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# COMPLETE ORDER AMENABILITY OF THE FOURIER ALGEBRA

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We define complete order amenability and first complete order cohomology groups for quantized Banach ordered algebras and show that the vanishing of the latter is equivalent to the operator amenability for the Fourier algebra.

**Key words :** Banach ordered algebras, Fourier algebra, operator amenability, complete order amenability, order tensor product

## 1. Introduction and Preliminaries

Operator spaces were introduced in the mid 1970's after the pioneering works of Stinespring and Arveson, and were presented to the mathematical community by Effros in his address to the ICM in 1986 [6]. The theory grew out of the analysis of completely positive and completely bounded mappings. It was shown by Effros and Choi that some properties shared by completely positive mappings can be taken over to the framework of operator systems [3]. One main idea behind [3] was to provide an abstract description of the order structure of self-adjoint unital subspaces of $C^*$-algebras [27]. Both the operator space and order structure of a $C^*$-algebra $A$ are naturally lifted to the dual space $A^*$. The situation is more interesting when the dual space $A^*$ is a Banach algebra. A typical example is the commutative $C^*$-algebra $A = C_0(G)$ of continuous functions vanishing at infinity on a locally compact group $G$ where $A^* = M(G)$ is the Banach algebra of regular bounded Borel measures on $G$ under the convolution product. A less trivial example (both in order and operator space structures) is provided by the group $C^*$-algebra $A = C^*(G)$, which is the enveloping $C^*$-algebra of the Banach algebra $L^1(G)$ of absolutely integrable (with respect to Haar measure) Borel functions on $G$. In this case, $A^* = B(G)$ is the Fourier-Stieltjes algebra. The Fourier algebra $A(G)$ and Fourier-Stieltjes algebra $B(G)$, introduced by Eymard in the 1960's [9], are commutative Banach algebras which have a natural order structure (given by the cone of positive-definite functions) and a quite relevant operator space structure. The last assertion is best exemplified by the celebrated result of Ruan which shows that $A(G)$ is operator amenable if and only if $G$ is amenable [23]. The significance of this result is best understood if it is contrasted to a later result of Forrest and Runde [11] asserting that $A(G)$ is amenable if and only if $G$ is abelian by finite (that is $G$ has an abelian subgroup of finite index). Note that by Leptin's theorem [15], the amenability of $G$ is equivalent to the weaker condition of existence of a bounded approximate identity in $A(G)$. There are also important facts which show the significance of the order structure of the Fourier algebra. For instance, it is shown by Arendt, Cannière that for locally compact groups $G_1$ and $G_2$, the Fourier algebras $A(G_1)$ and $A(G_2)$ are order isomorphic (in both positive-definite and pointwise orders) if and only if $G_1$ and $G_2$ are isomorphic and homeomorphic, whereas a similar result about usual Banach algebra isomorphism fails [1].

This paper discusses the role of the natural order in operator spaces and its relation to the topological structure of these spaces, especially in amenability theory. In particular, we introduce the concept of complete order amenability for a class of Banach algebras with order and show that it is equivalent to operator amenability for the Fourier algebra $A(G)$ of a locally compact group $G$. Unfortunately our theory does not work very well on $C^*$-algebras, because multiplication of two positive elements in a $C^*$-algebra is not in general positive, unless they commute.

The paper is organized as follows. Section 1 is dedicated to preliminaries, where in subsection 1.1 we introduce basic definitions and notations. Some of the concepts, such as (quantized) Banach ordered spaces (algebras) are introduced in this paper for the first time. In Section 2 we shall discuss complete order maps between operator spaces. In Section 3 we define (complete) order amenability and (complete) order tensor product for a class of (quantized) Banach ordered algebras.

There is a vast literature on operator spaces and operator algebras. We refer the reader to [8], [19] and [27]. More details about amenability could be found in [14], [17], and [24]. For the basic properties of the Fourier algebra, the reader is referred to [9] and [21].

1.1 *Basic definitions and notations*
Let $n \in \mathbb{N}$ and $E$ be a vector space. We denote the vector space of $n \times n$-matrices with entries from $E$ by $\mathbb{M}_n(E)$ and we put simply $\mathbb{M}_n := \mathbb{M}_n(\mathbb{C})$. The space $\mathbb{M}_n$ is equipped with the operator norm $|\cdot|_n$ from its canonical action on the $n$-dimensional Hilbert space $\mathbb{C}^n$. Clearly $\mathbb{M}_n$ acts on $\mathbb{M}_n(E)$ by matrix multiplication.

For normed spaces $E$ and $F$ the space of all bounded linear maps from $E$ to $F$ is denoted by $\mathcal{B}(E, F)$. An $L^\infty$-*matricial norm* on $E$ is a family $(\|\cdot\|_n)_{n=1}^{\infty}$ such that, for each $n \in \mathbb{N}$, $\|\cdot\|_n$ is a norm on $\mathbb{M}_n(E)$ satisfying

$$\|\lambda \cdot x \cdot \mu\|_n \leq |\lambda|_n \|x\|_n |\mu|_n, \quad \|x \oplus y\|_{n+m} = \max\{\|x\|_n, \|y\|_m\}$$

for $\lambda, \mu \in \mathbb{M}_n, x \in \mathbb{M}_n(E), y \in \mathbb{M}_m(E)$. The vector space $E$ equipped with an $L^\infty$-matricial norm $(\|\cdot\|_n)_{n=1}^{\infty}$ is called an $L^\infty$-*matricial normed space*. If moreover each space $(\mathbb{M}_n(E), \|\cdot\|_n)$ is a Banach space, $E$ is called an (abstract) *operator space*. The dual spaces of $C^*$-algebras and predual spaces of von Neumann algebras can be naturally equipped with an operator space structure [8].

Let $E$ and $F$ be operator spaces, and let $T : E \to F$ be a linear map. Then

$$T^{(n)} : \mathbb{M}_n(E) \to \mathbb{M}_n(F), \quad T^{(n)}([x_{ij}]) = [T(x_{ij})]$$

is the $n$th *amplification* of $T$. The map $T$ is said to be *completely bounded* if $\|T\|_{cb} := \sup \|T^{(n)}\| < \infty$, and a *complete contraction* if $\|T\|_{cb} \leq 1$. When each $T^{(n)}$ is an isometry, we say that $T$ is a *complete isometry*. The collection of all completely bounded maps from $E$ to $F$ is denoted by $\mathcal{CB}(E, F)$. This is a Banach space under $\|\cdot\|_{cb}$ and, for each $n \in \mathbb{N}$, we have the canonical identification $\mathbb{M}_n(\mathcal{CB}(E,F)) = \mathcal{CB}(E,\mathbb{M}_n(F))$. When $E$ and $F$ are dual operator spaces, $\mathcal{CB}^\sigma(E,F)$ denotes the space of all $w^*$-continuous, completely bounded linear maps from $E$ to $F$.

