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COCENTRALIZING DERIVATIONS AND NILPOTENT VALUES ON LIE IDEALS

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Let R be a prime ring with $charR \neq 2$, L a non-central Lie ideal of R, d, g non-zero derivations of R, $n \geq 1$ a fixed integer. We prove that if $(d(x)x - xg(x))^n = 0$ for all $x \in L$, then either $d = g = 0$ or R satisfies the standard identity s_4 and d, g are inner derivations, induced respectively by the elements a and b such that $a + b \in Z(R)$.

Key words : Prime rings, Differential identities, Generalized derivations.

1. Introduction

Throughout this paper R will represent a prime associative ring with center $Z(R)$ and extended centroid C (see [2]). For any $x, y \in R$, the symbol $[x, y]$ stands for the commutator $xy - yx$. Recall that an additive mapping

$d : R \to R$ is called derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. The study of derivations of prime rings was initiated by Posner [12]. He proved that if R is a prime ring and d is a nonzero commuting derivation of R, i.e., $[d(x), x] = 0$ for all $x, y \in R$, then R is commutative. A number of authors have extended this theorem in several ways. In [3] Bresar show that if R is prime ring, I is a nonzero left ideal of R, $d \neq 0$ and g are derivations of R such that $d(x)x - xg(x) \in Z(R)$ for all $x \in I$, then R is commutative. Later in [9] Lee and Wong prove that if L is a non-central Lie ideal of R and d, g non-zero derivations of R such that $ud(u) - g(u)u$ is central for all $u \in L$, then R satisfies the standard identity s_4. In [13] Wong extended the previous result to the case when the elements of the Lie ideal are replaced by all the evaluations of a multilinear polynomial. Finally in [10] Lee and Shiue generalize the theorem of Wong for any polynomial (without any assumption on multilinearity). More precisely they show that if d, g are derivations of R and $f(x_1, \ldots, x_n)$ a polynomial in $C\{x_1, \ldots, x_n\}$, which is not central on R, such that $d(f(r_1, \ldots, r_n))f(r_1, \ldots, r_n) - f(r_1, \ldots, r_n)g(f(r_1, \ldots, r_n)) \in C$ for all $r_1, \ldots, r_n \in R$, then either $d = g = 0$ or $d = -g$ and $f(x_1, \ldots, x_n)^2$ is central valued on R, except when $char(R) = 2$ and $dim_C RC = 4$.

Motivated by these results we shall prove the following theorem:

Theorem — *Let R be a prime ring with $charR \neq 2$, L a non-central Lie ideal of R, d, g non-zero derivations of R, $n \geq 1$ a fixed integer. If $(d(x)x - xg(x))^n = 0$ for all $x \in L$, then either $d = g = 0$ or R satisfies s_4 and d, g are inner derivations, induced respectively by the elements a and b such that $a + b \in Z(R)$.*

In all that follows, unless stated otherwise, R will be a prime ring. For any ring S, $Z(S)$ will denote its center. In addition s_4 will denote the the standard identity in 4 variables.

Remark 1 : We will make frequent use of the following result due to Kharchenko [7] (see also [8]) : Let R be a prime ring, d a nonzero derivation of R and I a nonzero two-sided ideal of R. Let $f(x_1, ..., x_n, d(x_1), ..., d(x_n))$ be a differential identity in I, that is

$$f(r_1, ..., r_n, d(r_1), ..., d(r_n)) = 0, \quad \text{for all} \quad r_1, ..., r_n \in I.$$

One of the following holds:

1. Either d is an inner derivation in Q, the Martindale quotient ring of

R, in the sense that there exists $q \in Q$ such that $d(x) = [q, x]$, for all $x \in R$, and I satisfies the generalized polynomial identity

$$f(r_1, ..., r_n, [q, r_1], ..., [q, r_n]) = 0;$$

2. or I satisfies the generalized polynomial identity

$$f(x_1, ..., x_n, y_1, ..., y_n) = 0.$$

2. The Result

We start by studying the 2×2 matrix case, more precisely we prove the following:

Lemma 1 — Let $R = M_2(F)$ be the ring of 2×2 matrices over the field F of characteristic different from 2, $a, b \in R$, $n \geq 1$ a fixed integer such that $([a, [x, y]][x, y] - [x, y][b, [x, y]])^n = 0$ for all $x, y \in R$. Then $a + b \in F$.

