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SUPER rpp SEMIGROUPS

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We study a class of special strongly rpp semigroups, namely, the class of super rpp semigroups. These super rpp semigroups are generalizations of both superabundant semigroups and Clifford semigroups within the class of rpp semigroups. In particular, we prove that a super rpp semigroup is a semilattice of $\mathcal{D}^{(l)}$-simple strongly rpp semigroups. Our result not only generalizes a well-known theorem of Clifford in the class of completely regular semigroups but also strengthens some structure theorems obtained by Ren-Shum for superabundant semigroups which

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are orthodox. Some special super rpp semigroups are considered and discussed.

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1. Introduction and Preliminaries

A semigroup S is said to be a *right principally projective semigroup,* in short, an *rpp semigroup* if every right principal ideal aS^1 of S, regarded as an S^1-system, is projective. According to the terminology of Fountain in [1], an rpp semigroup is just a semigroup in which every $\mathcal{L}^*$-class of S contains at least one idempotent, where $\mathcal{L}^*$ is the Green's *-relation which is a generalized Green's relation adopted by Fountain in [1] while studying the structure of rpp semigroups. We can dually define an lpp semigroup. Now, following the notations of Fountain in [1], we define an *abundant semigroup* as a semigroup which is both an rpp semigroup and an lpp semigroup. The structure of abundant semigroups and their special subclasses have already been described in [20].

It is well known that the class of completely regular semigroups forms an important subclass of the class of regular semigroups. This kind of semigroups has been extensively generalized by a number of authors in various directions, in particular, superabundant semigroups in the class of abundant semigroups. As a generalization of superabundant semigroups, Guo, Shum and others have considered the so called strongly rpp semigroups, see [11]. In fact, a *strongly rpp semigroup* S is an rpp semigroup in which for every $a \in S$, there exists a unique idempotent $a^\dagger$ which is $\mathcal{L}^*$-related to a such that $a^\dagger a = a$. Recently, there are several types of strongly rpp semigroups investigated by Guo, Guo, Ren, Zhang and Shum (see, [5]-[11] and [17]-[23]). In this paper, we concentrate on the structure and the properties of super rpp semigroups.

We first give the definition of a super rpp semigroup and then give some characterization theorems for such semigroups in Section 2. In Sections 3 and 4, we study the structure of a special super rpp semigroup whose idempotents form a band. Finally, we focus on the primitive strongly rpp semigroups. Some important properties of these semigroups are given.

Throughout this paper, we use most of the notations given in [2] and [3]. We first recall the following basic definitions of the Green $*$-relations on a semigroup S: for $a, b \in S$, we define

$$\begin{array}{rcl} a\mathcal{L}^*b & \Leftrightarrow & Kera_l = Kerb_l; \\ a\mathcal{R}^*b & \Leftrightarrow & Kera_r = Kerb_r; \\ \mathcal{D}^* & = & \mathcal{L}^* \vee \mathcal{R}^* \\ \mathcal{H}^* & = & \mathcal{L}^* \bigcap \mathcal{R}^* \\ a\mathcal{J}^*b & \Leftrightarrow & J^*(a) = J^*(b), \end{array}$$

where for a mapping φ of S into S itself, $Ker\varphi$ is the kernel of φ, that is, $Ker\varphi = \{(x, y) \in S \times S \mid \varphi(x) = \varphi(y)\}$; a_l (a_r) is the inner left (right) translation on S^1; $J^*(a)$ is the smallest ideal of S containing $a \in S$ and is both $\mathcal{L}^*$-saturated and $\mathcal{R}^*$-saturated, that is, $J^*(a)$ is the union of some $\mathcal{L}^*$-classes of S as well as the union of some $\mathcal{R}^*$-classes of S. Evidently, $\mathcal{L}^*$ is a right congruence on S while $\mathcal{R}^*$ is a left congruence on S. In general, $\mathcal{L} \subseteq \mathcal{L}^*$ and $\mathcal{R} \subseteq \mathcal{R}^*$ but in case if a and b are regular elements of S, then $a\mathcal{L}b$ if and only if $a\mathcal{L}^*b$.