We use the notations of [8] for tensor products, in particular, $E \otimes F$, $E \otimes^\gamma F$, and $E \widehat{\otimes}^{op} F$ denote the algebraic, projective, and operator projective tensor products. Also $E \bar{\otimes} F$ is used to denote the complete von Neumann algebra tensor product. For a normed space $X$, $X_1$ is the unit ball of $X$. When $X$ is a von Neumann algebra, its (unique) predual is denoted by $X_*$. Also we use the notations $\cong^{iso}$, $\cong^{iso}_{iso}$, and $\cong^{c.iso}$ for isomorphism, isometric isomorphism, and complete isometry, respectively.

If $E$ is a real vector space, then $E_{\mathbb{C}} = E + iE$ is the complexification of $E$ [26]. Each $\mathbb{R}$-linear map $T : E \to F$ has a unique $\mathbb{C}$-linear extension $\tilde{T} : E_{\mathbb{C}} \to F_{\mathbb{C}}$. A real vector space $E$ endowed with an order relation $\leq$ is called a *real ordered space* if, given $x, y \in E$, the relation $x \leq y$ implies $x + z \leq y + z$ and $\alpha x \leq \alpha y$, for all $z \in E$ and $\alpha \in \mathbb{R}_+$. The *positive cone* of $E$ is $E_+ = \{x : 0 \leq x\}$. We say that $E_+$ is *proper* if $E_+ \cap (-E_+) = \{0\}$. The complexification of a real ordered vector space is called an *ordered space*. Given ordered spaces $E_{\mathbb{C}}$ and $F_{\mathbb{C}}$, a $\mathbb{C}$-linear map $T : E_{\mathbb{C}} \to F_{\mathbb{C}}$ is *positive* if $T(E_+) \subseteq F_+$. An *involution* on a complex vector space $E$ is a conjugate linear map $* : E \to E$, $v \mapsto v^*$, such that $v^{**} = v$. A complex vector space $E$ endowed with an involution is called an *$*$-space*. An $*$-space $E$ which is also an ordered space with positive proper cone $E_+$ is called an *ordered $*$-space* if $E_+ \subseteq E_{sa}$, where $E_{sa}$ is the self-adjoint part of $E$. For an ordered $*$-space $E$, an element $e \in E_+$ is called an *order unit* if for each $v \in E_{sa}$, there is a real number $t > 0$, such that $-te \leq v \leq te$.

A (complex) Banach space $A$ is called a *Banach ordered space* if it is the complexification of a real ordered space $B$ which is also a real Banach space, such that:

(i) the inclusion map $i : B \to A$ is an isometry,

(ii) each element $a \in A$ can be written as $a = a_1 - a_2 + i(a_3 - a_4)$, where $a_1, ..., a_4$ are positive in $A$ and $\|a_j\| \leq \|a\|$, for $j = 1, ..., 4$.

For example, Banach lattices, $C^*$-algebras and their duals are Banach ordered spaces.

A linear map $f : A \to B$ between ordered spaces $A$ and $B$ is called an *order map* if it can be decomposed as $f = f_1 - f_2 + i(f_3 - f_4)$, where $f_1, ..., f_4$ are positive linear maps from $A$ to $B$. An *order isomorphism* between ordered spaces is an isomorphism which preserves the order structure of these spaces.

Each operator space $E$ can be considered as a norm-closed subspace of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ [8, Theorem 2.3.5]. Then $\mathbb{M}_n(E)$ has the order determined by the cone $\mathbb{M}_n(E)_+ = \mathbb{M}_n(E) \cap \mathcal{B}(\mathcal{H}^n)_+$. This is not an intrinsic order and it depends on the representation of $E$ in $\mathcal{B}(\mathcal{H})$, but since operator spaces are commonly given concretely, the order structure is usually transparent in the context. A *concrete operator system* on a Hilbert space $\mathcal{H}$ is a self-adjoint, norm-closed linear subspace $\mathcal{F} \subseteq \mathcal{B}(\mathcal{H})$, containing the identity operator. Operator systems can be characterized as $*$-spaces which have complete matrix norm and matrix ordering and order unit [3, Theorem 4.4]. The dual space of a $C^*$-algebra has complete matrix norm and matrix ordering [25, Corollary 3.2]. For more details about matrix ordering, see [27].

A linear map $\varphi : E_1 \to E_2$ between (concrete) operator spaces $E_1$ and $E_2$ is called *completely positive* if $\varphi^{(n)}$ is positive for all $n \in \mathbb{N}$. An order map $f : E \to F$ between operator spaces is called a *complete order map* if it can be written as $f = f_1 - f_2 + i(f_3 - f_4)$, where $f_1, ..., f_4$ are completely positive linear maps. We use the notation $L_{c.or}(E_1, E_2)$ for the space of all linear maps $T : E_1 \to E_2$ such that $T = T_1 - T_2 + i(T_3 - T_4)$, where $T_i$'s are completely bounded, completely positive linear maps.

The operator projective tensor product of two operator spaces [2] could be considered as an ordered space. In general there is no straightforward description of the positive cone of this space, but we know that for operator spaces $E$ and $F$, $(E \widehat{\otimes}^{op} F)^* \cong^{c.iso} \mathcal{CB}(E, F^*)$. The order structure that we use on $(E \widehat{\otimes}^{op} F)^*$ comes from the positive cone of the operator space $\mathcal{CB}(E, F^*)$. Note that there are two order structures on $\mathbb{M}_n(\mathcal{CB}(E, F^*))$. The first order comes from the trace pairing. A matrix $[f_{ij}] \in \mathbb{M}_n(\mathcal{CB}(E, F^*))$ is positive in this order if it is a positive map from $\mathbb{M}_n(E)$ to $F^*$, that is $\langle [f_{ij}], [x_{ij}] \rangle = \sum_{i,j} f_{ij}(x_{ij}) \geq 0$ in $F^*$, for each $[x_{ij}] \in \mathbb{M}_n(E)_+$. As seen on [8, page 6], in some respects this order structure is not natural for operator spaces. The second order comes from the identification $\mathbb{M}_n(\mathcal{CB}(E, F^*)) \cong^{iso}_{iso} \mathcal{CB}(E, \mathbb{M}_n(F^*))$. An element in $\mathbb{M}_n(\mathcal{CB}(E, F^*))$ is positive in this order, if it identifies with a completely positive and completely bounded map from $E$ to