Proof : Denote by e_{ij} the usual matrix unit with 1 in the (i, j)-entry and zero elsewhere. Let $a = \sum_{i,j=1}^{2} a_{ij}e_{ij}$ and $b = \sum_{i,j=1}^{2} b_{ij}e_{ij}$, for $a_{ij}, b_{ij} \in F$. If we choose $x = e_{12}, y = e_{21}$ then we get:

$$0 = ([a, [x, y]][x, y] - [x, y][b, [x, y]])^n = (2^n) \cdot \begin{bmatrix} 0 & a_{12} + b_{12} \\ a_{21} + b_{21} & 0 \end{bmatrix}^n.$$

In particular, if $n = 2t$ is even we have

$$(2^n) \cdot \begin{bmatrix} (a_{12} + b_{12})^t (a_{21} + b_{21})^t & 0 \\ 0 & (a_{12} + b_{12})^t (a_{21} + b_{21})^t \end{bmatrix} = 0$$

and, if $n = 2t + 1$ is odd,

$$(2^n) \cdot \begin{bmatrix} 0 & (a_{12} + b_{12})^{t+1} (a_{21} + b_{21})^t \\ (a_{12} + b_{12})^{t+1} (a_{21} + b_{21})^t & 0 \end{bmatrix} = 0.$$

In both cases it follows that at least one of the following holds

1. $a_{12} + b_{12} = 0$;

2. $a_{21} + b_{21} = 0$.

Our first aim is to show that $a+b$ is a diagonal matrix. To do this we assume that $a+b$ is not diagonal and prove that this leads to a contradiction.

Suppose first that $a_{21}+b_{21} \neq 0$, so that $a_{12}+b_{12}=0$. Let φ_{12} be the automorphism of $M_2(F)$ defined as $\varphi_{12}(x)=(1+e_{12})x(1-e_{12})$, for all $x \in M_2(F)$. Denote $c=\varphi_{12}(a+b)=\varphi_{12}(a)+\varphi_{12}(b)=\sum_{i,j=1}^{2} c_{ij}e_{ij}$, for $c_{ij} \in F$. Since

$$([\varphi_{12}(a),[x,y]][x,y]-[x,y][\varphi_{12}(b),[x,y]])^n=0$$

then as above at least one of the following holds

1. $c_{12}=0$;
2. $c_{21}=0$.

By calculations we have that

$$c=\varphi_{12}(a+b)=(a+b)+e_{12}(a+b)-(a+b)e_{12}-e_{12}(a+b)e_{12}=$$

$$\begin{bmatrix} (a_{11}+b_{11})+(a_{21}+b_{21}) & (a_{22}+b_{22})-(a_{11}+b_{11})-(a_{21}+b_{21}) \\ (a_{21}+b_{21}) & (a_{22}+b_{22})-(a_{21}+b_{21}) \end{bmatrix}.$$

Since $c_{21}=(a_{21}+b_{21}) \neq 0$, then $c_{12}=(a_{22}+b_{22})-(a_{11}+b_{11})-(a_{21}+b_{21})=0$.

On the other hand, choose $x=e_{11}, y=e_{21}+e_{12}$. By using the assumption that $a_{12}+b_{12}=0$ and the result that $(a_{22}+b_{22})-(a_{11}+b_{11})=(a_{21}+b_{21})$ we get :

$$0=([a,[x,y]][x,y]-[x,y][b,[x,y]])^n=(a_{21}+b_{21})^n(e_{11}-e_{12}-e_{21}-e_{22})^n$$

Since the latter matrix is invertible one concludes with the contradiction $a_{21}+b_{21}=0$.

A similar argument forces a contradiction, in case $a_{21}+b_{21} \neq 0$ and $a_{12}+b_{12}=0$ are initially assumed. Therefore both $a_{21}+b_{21}=0$ and $a_{12}+b_{12}=0$, that is $a+b$ is a diagonal matrix, as desired.

As above, for any φ automorphism of $M_2(F)$, the following holds

$$([\varphi(a),[x,y]][x,y]-[x,y][\varphi(b),[x,y]])^n=0.$$

This means that $\varphi(a)+\varphi(b)=\varphi(a+b)$ is a diagonal matrix. If we fix $\varphi(x)=(1+e_{12})x(1-e_{12})$, for all $x \in M_2(F)$, then

$$\varphi(a+b)=(a+b)+e_{12}(a+b)-(a+b)e_{12}-e_{12}(a+b)e_{12}=(a+b)+(a_{22}+b_{22}-a_{11}-b_{11})e_{12}$$

and it follows that $a_{11} + b_{11} = a_{22} + b_{22}$, that is $a + b$ is a central matrix.

Corollary 1 — Let $R = M_2(F)$ be the ring of 2×2 matrices over the field F with $char(F) \neq 2$, $n \geq 1$ a fixed integer and $a \in R$ such that $([a, [x, y]][x, y])^n = 0$ for all $x, y \in R$. Then $a \in F$.

Analogously if $([x, y][a, [x, y]])^n = 0$, for all $x, y \in R$, then $a \in F$.