Denote by $L^*(a)$ the smallest left ideal of S containing a which is $\mathcal{L}^*$-saturated, that is, $L^*(a)$ is the union of some $\mathcal{L}^*$-classes of S. It was observed by Fountain [3] that $x\mathcal{L}^*y$ if and only if $L^*(x) = L^*(y)$ for any x, y in a semigroup S.

The following lemma will be useful in the sequel.

Lemma 1.1 — [3] Let S be a semigroup. Then $b \in L^*(a)$ if and only if there exist $a_0, a_1, \cdots, a_n \in S$ with $a_0 = a, a_n = b$, and $x_1, x_2, \cdots, x_n \in S^1$ such that $(a_i, x_i a_{i-1}) \in \mathcal{L}^*$ *for* $i = 1, 2, \cdots, n$.

It was first observed by Petrich [15] that the Green's relations and congruences on a semigroup S are the most important concepts in the theory of completely regular semigroups. For rpp semigroups, Fountain showed that the Green's $*$-relations play a crucial role. For further generalizations of rpp semigroups, Guo, Zhang, Shum and Zhu and some others (see, [5]-[11] and [21]-[22]) extended the Green's $*$-relations (see also Pastijn [13]) to Green's (l)-relations. These relations are defined in the following way: for any $a, b \in S$, define

$$\begin{array}{rcl} a\mathcal{L}^{(l)}b & \Leftrightarrow & Kera_l = Kerb_l, \ i.e. \ ; \mathcal{L}^{(l)} = \mathcal{L}^*; \\ a\mathcal{R}^{(l)}b & \Leftrightarrow & Ima_l = Imb_l, \ i.e. \ \mathcal{R}^{(l)} = \mathcal{R} \\ \mathcal{D}^{(l)} & = & \mathcal{L}^{(l)} \bigvee \mathcal{R}^{(l)}; \\ \mathcal{H}^{(l)} & = & \mathcal{L}^{(l)} \bigcap \mathcal{R}^{(l)}; \\ a\mathcal{J}^{(l)}b & \Leftrightarrow & J^{(l)}(a) = J^{(l)}(b), \end{array}$$

where Ima_l (Imb_l) is the image of a_l (b_l); $J^{(l)}(a)$ is the smallest ideal of S containing a and is also $\mathcal{L}^{(l)}$-saturated.

A $\mathcal{D}$-class of S is called *regular* if each of its elements is regular. According to [12, Propositions 3.1 and 3.2, P. 44], a $\mathcal{D}$-class is regular if and only if it has a regular element. For the Green's (l)-relations, we have the following lemma.

Lemma 2.1 — [7, Lemma 1.2] The following statements hold on a semigroup S:
(A) $\mathcal{D}^{(l)} = \mathcal{L}^{(l)} \circ \mathcal{R}^{(l)} = \mathcal{R}^{(l)} \circ \mathcal{L}^{(l)}$;
(B) each $\mathcal{D}^{(l)}$-class contains at most one regular $\mathcal{D}$-class.

The following result in [11] is similar to Lemma 1.1.

Lemma 1.3 — Let S be a semigroup with $a, b \in S$. Then the following statements are equivalent:
(A) $b \in J^{(l)}(a)$;
(B) there exist $a_0, a_1, \ldots, a_n \in S$ and $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in S^1$ such that $a_0 = a, a_n = b$ and $(a_i, x_i a_{i-1} y_i) \in \mathcal{L}^{(l)}$ for $i = 1, 2, \ldots, n$.

The following lemma gives the basic property of strongly rpp semigroups.

Lemma 1.4 — [7, Theorem 2.3] Let S be a strongly rpp semigroup and $a \in S$. Then
(A) $D_a^{(l)}$ (the $\mathcal{D}^{(l)}$-class of S containing a) is a $\mathcal{D}^{(l)}$-simple strongly rpp semigroup.
(B) $RegD_a^{(l)}$ (the set of regular elements of $D_a^{(l)}$) is a completely simple semigroup.