$\mathbb{M}_n(F^*)$, where positive elements of $\mathbb{M}_n(F^*) \cong_{iso}^{iso} \mathcal{CB}(F, \mathbb{M}_n)$ are completely positive and completely bounded maps from $F$ to $\mathbb{M}_n$. In this paper, we adapt the second order structure, unless otherwise specified. In particular, a typical element $u = e \otimes f \in E \otimes F$ is positive if for each completely bounded and completely positive linear map $T : E \to F^*$, $T(e)(f) \geq 0$. We know that each $u \in E \otimes F$ can be written as $u = \alpha \cdot (x \otimes y) \cdot \beta$, where $x \in \mathbb{M}_n(E)$ and $y \in \mathbb{M}_m(F)$, $\alpha \in \mathbb{M}_{1,nm}$, $\beta \in \mathbb{M}_{nm,1}$, for some $m, n \in \mathbb{N}$ and $x \otimes y = [x_{ij} \otimes y_{kl}]_{mn}$. Therefore, for $\alpha \in \mathbb{M}_{1,n^2}$ and $x, y \in \mathbb{M}_n(E)_+$, $\alpha \cdot (x \otimes y) \cdot \alpha^*$ is a positive element in $E \otimes F$, where $\alpha^*$ is the conjugate transpose of $\alpha$. Finally, since there is a complete isometry from $\mathcal{CB}(E, F^*)$ onto $\mathcal{CB}(F, E^*)$ which identifies the corresponding positive cones, the order structures coming from $\mathcal{CB}(F, E^*)$ or $\mathcal{CB}(F, E^*)$ would be the same on $E \widehat{\otimes}^{op} F$.

An operator space $E$ is called a *matricial Banach ordered space* if for all $n \in \mathbb{N}$, $\mathbb{M}_n(E)$, is a Banach ordered space. A Banach algebra $A$ is called a *Banach ordered algebra* if it is a Banach ordered space such that $A_+ A_+ \subseteq A_+$. For example, commutative $C^*$-algebras, Fourier and Fourier-Stieltjes algebras are examples of Banach ordered algebras. A quantized algebra $A$ is a *quantized Banach ordered algebra* if it is a matricial Banach ordered space such that the diagonal map $\Delta : A \widehat{\otimes}^{op} A \longrightarrow A$; $a \otimes b \mapsto ab$, is a completely positive linear map.

For a normed space $E$ with an order and positive cone $E_+$, the natural order structure on the dual space $E^*$ is given by the cone $E^*_+ = \{f \in E^* : f(p) \geq 0 \ (p \in E_+)\}$. Note that the dual cone $E^*_+$ is always $w^*$-closed and hence norm-closed. In general there is a characterization of closed positive cones in normed spaces with order [25]. Indeed, for a normed space $E$ with an order the positive cone $E_+$ is norm-closed if and only if $x \in E_+$ whenever $x \in E$ and $f(x) \geq 0$, for all $f \in E^*_+$. Moreover, the norm closure of $E_+$ is given by $\overline{E_+} = E^{**}_+ \cap E$. We note that the positive cones of $C^*$-algebras and Banach lattices are closed. Throughout this paper, all real ordered spaces are assumed to have a *proper* positive cone $P$, namely $P \cap -P = \{0\}$.

1.2 *Operator and order structure of the Fourier algebra*

Let $G$ be a locally compact group. The *Fourier algebra* $A(G)$ consists of all coefficient functions of the left regular representation $\lambda$ of $G$,

$$A(G) = \{w = (\lambda\xi, \eta) \ : \ \xi, \eta \in L^2(G)\}.$$

This is a regular, commutative Banach algebra with pointwise multiplication and the norm $\|w\| = \inf\{\|\xi\|\|\eta\| : w = (\lambda\xi, \eta)\}$. As the predual of the (left) group von Neumann algebra $VN(G)$ [9], $A(G)$ has the canonical operator space structure given by

$$\|v\|_n = \sup\{[\langle v_{ij}, w_{kl}\rangle] : w \in \mathbb{M}_n(VN(G)), \|w\|_n \leq 1\}$$

for $n \in \mathbb{N}$ and $v \in \mathbb{M}_n(A(G))$. Moreover, $\mathcal{CB}^\sigma(VN(G), \mathbb{M}_n) \cong^{iso}_{iso} \mathbb{M}_n(A(G))$. The Fourier algebra $A(G)$ is a closed ideal of the Fourier-Stieltjes algebra $B(G)$, which is an operator space as the dual space of the group $C^*$-algebra $C^*(G)$ [9]. For more details on the operator space structure of $A(G)$, see [12].

The matrix order structure of $B(G)$ comes from its operator space structure [25, Corollary 3.2]. The positive cone of $B(G)$ is the set $P(G)$ of all continuous, positive-definite functions on $G$. Similarly the positive cone of the Fourier algebra $A(G)$ is $P(G) \cap A(G)$. It is easy to see that $P(G) \cap A(G)$ is a (closed) proper cone for $A(G)$, and hence the Fourier algebra with this order structure is an ordered space. To see that we have imposed a natural order structure on $A(G)$, one could observe that for a locally compact abelian group $G$, $A(G) \cong^{iso}_{iso} L^1(\widehat{G})$ through the Fourier transform [10]. If we consider $L^1(\widehat{G})$ with its pointwise order, then by a corollary of Bochner's theorem [10, Corollary 4.23], the above isomorphism (given by the Fourier transform) preserves order structures. A similar observation applies to $B(G)$, when the canonical order of the measure algebra $M(\widehat{G})$ is transferred to $B(G)$ via the Fourier–Stieltjes transform [10, Theorem 4.18].

Let $G$ be a locally compact group and for a function $f : G \to \mathbb{C}$ let $\check{f} : G \to \mathbb{C}$ be defined by $\check{f}(x) = f(x^{-1})$. We recall that, by [9, Theorem 3.10], for every $T \in VN(G)$ there exists a unique $\varphi_T \in A(G)^*$ such that $\varphi_T(f * g)\check{} = \varphi_T(\bar{g} * \check{f}) = (Tf, g)$ for all $f, g \in L^2(G)$ and, moreover, $T \mapsto \varphi_T$ defines an isometric isomorphism from $VN(G)$ onto $A(G)^*$. It is now easy to see that for each $T \in VN(G)$, $\varphi_T \in A(G)^*_+$ if and only if $T \in VN(G)_+$, that is the positive cone of the dual space $A(G)^*$ coincides with the positive cone of $VN(G)$ as a von Neumann algebra. In particular, given $u \in A(G)$, we have $u \in A(G)_+$ if and only if $\langle T, u\rangle \geq 0$, for each $T \in VN(G)_+$. The involution on $A(G)$ is given by $f \mapsto f^*$, where $f^*(x) = \overline{f(x^{-1})}$ for $x \in G$. This can be compared to the involution on $L^1(G)$, given by $f^*(x) = \triangle(x^{-1})\overline{f(x^{-1})}$, for $x \in G$ where $\triangle$ is the modular function of $G$. Both $A(G)$ and $L^1(G)$ are ordered $*$-spaces. When $G$ is abelian, the Fourier transform from $L^1(G)$ to $A(\widehat{G})$ preserves the $*$-operation.

