that $N = M/Rad(M)$, we have $Q_0 = L_0$, which forces $P_0 = 0$. It is impossible since $K \neq 0$. Therefore, the "Minimal Horseshoe Lemma" does not hold in this case.

Corollary 6.5 — Let $M = \bigoplus_{i\geq 0} M_i$ be a weakly (p,λ)-Koszul module. Using the notations of Theorem 4.7. Then $JU_{j-1} = U_{j-1} \cap JU_j$ for all $1 \leq j \leq t$, where $U_0 = 0$.

Proof : It is immediate from Lemmas 6.1 and 6.2.

***Theorem* 6.6** — *Let M be a weakly (p,λ)-Koszul module and $0 = U_0 \subset U_1 \subset U_2 \subset \cdots \subset U_{t-1} \subset U_t = M$ its natural submodule filtration. Setting $K_i := U_i/U_{i-1}$ for $i = 1, 2, \cdots, t$. Let $\mathcal{P}_* \to M \to 0$ and $\mathcal{P}^i_* \to K_i \to 0$ be the minimal graded projective resolutions of M and K_i's, respectively. Then for all $n \geq 0$, we have*

$$\mathcal{P}_n \cong \bigoplus_{i=1}^{t} \mathcal{P}^i_n.$$

Proof : Consider the following exact sequence

$$0 \longrightarrow U_1 \longrightarrow M \longrightarrow M/U_1 \longrightarrow 0.$$

By Lemma 6.2 (1), we have the following commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{P}_*^1 & \longrightarrow & \mathcal{P}_* & \longrightarrow & \mathcal{L}_*^1 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & U_1 & \longrightarrow & M & \longrightarrow & M/U_1 & \longrightarrow & 0, \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

where $\mathcal{P}_*^1$, $\mathcal{P}_*$ and $\mathcal{L}_*^1$ are the minimal graded projective resolutions of U_1, M and M/U_1, respectively. Clearly, $\mathcal{P}_* = \mathcal{P}_*^1 \oplus \mathcal{L}_*^1$. Setting $W = M/U_1$. Then $\langle W_{d_2} \rangle = U_2/U_1 = K_2$. Consider the following exact sequence

$$0 \longrightarrow K_2 \longrightarrow W \longrightarrow W/K_2 \longrightarrow 0.$$

By Lemma 6.2 (1) again, we have the following commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{P}_*^2 & \longrightarrow & \mathcal{L}_*^1 & \longrightarrow & \mathcal{L}_*^2 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K_2 & \longrightarrow & W & \longrightarrow & W/K_2 & \longrightarrow & 0, \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

where $\mathcal{P}_*^2$, $\mathcal{L}_*^1$ and $\mathcal{L}_*^2$ are the minimal graded projective resolution of K_2, W and W/K_2, respectively. Clearly, $\mathcal{L}_*^1 = \mathcal{P}_*^2 \oplus \mathcal{L}_*^2$. Repeating the above argument and by induction, we finally get the following commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{P}_*^{t-1} & \longrightarrow & \mathcal{L}_*^{t-1} & \longrightarrow & \mathcal{P}_*^t & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K_{t-1} & \longrightarrow & X & \longrightarrow & K_t & \longrightarrow & 0. \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

Therefore, we have $\mathcal{P}_n \cong \bigoplus_{i=1}^t \mathcal{P}_n^i$ for all $n \geq 0$.

6.2. *On the relationship of minimal graded projective resolutions between M and $\mathbf{G}(M)$.*

Lemma 6.7 — Let $0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$ be an exact sequence in $gr(A)$ such that each term has the same highest degree. Then

$$0 \longrightarrow \mathbf{G}(K) \longrightarrow \mathbf{G}(M) \longrightarrow \mathbf{G}(N) \longrightarrow 0$$

is exact if and only if, for all $k \geq 0$, $J^k K = K \cap J^k M$.