Lemma 2 — Let R be a prime ring with $char(R) \neq 2$, I a two-sided ideal of R, $n \geq 1$ a fixed integer and $a \in R$ such that $([x, y][a, [x, y]])^n = 0$ for all $x, y \in I$. Then $a \in Z(R)$.

PROOF : If R is commutative there is nothing to prove. Suppose that R is not commutative and $a \notin Z(R)$.

Set $G(X, Y) = ([X, Y][a, [X, Y]])^n$. Since $a \notin Z(R)$, then by [4] $G(X, Y)$ is a nontrivial generalized polynomial identity for I. By a theorem due to Beidar (Theorem 2 in [1]), $G(X, Y)$ is a nontrivial generalized polynomial identity for Q. Denote by F either the algebraic closure of C or C according as C is either infinite or finite, respectively. Then, by a standard argument (see, for instance [9], Proposition], $G(X, Y)$ is also a GPI for $Q \otimes_C F$. Since $Q \otimes_C F$ is centrally closed prime $F-$ algebra [[5], Theorem 2.5 and 3.5], by replacing R, C with $Q \otimes_C F, F$, respectively we may assume R is centrally closed over C, which is either finite or algebraically closed. By Martindale's theorem [11] R is a primitive ring having a non-zero socle H with C its associated division ring.

Thus R is isomorphic to a dense ring of linear transformations of a vector space V over C. Consider first the case when $dim_C V \geq 3$. Suppose that there exists $v \in V$ such that v, va are linearly C-independent. Then there exists $w \in V$ such that v, va, w are also linearly C-independent. By the density of R, there exist $x, y \in R$ such that

$$vx = v, \qquad vy = v$$

$$vax = va, \qquad vay = w$$

$$wx = 0, \qquad wy = v.$$

Therefore we get the following contradiction

$$0 = w([x, y][a, [x, y]])^n = (-1)^n w \neq 0.$$

Thus $\{v, va\}$ are linearly C-dependent for all $v \in V$, that is a is central in R.

If $dim_C V = 2$, then $R \cong M_2(C)$ and, by corollary 1, again we have $a \in C$.

Symmetrically the following holds:

Lemma 3 — Let R be a prime ring with $char(R) \neq 2$, I a two-sided ideal of R, $n \geq 1$ a fixed integer and $a \in R$ such that $([a, [x, y]][x, y])^n = 0$ for all $x, y \in I$. Then $a \in Z(R)$.

Proposition 1 — Let R be a prime ring with $char(R) \neq 2$, I a two-sided ideal of R, $n \geq 1$ a fixed integer and $a, b \in R$ such that $([a, [x, y]][x, y] - [x, y][b, [x, y]])^n = 0$ for all $x, y \in I$. Then either a and b fall in $Z(R)$ or R satisfies s_4 and $a + b \in Z(R)$.

PROOF : If either $a \in Z(R)$ or $b \in Z(R)$, the conclusion follows respectively from Lemma 2 and Lemma 3.

Thus consider the case when both a and b are non-central elements of R. By the hypothesis we have

$$([a, [x, y]][x, y] - [x, y][b, [x, y]])^n =$$

$$([[a, x], y][x, y] + [x, [a, y]][x, y] - [x, y][[b, x], y] - [x, y][x, [b, y]])^n = 0$$

for all $x, y \in I$. Since by Theorem 2 in [1] I and Q satisfy the same generalized identities we have

$$([[a, x], y][x, y] + [x, [a, y]][x, y] - [x, y][[b, x], y] - [x, y][x, [b, y]])^n = 0$$

for all $x, y \in Q$. In particular this one holds in R and so R satisfies a nontrivial generalized polynomial identity, because $a, b \notin Z(R)$. Moreover, since Q remains prime by the primeness of R , replacing R by Q we may assume that the extended centroid $C = Z(Q)$ is just the center of R. Note that R is centrally closed $C-$ algebra in the present situation [5],i.e., $RC = R$. As remarked in Lemma 2, by Martindale's theorem in [11], we may assume that R is a primitive ring which is isomorphic to a dense ring of linear transformations of a vector space V over C. Denote

$$K^n = ([[a, x], y][x, y] + [x, [a, y]][x, y] - [x, y][[b, x], y] - [x, y][x, [b, y]])^n = 0.$$

Consider first the case when $dim_C V \geq 3$. Suppose that there exists $v \in V$ such that v, va are linearly $C-$independent. Then there exists $w \in V$ such that v, va, w are also linearly C-independent. By the density of R, there exist $x, y \in R$ such that

$$vx = v, \qquad vy = v$$

$$vax = va, \qquad vay = w$$

$$wx = 0, \qquad wy = v.$$

These imply that $0 = vK^n = (-1)^n v \neq 0$, which is a contradiction so we conclude that v, va are linearly $C-$ dependent, for all $v \in V$ and, as above, standard arguments show that $a \in C$. By the hypothesis we get $([x, y][b, [x, y]])^n = 0$ for all $x, y \in R$ so Lemma 2 forces b to be central.