For locally compact groups $G$ and $H$, we have

$$\mathcal{B}(A(G), VN(H)) \cong_{iso}^{iso} (A(G) \otimes^{\gamma} A(H))^*.$$

Note that the right-hand side is not the same as $VN(G\times H)$, in general. On the other hand, we have $A(G)\widehat{\otimes}^{op}A(H) \cong^{c.iso} A(G \times H)$ [8, Theorem 7.2.4], and there is a bounded contraction $\Psi : A(G) \otimes^{\gamma} A(H) \to A(G)\widehat{\otimes}^{op}A(H)$ with dense range, given by $\Psi(f \otimes g)(x, y) = f(x)g(y)$, $f \in A(G)$, $g \in A(H)$, $x \in G$, $y \in H$ [16].

## 2. Decomposition of Complete Order Maps

In this section we study (complete) boundedness of (complete) order maps on ordered (operator) spaces. This is motivated by the fact that positive maps between Banach lattices are continuous [26, Theorem II.5.3]. We also discuss the decomposition of maps on ordered spaces into linear combinations of positive maps. We know that each bounded linear functional on a Banach lattice is a linear combination of positive linear maps [26, Propositions 1.4, 5.5]. Similar decompositions for bounded linear maps from a $C^*$-algebra to a von Neumann algebra have been discussed in [13].

### 2.1 *Complete order maps*

The following proposition is a known result in Banach lattice theory, which is reformulated in the Banach ordered spaces setting.

*Proposition* 2.1.1 — Let $E$ be a Banach ordered space with a closed positive cone $E_+$, and let $F$ be a normed space equipped with an order satisfying $\|x\| \leq \|y\|$, for all $x, y \in F$ with $0 \leq x \leq y$. Then every positive linear map from $E$ into $F$ is continuous.

Proof : Assume towards a contradiction that $T : E \to F$ is a positive linear map that is not continuous. Then $T$ must be unbounded on the unit ball $U$ of $E$, and hence on $U_+ := U \cap E_+$, since $U \subseteq U_+ - U_+ + i(U_+ - U_+)$. This implies that there exists a sequence $(x_n)_{n=1}^{\infty}$ in $U_+$ such that $\|Tx_n\| \geq n^3$, for each $n \in \mathbb{N}$. Since $E_+$ is closed, $z := \sum x_n/n^2$ is in $E_+$. Therefore $Tz \geq Tx_n/n^2 \geq 0$, for each $n \in \mathbb{N}$. Hence $\|Tz\| \geq n$, for each $n \in \mathbb{N}$. This is a contradiction.

Note that all Banach lattices, $C^*$-algebras and their dual spaces satisfy the condition mentioned in the above proposition for $F$. Concrete examples include $A(G)$ and $VN(G)$, for a locally compact group $G$.

*Corollary* 2.1.2 — Let $X$ and $Y$ be measure spaces such that $Y$ is $\sigma$-finite. Then each linear map $T : L^1(X) \to L^\infty(Y)$ is an order map if and only if it is completely bounded.

PROOF : Sufficiency follows from [26, IV, Theorem 1.5]. For the necessity, we may suppose that $T$ is a positive linear map. Then $T$ is a bounded linear map by Proposition 2.1.1. Hence $T$ is completely bounded, as $L^\infty(Y)$, being a commutative $C^*$-algebra, carries a minimal operator space structure.

*Proposition* 2.1.3 — Let $G$ be a locally compact group.

(i) If $G$ is abelian by finite then each order map from $A(G)$ into $VN(G)$ is completely bounded.

(ii) Each bounded linear map $T : A(G) \to VN(G)$ is completely bounded with $\|T\| = \|T\|_{cb}$ if and only if $G$ is abelian.

PROOF : (i) Let $T : A(G) \longrightarrow VN(G)$ be an order map. Without loss of generality we may suppose that $T$ is positive, and so by Proposition 2.1.1 it is bounded. Thus $T$ is completely bounded by [12, Theorem 4.5].

(ii) The sufficiency is clear. For the necessity, consider $T$ as a bounded linear map from $A(G)$ into $VN(G)$. Hence $T$ is completely bounded with $\|T\| = \|T\|_{cb}$ by assumption. This means that the identity map $i : \mathcal{CB}(A(G), VN(G)) \longrightarrow \mathcal{B}(A(G), VN(G))$ is a surjective isometry. A simple verification shows that $\Psi^* = i$. Hence $\Psi$ is also a surjective isometry. Therefore the maximal dimension of irreducible representations of $G$ is 1, by [16, Theorem 1]. Therefore $G$ is abelian.

***Theorem* 2.1.4** — *Let $\mathcal{M}$ and $\mathcal{N}$ be von Neumann algebras. Then each completely bounded linear map from $\mathcal{M}_*$ to $\mathcal{N}$ is a linear combination of completely bounded completely positive linear maps.*

PROOF : We first note that $\mathcal{CB}(\mathcal{M}_*, \mathcal{N}) \cong^{c.iso} (\mathcal{M}_* \widehat{\otimes}^{op} \mathcal{N}_*)^* \cong^{c.iso} \mathcal{M} \bar{\otimes} \mathcal{N}$ [8, Theorems 7.2.4, 7.1.2]. Since $\mathcal{M} \bar{\otimes} \mathcal{N}$ is a von Neumann algebra, each completely bounded linear map $T : \mathcal{M}_* \longrightarrow \mathcal{N}$ is a linear combination of positive elements (as an element of $\mathcal{M} \bar{\otimes} \mathcal{N}$). Let us observe that the above identifications send positive elements of $\mathcal{M}\bar{\otimes}\mathcal{N}$ to completely bounded and completely positive maps from $\mathcal{M}_*$ to $\mathcal{N}$. Since the positive cone of the von Neumann algebra tensor product is the norm closure of the positive cone of $\mathcal{M}\otimes\mathcal{N}$ as its $*$-algebra structure, we need only to check this for a typical positive element $\Lambda = \sum_{r=1}^{s} \Phi_r \otimes \Psi_r$ in the algebraic tensor product $\mathcal{M}\otimes\mathcal{N}$ (see [22, Definition 3.2.4]). Consider positive matrices $f = [f_{ij}]$ and $g = [g_{kl}]$ in $\mathbb{M}_n(\mathcal{M}_*)_+$ and $\mathbb{M}_m(\mathcal{N}_*)_+$, respectively, then we need to show that the scalar matrix $[\sum_r \Phi_r(f_{ij})\Psi_r(g_{kl})]$ is positive. Now note that $f\otimes g \in \mathbb{M}_{mn}\left((\mathcal{M}_*)_+ \otimes (\mathcal{N}_*)_+\right)$ and $\mathbb{M}_{mn}(\Lambda) \in \mathbb{M}_n(\mathcal{M})_+ \otimes \mathbb{M}_m(\mathcal{N})_+$ and the above scalar matrix is canonically identified with the positive matrix $M_{mn}(\Lambda)(f\otimes g)$. Now by the closedness of the positive cones and complete isomorphism between them claim will be proved. Hence $T$ is a linear combination of completely bounded completely positive linear maps from $\mathcal{M}_*$ to $\mathcal{N}$.