Proof : The necessity is proved by the exact sequence

$$0 \longrightarrow J^k K/J^{k+1}K \longrightarrow J^k M/J^{k+1}M \longrightarrow J^k N/J^{k+1}N \longrightarrow 0,$$

which is induced from $0 \longrightarrow \mathbf{G}(K) \longrightarrow \mathbf{G}(M) \longrightarrow \mathbf{G}(N) \longrightarrow 0$.

The sufficiency is proved by the exact sequence

$$0 \longrightarrow J^k K/J^{k+1}K \longrightarrow J^k M/J^{k+1}M \longrightarrow J^k N/J^{k+1}N \longrightarrow 0,$$

which is induced from the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J^{k+1}K & \longrightarrow & J^{k+1}M & \longrightarrow & J^{k+1}N & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & J^k K & \longrightarrow & J^k M & \longrightarrow & J^k N & \longrightarrow & 0. \end{array}$$

***Theorem* 6.8** — *Let M be a weakly (p,λ)-Koszul module and $\mathbf{G}(M)$ be its associated graded module. Let*

$$\cdots \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \longrightarrow 0$$

and

$$\cdots \longrightarrow Q_2 \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow \mathbf{G}(M) \longrightarrow 0$$

be the minimal graded projective resolutions. Then for all $i \geq 0$, we have

$$Q_i \cong \mathbf{G}(P_i)[\delta_\lambda^p(i)].$$

Proof : Let M be a weakly (p,λ)-Koszul module. Then for all $i \geq 0$, we have the following exact sequences

$$0 \longrightarrow \ker d_i \longrightarrow J^{\delta_\lambda^p(i+1)-\delta_\lambda^p(i)} P_i \longrightarrow J^{\delta_\lambda^p(i+1)-\delta_\lambda^p(i)} \ker d_{i-1} \longrightarrow 0,$$

where $\ker d_{-1} = M$. By the definition of weakly (p,λ)-Koszul modules, we have

$$J^k \ker d_i = \ker d_i \cap J^{\delta_\lambda^p(i+1)-\delta_\lambda^p(i)+k} P_i = \ker d_i \cap J^k(J^{\delta_\lambda^p(i+1)-\delta_\lambda^p(i)} P_i).$$

By Lemma 6.7, for all $i \geq 0$, we have the following exact sequences

$$0 \longrightarrow \mathbf{G}(\ker d_i) \longrightarrow \mathbf{G}(J^{\delta_\lambda^p(i+1)-\delta_\lambda^p(i)} P_i) \longrightarrow \mathbf{G}(J^{\delta_\lambda^p(i+1)-\delta_\lambda^p(i)} \ker d_{i-1}) \longrightarrow 0,$$

which imply the following exact sequences

$$0 \longrightarrow \mathbf{G}(\ker d_i)[\delta_\lambda^p(i+1) - \delta_\lambda^p(i)] \longrightarrow \mathbf{G}(P_i) \longrightarrow \mathbf{G}(\ker d_{i-1}) \longrightarrow 0.$$

Putting all the above exact sequences together, we have the following minimal graded projective resolution of $\mathbf{G}(M)$,

$$\cdots \longrightarrow \mathbf{G}(P_1)[\delta_\lambda^p(1)] \longrightarrow \mathbf{G}(P_0)[\delta_\lambda^p(0)] \longrightarrow \mathbf{G}(M) \longrightarrow 0.$$

Thus, we complete the proof since all the minimal projective resolutions of a module are isomorphic. □