Let S be a strongly rpp semigroup. For any $a, b \in S$, define

$$a\overline{\mathcal{R}}b \Leftrightarrow a^\dagger \mathcal{R} b^\dagger$$

and

$$\overline{\mathcal{H}} = \overline{\mathcal{R}} \bigcap \mathcal{L}^{(l)}.$$

Evidently, $\overline{\mathcal{R}} \subseteq \mathcal{D}^{(l)}$. Also, $a\overline{\mathcal{H}}b$ if and only if $a^\dagger = b^\dagger$.

Let I and Λ be arbitrary non-empty sets, M a left cancellative monoid and let $P = (p_{\lambda i})$ be a $\Lambda \times I$-matrix with entries $p_{\lambda i}$ which are the units of M. Define a multiplication on $S = M \times I \times \Lambda$ by

$$(a, i, \lambda)(b, j, \mu) = (ap_{\lambda j}b, i, \mu).$$

Then it can be easily verified that, under the above multiplication, S becomes a semigroup. We now call this semigroup S the *Rees matrix semigroup over M with a sandwich matrix P* and is denoted by $S = \mathcal{M}(M; I, \Lambda; P)$.

By a *$\mathcal{D}^{(l)}$-simple semigroup*, we mean a semigroup having exactly one $\mathcal{D}^{(l)}$-class. The following result about $\mathcal{D}^{(l)}-$simple strongly rpp semigroups has recently been proved by the authors in [7].

Lemma 1.5 — [7, Theorem 3.4] A semigroup is a $\mathcal{D}^{(l)}$-simple strongly rpp semigroup if and only if it is isomorphic to some Rees matrix semigroup $\mathcal{M}(M; I, \Lambda; P)$ over a left cancellative monoid M with the sandwich matrix P.

For our purpose, we need the following results about Rees matrix semigroups.

Lemma 1.6 — Let $S = \mathcal{M}(M; I, \Lambda; P)$ be a Rees matrix semigroup over the left cancellative monoid M with the sandwich matrix P.

(A) $(a, i, \lambda)\mathcal{L}^{(l)}(b, j, \mu)$ if and only if $\lambda = \mu$.
(B) $(a, i, \lambda)\overline{\mathcal{R}}(b, j, \mu)$ if and only if $i = j$.
(C) $(a, i, \lambda)\overline{\mathcal{H}}(b, j, \mu)$ if and only if $i = j$ and $\lambda = \mu$.

Proof : (A) The proof of this part follows from [7, Lemma 2.1].

(B) By (A), a simple computation shows that $(a, i, \lambda)^\dagger = (p_{\lambda i}^{-1}, i, \lambda)$ and $(b, j, \mu)^\dagger = (p_{\mu j}^{-1}, j.\mu)$. Hence by the definition of $\overline{\mathcal{R}}$, $(a, i, \lambda)\overline{\mathcal{R}}(b, j, \mu)$ if and only if $(p_{\lambda i}^{-1}, i, \lambda)\mathcal{R}(p_{\mu j}^{-1}, j.\mu)$; if and only if $i = j$.

(C) This part follows from (A) and (B). □

By using the above notation, we can formulate the following lemma for strongly rpp semigroups.

Lemma 1.7 — Let S be a strongly rpp semigroup with $x, y \in S$. Then the following statements hold:

(A) If $x\mathcal{D}^{(l)}y$, then $xy \in \overline{R}_x \bigcap L_y^{(l)}$, where $\overline{R}_x$ $[L_x^{(l)}]$ is the $\overline{\mathcal{R}}$- $[\mathcal{L}^{(l)}]$-class of S containing x.

(B) $\overline{H}_x$ (the $\overline{\mathcal{H}}$-class of S containing x) is a left cancellative monoid.

(C) $\mathcal{D}^{(l)} = \overline{\mathcal{R}} \circ \mathcal{L}^{(l)}$.