*Corollary* 2.1.5 — Let $G$ and $H$ be locally compact groups. Then:

(i) each completely bounded map from $M(G)$ to $C_0(H)^{**}$ is a linear combination of completely bounded completely positive linear maps;

(ii) each completely bounded map from $B(G)$ to $B(H)^*$ is a linear combination of completely bounded completely positive linear maps;

(iii) each completely bounded map from $A(G)$ to $VN(H)$ is a linear combination of completely bounded completely positive linear maps;

***Theorem* 2.1.6** — *If one of the locally compact groups $G$ and $H$ is abelian by finite, then each bounded linear map $T$ from $A(G)$ to $VN(H)$ is a complete order map.*

PROOF : It is enough to recall that in this case, $A(G) \otimes^{\gamma} A(H) \cong^{iso} A(G)\widehat{\otimes}^{op} A(H) \cong^{c.iso} A(G \times H)$ [16, Theorem 1], [8, Theorem 7.2.4]. Note that the first isomorphism between two Banach spaces is not necessarily an isometry, whereas the second one is a complete isometry between two operator spaces. Now the result follows by going to the dual spaces.

*Proposition* 2.1.7 — Let $G$ and $H$ be locally compact groups, and let $\varphi : A(G)\widehat{\otimes}^{op} A(H) \to \mathbb{C}$ be a positive linear map with respect to the natural order structure of $A(G)\widehat{\otimes}^{op} A(H)$ as an operator space. Then the canonical induced map $\widetilde{\varphi} : A(G) \to VN(H)$ is completely positive.

PROOF : Let $\varphi : A(G)\widehat{\otimes}^{op} A(H) \to \mathbb{C}$ be a positive linear map. We must show that, for each $n \geq 1$, the $n$-th amplification $\tilde{\varphi}^{(n)} : \mathbb{M}_n(A(G)) \to$

$\mathbb{M}_n(VN(H))$ is a positive map. Let $x = [x_{ij}]$ be in $\mathbb{M}_n(A(G))_+$ and $y = [y_{kl}]$ be in $\mathbb{M}_n(A(H))_+$. We need to check that $\tilde{\varphi}^{(n)}(x)(y) = [\tilde{\varphi}(x_{ij})][y_{kl}] = [\varphi(x_{ij} \otimes y_{kl}]$ is a positive scalar matrix. Let $\alpha = (c_1, \ldots, c_{mn})$ be a scalar row vector and put $u = \alpha \cdot (x \otimes y) \cdot \alpha^*$. Then $u$ belongs to $(A(G)\widehat{\otimes}^{op}A(H))^*_+$. Therefore $\langle \tilde{\varphi}^{(n)}(x)(y)\alpha, \alpha\rangle = \varphi(\alpha \cdot (x \otimes y) \cdot \alpha^*) = \varphi(u) \geq 0$.

## 3. Complete Order Amenability

3.1 *Order approximate diagonals*
In this section we introduce the new concept of order amenability for Banach ordered algebras which takes their order structure into account. An operator version of this concept would be defined for quantized Banach ordered algebras. This is to give a role to the order structure of ordered (operator) algebras in the theory of amenability. In classical amenability theory, one of the equivalent definitions of amenability for a Banach algebra $A$ is through the assumption of the existence of a net inside the projective tensor product $A \otimes^\gamma A$, called the approximate diagonal. A similar object in $A\widehat{\otimes}^{op}A$ is related to operator amenability, when $A$ has an additional operator space structure [23]. Here we define order amenability through the introduction of an order approximate diagonal. In the next section, the relation with (more familiar) approach based on derivations will be discussed. For a review of the basic results in the classical amenability theory of Banach algebras, interested reader is referred to [4], [17], [20], [21], and [24].

In order to define order approximate diagonals, we need an appropriate notion of order tensor products of ordered spaces.

Let $E$ and $F$ be two ordered spaces which are normed spaces as well, and let $L_{or}(E, F)$ be the space of all bounded linear maps $T : E \to F$ such that $T = T_1 - T_2 + i(T_3 - T_4)$ for some positive bounded linear maps $T_i$, $i = 1, ..., 4$. Consider the seminorm $\|\cdot\|_{or}$ defined on the algebraic tensor product $E \otimes F$ by

$$\|\sum_{i=1}^{n} a_i \otimes b_i\|_{\{or\}} := \sup\{|\sum_{i=1}^{n} T(a_i)(b_i)| : T \in L_{or}(E, F^*), \|T\| \leq 1\}.$$

Then $\|\cdot\|_{or}$ is a norm on $E \otimes F$ if and only if $L_{or}(E, F^*)$ is a separating family, considered as a family of linear functionals on $E\otimes F$. If this is the case (as we shall see later in several examples), we define $E\otimes^{or}F$ as the completion of $E \otimes F$ with respect to $\|\cdot\|_{or}$ and we call it the *order tensor product* of $E$ and $F$. Recall that, for operator spaces $E$ and $F$, $L_{c.or}(E,F)$ is the space of all bounded linear maps $T : E \to F$ such that $T = T_1 - T_2 + i(T_3 - T_4)$ for some completely positive completely bounded linear maps $T_i$, $i = 1,...,4$. We define the seminorm $\|\cdot\|_{c.or}$ on $E \otimes F$ by

$$\|\sum_{i=1}^{n} a_i \otimes b_i\|_{\{c.or\}} := \sup\{|\sum_{i=1}^{n} T(a_i)(b_i)| : T \in L_{c.or}(E, F^*), \|T\|_{cb} \leq 1\}.$$

Again this is a seminorm, and whenever it is a norm, we define $E \otimes^{c.or} F$ as the completion of $E \otimes F$ with respect to $\|\cdot\|_{c.or}$ and call it the *complete order tensor product* of $E$ and $F$. In this paper whenever we talk about (complete) order tensor products, we assume $\|\cdot\|_{or}$ (respectively, $\|\cdot\|_{c.or}$) is a norm.

For two normed spaces $E$ and $F$, we say that a norm $\|\cdot\|'$ on $E \otimes F$ is a *cross norm* if for each $x \in E$ and $y \in F$, $\|x \otimes y\|' = \|x\|\|y\|$. We note that for two ordered spaces $E$ and $F$, $E \otimes F$ is an ordered space with $co\{a \otimes b, a \in E_+, b \in F_+\}$, the convex hull of the set $\{a \otimes b, a \in E_+, b \in F_+\}$, as its positive cone [5]. For simplicity we denote this convex set by $E_+ \otimes F_+$. Since the closure of this cone in $E \otimes^{or} F$ is not necessarily a proper cone, in the sequel we consider $E \otimes^{or} F$ as an ordered space with the same positive cone $co\{a \otimes b, a \in E_+, b \in F_+\}$, i.e. an arbitrary element $u \in E \otimes^{or} F$ is positive if and only if $u \in E_+ \otimes F_+$.