6.3 *On the finite generation of $\mathcal{E}(M)$.*

***Theorem* 6.9** — *Let $M \in \mathcal{WK}_\lambda^p(A)$. Then the Koszul dual of M, $\mathcal{E}(M) \in gr_0(E(A))$.*

Proof : First we will show that $\mathcal{E}(M)$ is finitely generated as a graded $E(A)$-module. Suppose that the generators of M lies in the degrees $k_0 < k_1 < \cdots < k_t$ parts. we will do it by induction. If $t = 0$, then M is pure and a (p,λ)-Koszul module. Then obviously $\mathcal{E}(M) \in gr(E(A))$. Assume that the statement holds for less than t. Since M is a weakly (p,λ)-Koszul module, by Theorem 4.7, M admits a chain of graded submodules $0 \subset U_1 \subset U_2 \subset \cdots \subset U_t = M$, such that all U_i/U_{i-1} are (p,λ)-Koszul modules. Consider the following exact sequence, $0 \to U_0 \to M \to M/U_0 \to 0$. From the proof of Lemma 4.5, we get the following exact sequences $0 \to \Omega^n(U_0) \to \Omega^n(M) \to \Omega^n(M/U_0) \to 0$ for all $n \geq 0$, which imply the exact sequences for all $n \geq 0$,

$$0 \to \mathrm{Hom}_A(\Omega^n(U_0), A_0) \to \mathrm{Hom}_A(\Omega^n(M), A_0) \to \mathrm{Hom}_A(\Omega^n(M/U_0), A_0) \to 0.$$

Thus, we have the following exact sequences for all $n \geq 0$

$$0 \to \mathrm{Ext}_A^n(M/U_0, A_0) \to \mathrm{Ext}_A^n(M, A_0) \to \mathrm{Ext}_A^n(U_0, A_0) \to 0.$$

Applying the exact functor "$\bigoplus$" to the exact sequence above exact sequence, we get the following exact sequence

$$0 \to \bigoplus_{n\geq 0} \mathrm{Ext}_A^n(M/U_0, A_0) \to \bigoplus_{n\geq 0} \mathrm{Ext}_A^n(M, A_0) \to \bigoplus_{n\geq 0} \mathrm{Ext}_A^n(U_0, A_0) \to 0.$$

That is, we have the exact sequence $0 \to \mathcal{E}(M/U_0) \to \mathcal{E}(M) \to \mathcal{E}(U_0) \to 0$. It is evident that $\mathcal{E}(U_0) \in gr(E(A))$ and the number of the generating spaces of M/U_0 is less than t, by induction assumption, we have that $\mathcal{E}(M/U_0) \in gr(E(A))$. Thus $\mathcal{E}(M) \in gr(E(A))$.

Now we will show that $\mathcal{E}(M)$ is generated in degree 0 as a graded $E(A)$-module. Without loss of generality, assume that M is a weakly (p,λ)-Koszul module generated by homogeneous generators of degrees $d_0 = 0$ and $d_1 = 1$. Similarly, we have the exact sequence

$$0 \to \mathcal{E}(U_1/U_0) \to \mathcal{E}(U_1) \to \mathcal{E}(U_0) \to 0$$

in $gr(E(A))$. It is trivial that $\mathcal{E}(U_0) \in Gr_0(E(A))$ and $\mathcal{E}(U_1/U_0) \in Gr_0(E(A))$ since U_0 and U_1/U_0 are (p,λ)-Koszul modules.

Therefore $\mathcal{E}(M) \in gr_0(E(A))$. □

6.4 *On the finitistic dimension conjecture.* It is well-known that the *finitistic dimension conjecture*, which states that there should be a bound on the projective dimension of modules of finite projective dimension, i.e.,

$$\sup\{pd_A(M)|pd_A(M) < \infty\} < \infty,$$

which is one of the most intriguing problems for artin algebras. The following lemma suggests that the finitistic dimension conjecture holds in the category $\mathcal{K}_\lambda^p(A)$ in a special case and the proof is immediate from (Theorem 4.5, 4.5 [7]).

Lemma 6.10 — Let A be a finite dimensional (p,λ)-Koszul algebra. Then the finitistic dimension conjecture holds in the category $\mathcal{K}_\lambda^p(A)$.