PROOF : By Lemmas 1.4 and 1.5, $D_x^{(l)}$ (the $\mathcal{D}^{(l)}$-class of S containing x) is a $\mathcal{D}^{(l)}$-simple strongly rpp semigroup which is isomorphic to some Rees matrix semigroup $\mathcal{M} := \mathcal{M}(M; I, \Lambda; P)$ over a left cancellative monoid M with a sandwich matrix P. By Lemma 1.6, we have

- $(a, i, \lambda)^\dagger = (p_{\lambda i}^{-1}, i, \lambda)$;
- $(a, i, \lambda)\overline{\mathcal{R}}(b, j, \mu)$ if and only if $i = j$; and
- $(a, i, \lambda)\overline{\mathcal{H}}(b, j, \mu)$ if and only if $i = j$ and $\lambda = \mu$,

for all $(a, i, \lambda), (b, j, \mu) \in \mathcal{M}$. By the multiplication of the Rees matrix semigroups, (A) holds obviously. On the other hand, by using the foregoing proof, we have $\overline{H}_{(a,i,\lambda)} = \{(u, i, \lambda) : u \in M\}$. It can be easily verified that the mapping $\theta : \overline{H}_{(a,i,\lambda)} \to M$; $(u, i, \lambda) \mapsto up_{\lambda i}$ is a semigroup isomorphism and whence, (B) is proved. We now prove (C). For this purpose, let $x, y \in S$ with $x\mathcal{D}^{(l)}y$. Then by (A) , we have $xy \in \overline{R}_x \bigcap L_y^{(l)}$, and whence $x\overline{\mathcal{R}}xy\mathcal{L}^{(l)}y$. Therefore, $\mathcal{D}^{(l)} \subseteq \overline{R} \circ \mathcal{L}^{(l)}$ and the reverse inclusion is clear. Thus, (C) is proved. □

Let $E(S)$ be the set of idempotents of a semigroup S. Then an idempotent $e \in E(S)$ is said to be *primitive* if for any $f \in E(S)$, $f \leq e$, *i.e.*, $f = fe = ef$, implies $e = f$ or f is the zero element if the zero element exists. Now, we call e a *$\mathcal{J}$-primitive idempotent* if for any $f \in E(S) \bigcap J_e$, $f \leq e$ implies $e = f$, where J_e denotes the $\mathcal{J}$-class containing $e \in E(S)$. The semigroup S is said to be *primitive* [*$\mathcal{J}$-primitive*] if all the idempotents of S are primitive [$\mathcal{J}$-primitive].

Lemma 1.8 — Any strongly rpp semigroup is $\mathcal{J}$-primitive.

PROOF : Suppose that e and f are idempotents in S with $f \in J_e$. Then, by the definition of J_e, we have $e \in SfS$ and so $e = zft$, for some $z, t \in S$. If we let $x = ezf$ and $y = fte$, then we can see immediately that $ex = x = xf$. Furthermore, we also have

$$xfy = (ezf)f(fte) = e(zft)e = e^3 = e.$$

On the other hand, since the semigroup S is strongly rpp, for every $x \in S$, there exists a unique idempotent $x^\dagger \in E(S)$ such that $x^\dagger \mathcal{L}^* x$ and $x^\dagger x = x$ so that $x^\dagger x(fy) = xfy$, that is, $x^\dagger e = e$.

Now, we suppose that $f \leq e$. Then $f = ef = fe$ and it follows that

$$x^\dagger f = x^\dagger(ef) = (x^\dagger e)f = ef = f.$$

Since $x^\dagger \mathcal{L}^{(l)} x$, we can easily see that $xf = x$ is equivalent to $x^\dagger f = x^\dagger$, and by the above proof, we have $x^\dagger f = f$ and $x^\dagger = f$. This leads to $f = fe = x^\dagger e = e$, and hence e is $\mathcal{J}$-primitive. This shows that S is $\mathcal{J}$-primitive.

For terminologies and definitions not given in this paper, the reader is referred to the texts and survey article of J.M. Howie [12], M. Petrich [14] and [22].

2. Definitions and Characterizations

To begin with, we define a super rpp semigroup.

Definition 2.1 — A strongly rpp semigroup S is called **super rpp** if $\overline{\mathcal{R}}$ is a left congruence on S.

For the sake of convenience, we denote the restriction of ρ to U by $\rho|_U$ for an equivalence ρ on the semigroup S and a subset U of S. We now give a characterization theorem for super rpp semigroups.