Now, let $G$ and $H$ be locally compact groups. As in the proof of Proposition 2.1.7, the complete isomorphism $A(G)\widehat{\otimes}^{op} A(H) \cong^{c \cdot iso} A(G \times H)$ preserves positive cones, and so $\left(A(G)\widehat{\otimes}^{op} A(H)\right)_+$ is a closed proper cone. Therefore

$$(A(G) \otimes^{or} A(H))_+ = A(G)_+ \otimes A(H)_+ \subseteq \left(A(G)\widehat{\otimes}^{op} A(H)\right)_+.$$

***Theorem* 3.1.1** — *Let $E$ and $F$ be ordered spaces which are Banach spaces. If $\|\cdot\|_{or}$ is a cross norm on $E \otimes F$, then $L_{or}(E, F^*) \cong^{iso}_{iso} L_{or}(E \otimes^{or} F, \mathbb{C})$.*

Proof : For each $T \in L_{or}(E, F^*)$, let $\Phi(T) : E \otimes F \to \mathbb{C}$ be the linear map associated to the bilinear form $(a, b) \to T(a)(b)$ on $E \times F$. Then

$$\|\Phi(T)\| = \sup_{\|\sum a_i \otimes b_i\|_{or} \leq 1} \|\Phi(T)(\sum a_i \otimes b_i)\| = \sup_{\|\sum a_i \otimes b_i\|_{or} \leq 1} |\sum T(a_i)(b_i)| \leq \|T\|.$$

Hence $\Phi : L_{or}(E, F^*) \longrightarrow (E \otimes F)^*$ is a contraction which clearly maps positive elements to positive linear functionals, therefore it defines a linear map from $L_{or}(E, F^*)$ into $L_{or}(E \otimes F, \mathbb{C})$. On the other hand, since $\|\cdot\|_{or}$ is assumed to be a cross norm,

$$\|T\| = \sup_{\|a\|\leq 1, \|b\|\leq 1} |T(a)(b)| = \sup_{\|a\otimes b\|_{or}\leq 1} \|\Phi(T)(a \otimes b)\| \leq \|\Phi(T)\|,$$

hence $\Phi$ is an isometry. Conversely, it can be easily verified that, for each $S \in L_{or}(E \otimes F, \mathbb{C})$, the linear map $\Psi(S) : E \to F^*$ defined by $\Psi(S)(a)(b) = S(a \otimes b)$, $a \in E$, $b \in F$, is an element of $L_{or}(E, F^*)$, with $\Phi(\Psi(S)) = S$. Therefore $\Phi$ is a surjective linear isometry.

*Proposition* 3.1.2 — Let $G$ and $H$ be locally compact groups. Then $A(G)\widehat{\otimes}^{op} A(H) \cong^{iso}_{iso} A(G) \otimes^{c.or} A(H)$.

Proof : Recall that $(A(G)\widehat{\otimes}^{op} A(H))^* \cong^{c.iso} \mathcal{CB}(A(G), VN(H))$, which by the Hahn-Banach theorem implies that, for each $u \in A(G) \otimes A(H)$,

$$\|u\|_{op} = \sup\{|\sum_{i=1}^{n} T(a_i)(b_i)| : T \in \mathcal{CB}(A(G), A(H)^*), \|T\|_{cb} \leq 1\}.$$

Now since by Corollary 2.1.5 (iii), $\mathcal{CB}(A(G), A(H)^*) = L_{c.or}(A(G), A(H)^*)$ it follows from the above equality that $\|\cdot\|_{op} = \|\cdot\|_{c\cdot or}$ on the algebraic tensor product.

Before we turn to (quantized) Banach ordered algebras, we feel obliged to present some examples in which the seminorms introduced in this section are (cross) norms. We also give some negative examples, for which these are not norms.

Let $A$ be a $C^*$-algebra and $\mathcal{M}$ be an injective von Neumann algebra. Then by [18, Corollary 2.6] or [13, Theorem 2.1] we have

$$(A\widehat{\otimes}^{op}\mathcal{M}_*)^* \cong^{c.iso} \mathcal{CB}(A, \mathcal{M}) = L_{c.or}(A, \mathcal{M}) \subseteq L_{or}(A, \mathcal{M}),$$

which implies that $\|\cdot\|_{or}$ and $\|\cdot\|_{c.or}$ are cross norms on $A \otimes \mathcal{M}_*$. Finally, for each locally compact group $G$, by Corollary 2.1.5 (iii),

$$\mathcal{CB}(A(G), VN(G)) = L_{c.or}(A(G), VN(G)) \subseteq L_{or}(A(G), VN(G)) \subseteq \mathcal{B}(A(G), VN(G)),$$

hence $\|.\|_{op} = \|.\|_{c.or} \leq \|.\|_{or} \leq \|.\|_{\gamma}$, on $A(G) \otimes A(G)$, and in particular $\|\cdot\|_{or}$ and $\|\cdot\|_{c.or}$ are cross norms. When $G$ is abelian by finite, by Proposition 2.1.3 (i) and [8, Proposition 7.1.2], $(A(G)\widehat{\otimes}^{op}A(G))^* \cong^{iso} L_{or}(A(G), VN(G))$, and so $\|\cdot\|_{op} \leq \|\cdot\|_{or} \leq d\|\cdot\|_{op}$, where $d$ is the maximal dimension of irreducible representations of $G$. In this case, $L_{or}(A(G), VN(G)) \cong^{iso}_{iso} \mathcal{CB}(A(G), VN(G))$ if and only if $G$ is abelian.

We recall that the complete boundedness of order maps between two operator spaces really depends on the operator space structure and one could imagine much more general cases where all order maps are completely bounded. For an arbitrary Banach space $E$, consider the Max operator space structure, defined as follows: for $[a_{ij}] \in \mathbb{M}_n(E)$, let $\|[a_{ij}]\|_{\mathbb{M}_n(\mathrm{Max}(E))} = \sup \|[u(a_{ij})]\|_{\mathbb{M}_n(\mathcal{B}(H_u))}$, where the supremum is taken over all contractive representations $u : E \to \mathcal{B}(H_u)$ of $E$ on a Hilbert space $H_u$. Then $(E \otimes^{\gamma} F)^* \cong^{iso}_{iso} (\mathrm{Max}(E)\widehat{\otimes}^{op}F)^*$, for each operator space $F$ [8, Relation 3.3.9]. Hence

$$L_{c.or}(\mathrm{Max}(E), F^*) \subseteq L_{or}(\mathrm{Max}(E), F^*) \subseteq \mathcal{B}(E, F^*) \subseteq \mathcal{CB}(\mathrm{Max}(E), F^*).$$

There are cases that there is no non-zero, order-preserving, continuous linear functional on a Banach space with order. Let $\mathcal{H}$ be a Hilbert space, consider the singly generated closed subspace $A = \langle T \rangle$, of $\mathcal{B}(H)$ generated by a non-normal bounded operator $T$. If $T = (T_1 - T_2) + i(T_3 - T_4)$, where $T_i$'s are positive operators, then since $T^*$ does not belong to $A$, at least one of $T_i$'s does not belong to $A$. But $A$ is a $w^*$-closed subspace of $\mathcal{B}(H)$, therefore there is a Banach space $X$ such that $X \subseteq \mathcal{B}(H)$ and $X^* \cong^{c.iso} A$, and the order structure of $X^*$ and $A$ are the same. Now the above observation means that there is no nontrivial continuous functional on $X$ which is an order map, that is $L_{or}(X, \mathbb{C}) = \{0\}$.