***Theorem* 6.11** — *Let A be a finite dimensional (p,λ)-Koszul algebra. Then the finitistic dimension conjecture holds in the category $\mathcal{WK}_\lambda^p(A)$.*

PROOF : Observe that $M \in \mathcal{WK}_\lambda^p(A)$, by Theorem 4.7, there exists a chain of graded submodules $0 \subset U_1 \subset U_2 \subset \cdots \subset U_t = M$ such that all

U_i/U_{i-1} are (p,λ)-Koszul modules. Consider the exact sequences $0 \to U_0 \to U_1 \to U_1/U_0 \to 0$, we have $pd_A(M) \leq \max\{\mathrm{pd_A(U_0)},\ \mathrm{pd_A(U_1/U_0)}\}$. By an induction, we have

$$pd_A(M) \leq \max\{\mathrm{pd_A(U_0)},\ \mathrm{pd_A(U_1/U_0)},\ \mathrm{pd_A(U_2/U_1)}, \cdots,\ \mathrm{pd_A(U_p/U_{p-1})}\}.$$

Therefore, $\sup\{pd_A(M)|\ M \in \mathcal{WK}^p_\lambda(A)\} \leq \sup\{\max\{\mathrm{pd_A(U_0)}, \mathrm{pd_A(U_1/U_0)}, \mathrm{pd_A(U_2/U_1)}, \cdots, pd_A(U_p/U_{p-1})\}|U_i/U_{i-1}[-d_i] \in \mathcal{PK}(A)\}$, which is finite by Lemma 6.10. □

Acknowledgment

The author would like to give his thanks to the referee for his/her many valuable suggestions, which improve the quality of the paper a lot. This work was supported by Zhejiang Innovation Project (Grant No. T200905).

References

1. R. Berger, Koszulity for nonquadratic algebras, *J. Alg.*, **239** (2001), 705-734.

2. A. Beilinson, V. Ginszburg and W. Soergel, Koszul duality patterns in representation theory, *J. Amer. Math. Soc.*, **9** (1996), 473-525.

3. S. Brenner, M. C. R. Butler and A. D. King, Periodic algebras which are almost Koszul, *Alg. Represent. Theory*, **5** (2002), 331-367.

4. E. L. Green, E. N. Marcos, R. Martínez-Villa and P. Zhang, D-Koszul algebras, *J. Pure Appl. Alg.*, **193** (2004), 141-162.

5. E. L. Green and E. N. Marcos, δ-Koszul algebras, *Comm. Alg.*, **33** (2005), 1753-1764.

6. E. L. Green and R. Martínez-Villa, Koszul and Yoneda algebras, Representation theory of algebras (Cocoyoc, 1994), CMS Conference Proceedings, *American Mathematical Society*, Providence, RI, **18** (1996), 247-297.

7. E. L. Green, R. Martínez-Villa, I. Reiten, ϕ. Solberg and D. Zacharia, On modules with linear presentations, *J. Alg.*, **205** (1998), 578-604.

8. J.-W. He and D.-M. Lu, Higher Koszul algebras and A-infinity algebras, *J. Alg.*, **293** (2005), 335-362.

9. B. Keller, *A-infinity algebras in representation theory*, Contribution to the proceedings of ICRA IX, Beijing 2000.

10. J.-F. Lü, J.-W. He and D.-M. Lu, Piecewise-Koszul algebras, *Sci. China, Ser. A*, **50** (2007), 1795-1804.

11. J.-F. Lü, On an example of δ-Koszul algebras, *Proc. Amer. Math. Soc.*, (2010), in press.

12. R. Martínez-Villa and D. Zacharia, Approximations with modules having linear resolutions, *J. Alg.*, **266** (2003), 671-697.

13. S. Priddy, Koszul resolutions, *Trans. Amer. Math. Soc.*, **152** (1970), 39-60.

14. A. Polishchuk and L. Positselski, *Quadratic algebras*, University Lectures Series, Vol. 37, American Mathematics Society, Providence, (2005).

15. C. A. Weibel, *An Introduction to Homological Algebra*, Cambridge Studies in Avanced Mathematics, Vol. 38, Cambridge Univ. Press, (1995).