Theorem 2.2 — *Let S be a strongly rpp semigroup. Then the following statements are equivalent:*

(A) S is a super rpp semigroup;
(B) for every $a \in S, J^{(l)}(a) = Sa^\dagger S$;
(C) $\mathcal{J}^{(l)}|_{E(S)} = \mathcal{J}|_{E(S)}$;
(D) $\mathcal{J}^{(l)} = \mathcal{D}^{(l)}$;
(E) $\mathcal{D}^{(l)}$ is a semilattice congruence.

Proof : (A)$\Rightarrow$(B) Let S be a super rpp semigroup. Then $\overline{\mathcal{R}}$ is a left congruence on S. Consider $a \in S$. Obviously,$e = a^\dagger \in J^{(l)}(a)$, and hence

$SeS \subseteq J^{(l)}(a)$. In order to show (B), it suffices to prove that SeS is $\mathcal{L}^{(l)}$-saturated. For this purpose, let $b = xey \in SeS$ and $k = (xe)^{\dagger}$. Since $xe\mathcal{L}^{(l)}k$ and $\mathcal{L}^{(l)}$ is a right congruence, $xey\mathcal{L}^{(l)}ky$. If $h = (ky)^{\dagger}$. then it is obvious that $h\overline{\mathcal{R}}ky$. Together with the fact that $\overline{\mathcal{R}}$ is a left congruence on S, we have $kh\overline{\mathcal{R}}ky$, and thereby $h\overline{\mathcal{R}}kh$. By Lemma 1.6$(C)$, we deduce that $h\mathcal{D}^{(l)}kh$, and hence by Lemma 1.2(B), $h\mathcal{D}(kh)^{\dagger}$. On the other hand, by $kh = khh$, we can deduce that $(kh)^{\dagger} = (kh)^{\dagger}h$. Since $kh\mathcal{L}^{(l)}(kh)^{\dagger}$, by easy computation, we can see immediately that $h(kh)^{\dagger} \in E(S), h(kh)^{\dagger} \leq h$ and $h(kh)^{\dagger}\mathcal{L}(kh)^{\dagger}$. From the above relations and by the fact that $h\mathcal{D}(hk)^{\dagger}$, we have $h\mathcal{D}h(kh)^{\dagger}$. But since S is $\mathcal{J}$-primitive (by Lemma 1.7), $h(kh)^{\dagger} \leq h$ implies $h(kh)^{\dagger} = h$. Together with the foregoing fact that $(kh)^{\dagger} = (kh)^{\dagger}h$, we obtain $h\mathcal{L}(kh)^{\dagger}$. Thus, $h\mathcal{L}^{(l)}kh$ and consequently , $h\overline{\mathcal{H}}kh$ because by the above proof, we have already shown that $h\overline{\mathcal{R}}kh$. It now shows that h is the identity in $\overline{H}_{kh}$. This implies $kh = hkh$, and whence $kh = (kh)^2$ since $k \in E(S)$. However, because $\overline{H}_{kh}$ is a left cancellative monoid and $(kh)h = kh = (kh)^2$, we have $kh = h$. By $xe = (xe)e$ and since $k = (xe)^{\dagger}\mathcal{L}^{(l)}xe$, we have $k = ke$ and $h = kh = keh \in SeS$. Since $xey\mathcal{L}^{(l)}ky\overline{\mathcal{R}}h$, we have $xey\mathcal{D}^{(l)}h$. Hence,we obtain $(xey)^{\dagger}\mathcal{D}^{(l)}h$ so that $(xey)^{\dagger}\mathcal{D}h$. Accordingly, we can find $u, v \in S$ such that $(xey)^{\dagger} = uhv$. This shows that

$$(xey)^{\dagger} = uhv = u(keh)v = (uk)e(hv) \in SeS,$$

and so for any $c \in L^{(l)}_{xey}$, we have $c = c(xey)^{\dagger} \in SeS$. Hence, it follows that $L^{(l)}_{xey} \subseteq SeS$, and therefore SeS is $\mathcal{L}^{(l)}$-saturated,as required.