*Proposition* 3.1.3 — Let $G$ be a locally compact group. Then the Fourier algebra $A(G)$ is a quantized Banach ordered algebra.

PROOF : Consider the canonical complete contraction $\triangle : A(G)\widehat{\otimes}^{op}A(G) \to A(G)$. We shall show that $\triangle$ is completely positive. But complete positiveness of $\triangle$ is equivalent to the complete positiveness of its adjoint map $\triangle^* : VN(G) \to VN(G \times G)$. For $x, y \in A(G)$, $\triangle^*(\lambda_e)(x \otimes y) = \lambda_e(xy) = \lambda_{(e,e)}(x \otimes y)$, where $e$ is the identity of $G$. Therefore the complete contraction $\triangle^*$ maps identity to identity, and hence is completely positive by [8, Corollary 5.1.2].

In order to define order-approximate diagonals, we need to have an appropriate notion of order Banach modules and show that there is a canonical action of $A$ on $A \otimes^{or} A$.

Let $A$ be a Banach ordered algebra, and $X$ be a Banach space which is an ordered space. Then $X$ is called a *left Banach ordered $A$-module* if it is a left Banach $A$-module such that the module action $A \times X \to X$ is a positive map. For a quantized Banach ordered algebra $A$, an operator space $X$ is a *left Banach completely ordered $A$-module* if it is a left operator $A$-module and the module action $A \widehat{\otimes}^{op} X \to X$ is a completely positive map. The right Banach (completely) ordered $A$-module and Banach (completely) ordered $A$-bimodule are defined similarly. It is easy to see that if $X$ is a left Banach (completely) ordered $A$-module, then $X^*$ is a right Banach (completely) ordered $A$-module, under the canonical actions. Clearly, each quantized order Banach algebra $A$, is Banach completely ordered $A$-bimodule.

Let $A$ be a Banach ordered algebra. Then the isometric inclusion $L_{or}(A, A^*) \subseteq \mathcal{B}(A, A^*)$ induces a canonical action of $A$ on $L_{or}(A, A^*)$. Also there is a canonical module actions of $A$ on $A \otimes A$ such that $x \cdot (a \otimes b) = xa \otimes b$, for $x, a, b \in A$. Moreover for $x, a_i, b_i \in A$, $i = 1, ..., n$,

$$\|x \cdot \sum_{i=1}^{n} a_i \otimes b_i\|_{or} = \|\sum_{i=1}^{n} xa_i \otimes b_i\|_{or} = \sup_{T \in L_{or}(A,A^*), \|T\| \leq 1} |\sum_{i=1}^{n} T(xa_i)b_i|$$
$$\leq \|x\| \sup_{T \in L_{or}(A,A^*), \|T\| \leq 1} |\sum_{i=1}^{n} T(a_i)(b_i)|.$$

Therefore, the latter action extends to a continuous action of $A$ on $A \otimes^{or} A$ such that $\|x \cdot u\|_{or} \leq \|x\| \|u\|_{or}$, for each $x \in A$, $u \in A \otimes^{or} A$. Also from $A_+ A_+ \subseteq A_+$, it follows that the module map is positive. A similar observation for the right actions shows that $A \otimes^{or} A$ is a Banach order $A$-bimodule. This induces a module structure on $(A \otimes^{or} A)^*$, hence $L_{or}(A, A^*)$ is also a Banach order $A$-bimodule. Unfortunately, for a quantized Banach ordered algebra $A$, we are not aware of any natural operator space structure on $A \otimes^{c.or} A$, but for a locally compact group $G$, Proposition 3.1.2 implies that $A(G) \otimes^{c.or} A(G)$ has a canonical operator space structure, and it is straightforward to check the following proposition.

*Proposition* 3.1.5 — Let $G$ be a locally compact group. Then $A(G) \otimes^{c.or} A(G)$ is a Banach completely ordered $A(G)$-bimodule.

*Definition* 3.1.5 — (i) Let $A$ be a Banach ordered algebra with a bounded diagonal operator $\triangle : A \otimes^{or} A \to A$. We say that $A$ is *order amenable* if there is a bounded net $\{m_\alpha\}$ in $A \otimes^{or} A$ such that $a.m_\alpha - m_\alpha.a \to 0$ and $a.\triangle(m_\alpha) \to a$, as $\alpha \to \infty$, for each $a \in A$. We call $\{m_\alpha\}$ an *order approximate diagonal* for $A$.

(ii) Let $A$ be a quantized Banach ordered algebra with a bounded diagonal operator $\triangle : A \otimes^{c.or} A \to A$ such that $A \otimes^{c.or} A$ is a Banach completely ordered $A$-bimodule. We say that $A$ is *complete order amenable* if there is a bounded net $\{m_\alpha\}$ in $A \otimes^{c.or} A$ such that $a.m_\alpha - m_\alpha.a \to 0$ and $a.\triangle(m_\alpha) \to a$, as $\alpha \to \infty$, for each $a$ in $A$. We call $\{m_\alpha\}$ a *complete order approximate diagonal* for $A$.

Note that the diagonal map $\triangle$ from the algebraic tensor product $A \otimes A$ into $A$ is not necessarily bounded in the (complete) order norm and so it does not extend to an operator on $A \otimes^{or} A$ (on $A \otimes^{c.or} A$, respectively) in general. We have defined (complete) order amenability for the class of those (quantized) Banach ordered algebras for which such an extension exists. The Fourier algebra is a typical example for which the diagonal map is bounded in both $\|.\|_{or}$ and $\|.\|_{cor}$.

Clearly each order amenable Banach ordered algebra has an approximate identity. Some of the other well known facts about amenability also hold for order amenability. For instance, if $A$ and $B$ are Banach ordered algebras and $\theta : A \to B$ is a bounded positive homomorphism with dense range, then the order amenability of $A$ implies the order amenability of $B$. Indeed if $\{m_\alpha\}$ is an order approximate diagonal for A, then $\{(\theta \otimes \theta)(m_\alpha)\}$ is an order approximate diagonal for $B$.