Finally consider the case when $dim_C V = 2$, then $R \cong M_2(C)$ and, by Lemma 1, we have $a + b \in C$.

Now we are ready to give the proof of our main result:

2.1 The proof of Theorem. Since we assume that $char(R) \neq 2$, it follows from Herstein [6] that $L \supseteq [I, R]$ for some $I \neq 0$, an ideal of R, and also L is not commutative. Therefore we will assume throughout $L \supseteq [I, R]$. Without loss of generality we can assume $L = [I, I]$. Then we have

$$(d[x, y][x, y] - [x, y]g[x, y])^n = 0 \qquad \text{for all} \quad x, y \in I. \qquad (1)$$

Case 1: Let d and g be $Q-$ inner, namely, $d = ad(a)$ and $g = ad(b)$ for some $a, b \in Q$. In this case the conclusion follows from Proposition 1.

Case 2 : Let $Der(Q)$ be the set of all derivations on Q. Now we denote by D_{int} the C-subspace of $Der(Q)$ consisting of all inner derivations on Q. Suppose d and g are $C-$independent module D_{int}. From (1) we have

$$0 = ([d(x), y][x, y] + [x, d(y)][x, y] - [x, y][g(x), y] - [x, y][x, g(y)])^n.$$

By Kharchenko's theorem I satisfies the polynomial identity

$$([z, y][x, y] + [x, w][x, y] - [x, y][u, y] - [x, y][x, v])^n.$$

Since I and R satisfy the same polynomial identities, then for $w = z = v = 0$ and $u = x$, R satisfies $[u, y]^{2n}$. In this case it is well known that there exists a field F such that R and F_m, the ring of all $m \times m$ matrices over F, satisfy the same identities. In particular F_m satisfies $[u, y]^{2n} = 0$. Suppose $m \geq 2$ and choose $u = e_{12}, y = e_{21}$. This leads to the contradiction $0 = [e_{12}, e_{21}]^{2n} = e_{11} + e_{22}$. Thus $m = 1$ and R is commutative, a contradiction again, since L is not central.

Case 3: Suppose d and g are C-dependent module D_{int}. Then $g = \beta d + f_b$ for some $\beta \in C$ and f_b the inner derivation induced by the element $b \in Q$. Notice that if d is inner,then g is also and we are finished by Case 1.

Therefore consider the case when d is not inner and $\beta \neq 0$. Then $([d(x), y][x, y] + [x, d(y)[x, y] - [x, y][\beta d(x) + [b, x], y] - [x, y][x, \beta d(y) + [b, y])^n = 0$ and so the Kharchenko's theorem provides that I satisfies $([w, y][x, y] + [x, z][x, y] - [x, y][\beta w + [b, x], y] - [x, y][x, \beta z + [b, y])^n = 0$. Setting $w = z = 0$, we obtain

$$([x, y][[b, x], y] + [x, y][x, [b, y]])^n = 0 \text{ for all } x, y \in I.$$

As remarked in Lemma 2, by Theorem 2 in [1] I and Q satisfy the same generalized polynomial identities. Thus $([x, y][f_b(x), y] + [x, y][x, f_b(y)])^n = ([x, y]f_b[x, y])^n = 0$ for all $x, y \in Q$, then, by Lemma 2, $b \in C$.

This implies that $g = \beta d$ and by the main assumption we have

$$0 = (d[x, y][x, y] - \beta[x, y]d[x, y])^n =$$

$$(([d(x), y] + [x, d(y)])[x, y] - \beta[x, y]([d(x), y] + [x, d(y)]))^n .$$

Since d is not inner, again by Kharchenko's result we have that Q satisfies the following polynomial identity

$$(([u, y] + [x, v])[x, y] - \beta[x, y]([u, y] + [x, v]))^n . \qquad (2)$$

There exists a field F such that Q and $M_k(F)$, the ring of all $k \times k$ matrices over F, satisfy the same polynomial identities. In particular $M_k(F)$ satisfies (2). Denote by e_{ij} the usual matrix unit and suppose $k \geq 2$. Fix $v = 0$, $x = e_{21}$, $y = e_{11}$, $u = e_{12} + e_{21}$. Thus one has the following contradiction:

$$0 = ([u, y][x, y] - \beta[x, y][u, y])^n =$$

$$([e_{12} + e_{21}, e_{11}][e_{21}, e_{11}] - \beta[e_{21}, e_{11}][e_{12} + e_{21}, e_{11}])^n =$$

$$(-1)^n e_{11} + \beta^n e_{22} \neq 0.$$

Therefore $k = 1$ and Q is commutative, as well as R, which contradicts the fact that L is non-central.

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