***Theorem* 3.1.6** — *Let $G$ be a locally compact group. Then $A(G)$ is operator amenable if and only if it is complete order amenable.*

PROOF : As we noted earlier, $\| \cdot \|_{op} = \| \cdot \|_{c \cdot or}$ on the algebraic tensor product $A(G) \otimes A(G)$. Hence the diagonal map $\triangle : A(G) \otimes^{c \cdot or} A(G) \to A(G)$ is completely bounded. On the other hand $A(G) \otimes^{c \cdot or} A(G) \cong^{iso}_{iso} A(G) \otimes^{op} A(G)$, as Banach $A(G)$-modules by Proposition 3.1.2, and we know that the operator amenability of a quantized Banach algebra is equivalent to the existence of an operator bounded approximate diagonal.

We do not know if amenability of $A(G)$ is equivalent to its order amenability.

3.2. *Order derivations*

In this section we briefly study cohomological aspects of (complete) order amenability by introducing (complete) order derivations. This is more in line with the original idea of amenability [14]. For a Banach ordered algebra $A$ and Banach ordered $A$-bimodule $X$, let $Z^1_{or}(A,X) := L_{or}(A,X) \cap Z^1(A,X)$ and $B^1_{or}(A,X) := L_{or}(A,X) \cap B^1(A,X)$. These are called *order inner derivations* and *order derivations*, and consist of those derivations and inner derivations from $A$ to $X$ which are order maps, respectively. Let $H^1_{or}(A,X) := B^1_{or}(A,X)/Z^1_{or}(A,X)$ be the *first order cohomology group* of $A$ with coefficients in $X$. Similarly, for a quantized Banach ordered algebra $A$ and Banach complete ordered $A$-bimodule $X$, we could define *complete order derivations* $Z^1_{c.or}(A,X)$ and *complete order inner derivations* $B^1_{c.or}(A,X)$ and form the *first complete order cohomology* $H^1_{c.or}(A,X)$ of $A$ with coefficients in $X$.

*Proposition* 3.2.1 — Let $A$ be a Banach ordered algebra with a positive bounded approximate identity, and let the diagonal operator $\triangle : A \otimes^{or} A \to A$ be bounded. If $H^1_{or}(A,X^*) = 0$ for all ordered $A$-bimodule $X$, then $A$ is order amenable.

Proof : Let $\{e_\alpha\}$ be a positive bounded approximate identity of $A$. Then $\{e_\alpha \otimes e_\alpha\}$ is a bounded net in $A \otimes^{or} A$, and since the positive cone of a dual space is always $w^*$-closed, it has a positive $w^*$-cluster point $E$ in $(A \otimes^{or} A)^{**}$. Let $\triangle^{**} : (A \otimes^{or} A)^{**} \to A^{**}$ be the second conjugate of $\triangle$. Then $\ker(\triangle^{**}) = (\ker(\triangle))^{**}$, hence $\ker(\triangle^{**})$ is a Banach ordered $A$-bimodule. Given $N \in (A \otimes^{or} A)^{**}$, let $ad_N : A \to (A \otimes^{or} A)^{**}$ be defined by $ad_N(x) = N \cdot x - x \cdot N$, $x \in A$. Then $ad_E(A) \subseteq \ker(\triangle^{**})$ and since the order cohomology is assumed to be trivial and $ad_E$ is an order derivation, there is $N \in \ker(\triangle^{**})$ such that $ad_E = ad_N$. Let $\{m_\alpha\}$ be a bounded net in $A \otimes^{or} A$, $w^*$-converging to $E - N$ in $(A \otimes^{or} A)^{**}$. Then $\{m_\alpha\}$ is an order-approximate diagonal for $A$.

*Proposition* 3.2.3 — Let $A$ be a (quantized) Banach ordered algebra with a bounded diagonal operator $\triangle$ whose first (complete) order cohomology with coefficients in $X^*$ vanishes for all Banach (complete) ordered $A$-bimodules $X$. Then $A$ has a bounded approximate identity.

Proof : Consider the canonical left module action $m : A \times A^{**} \to A^{**}$

and zero right module action $m' : A^{**} \times A \to A^{**}$. Since the right action is trivial, the embedding $D : A \to A^{**}$ is a (completely) positive order derivation on A, therefore there is $E \in A^{**}$ such that $D(a) = a \cdot E$. Let $\{x_\alpha\}$ be a net in $A$, $w^*$-converging to $E$. Then $\{a \cdot x_\alpha\}$ is $w^*$-convergent to $a \cdot E = a$, and appropriate convex combinations of $\{x_\alpha\}$'s provide a bounded approximate identity for $A$.

*Proposition* 3.2.3 — Suppose that $H^1_{c.or}(A(G), X^*) = 0$ for each Banach completely ordered $A$-bimodule $X$. Then $A(G)$ is complete order amenable.

PROOF : Let $\{e_\alpha\}$ be a positive bounded approximate identity of $A(G)$. Then $\{e_\alpha \otimes e_\alpha\}$ is a bounded net in $A(G) \otimes^{c.or} A(G)$ which has a $w^*$-cluster point $E$ in $(A(G) \otimes^{c.or} A(G))^{**}_+$. Consider the completely bounded and completely positive diagonal operator $\triangle : A(G) \otimes^{c.or} A(G) \to A(G)$. Then $\ker(\triangle^{**}) = (\ker(\triangle))^{**}$ is a Banach completely ordered $A(G)$-bimodule and $ad_E(A(G)) \subseteq \ker(\triangle^{**})$, where $ad_E$ is as in Proposition 3.2.1. Note that $x \mapsto E \cdot x$ is a completely positive linear map on $A(G)$, indeed for each $[T_{ij}] \in \mathbb{M}_m(\mathcal{CB}(A(G), VN(G)))_+$, $[E \cdot a_{ij}][Tij] = E(\langle\langle [a_{ij}], [T_{kl}]\rangle\rangle)$, which is positive. Thus $[E \cdot a_{ij}] \in \mathcal{CB}(\mathcal{CB}(A(G), VN(G)), \mathbb{M}_n)_+$, for each $n \in \mathbb{N}$ and $[a_{ij}] \in \mathbb{M}_n(A(G))_+$. Hence $ad_E$ is a complete order derivation, and therefore, there is $N \in \ker(\triangle^{**})$ such that $ad_E = ad_N$. Let $\{m_\alpha\}$ be a bounded net in $A(G) \otimes^{c.or} A(G)$, $w^*$-converging to $E - N$ in $(A(G) \otimes^{c.or} A(G))^{**}$, then $\{m_\alpha\}$ is a complete order-approximate diagonal for $A(G)$.

*Corollary* 3.2.4 — The Fourier algebra $A(G)$ is operator amenable if and only if all complete order derivations from $A(G)$ to any dual Banach completely ordered $A(G)$-bimodule are inner.


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