$(B) \Rightarrow (C)$ Suppose that (B) holds . Then for any $e, f \in E(S)$,

$$\begin{aligned} e\mathcal{J}^{(l)}f \quad & \text{if and only if} \quad J^{(l)}(e) = J^{(l)}(f); \\ & \text{if and only if} \quad SeS = SfS; \qquad \text{(by (B))} \\ & \text{if and only if} \quad e\mathcal{J}f. \end{aligned}$$

This shows that $\mathcal{J}^{(l)}|_{E(S)} = \mathcal{J}|_{E(S)}$. and (C) holds.

$(C) \Rightarrow (D)$ Assume that $\mathcal{J}^{(l)}|_{E(S)} = \mathcal{J}|_{E(S)}$. Observe that $a\mathcal{D}^{(l)}(\mathcal{J}^{(l)})b$ if and only if $a^{\dagger}\mathcal{D}^{(l)}(\mathcal{J}^{(l)})b^{\dagger}$. Hence, in order to show (D) holds, it suffices to prove that $\mathcal{D}|_{E(S)} = \mathcal{J}|_{E(S)}$. Obviously, $\mathcal{D}|_{E(S)} \subseteq \mathcal{J}|_{E(S)}$. Conversely, if we let $e, f \in E(S)$ and $e\mathcal{J}f$, then there exist $x, y \in S^1$ such that $e = xfy$. Write $g = fyexf$. Then, we can easily see that $g \in E(S)$, and so $g \leq f$. On the other hand, since $(fye)(exf)(fye) = fye$ and $(exf)(fye)(exf) = exf$, we can easily see that exf is an inverse element of fye. Now, together with the facts that $e = xfy = (exf)(fye)$ and $g = fyexf = (fye)(exf)$, we can derive that $e\mathcal{D}g$ by [12, Proposition 3.6 , p.47]. Because we have already proved that $g\mathcal{J}f$, and hence, by $\mathcal{J}$-primitiveness of S, $g \leq f$ implies $g = f$. Thus $e\mathcal{D}f$, and therefore $\mathcal{J}|_{E(S)} \subseteq \mathcal{D}|_{E(S)}$ and so $\mathcal{D}|_{E(S)} = \mathcal{J}|_{E(S)}$, as required.

$(D) \Rightarrow (E)$ Let $\mathcal{D}^{(l)} = \mathcal{J}^{(l)}$. It suffices to prove that $\mathcal{J}^{(l)}$ is a semilattice congruence on S. For this purpose, let $a \in S$. Then $a\mathcal{L}^{(l)}a^\dagger$ and $a = a^\dagger a$. Now, we can derive that $a^2\mathcal{L}^{(l)}a^\dagger a = a$ since $\mathcal{L}^{(l)}$ is a right congruence on S, and so $a\mathcal{J}^{(l)}a^2$. This shows that

$$J^{(l)}(ab) = J^{(l)}((ab)^2) = J^{(l)}(abab) \subseteq J^{(l)}(ba),$$

for all $b \in S$. Similarly, $J^{(l)}(ba) \subseteq J^{(l)}(ab)$. Thus, we obtain that $J^{(l)}(ab) = J^{(l)}(ba)$ and hence $ab\mathcal{J}^{(l)}ba$.

We now proceed to prove that $\mathcal{J}^{(l)}$ is a right congruence on S. Indeed, if $a\mathcal{J}^{(l)}b$, then $a^\dagger\mathcal{J}^{(l)}b^\dagger$ and hence $a^\dagger\mathcal{D}^{(l)}b^\dagger$. Because $a^\dagger$ and $b^\dagger$ are idempotents , $a^\dagger\mathcal{D}b^\dagger$, and so there exist $u, v \in S^1$ such that $a^\dagger = ub^\dagger v$. It is clear that for any $c \in S$, $ac\mathcal{L}^{(l)}a^\dagger c$ since $\mathcal{L}^{(l)}$ is a right congruence on S, and hence $ac\mathcal{J}^{(l)}a^\dagger c$. Thus, by the above proof, we deduce that

$$\begin{aligned} J^{(l)}(ac) &= J^{(l)}(a^\dagger c) = J^{(l)}(ub^\dagger vc) \\ &\subseteq J^{(l)}(b^\dagger vc) = J^{(l)}(vcb^\dagger) \\ &\subseteq J^{(l)}(cb^\dagger) = J^{(l)}(b^\dagger c) = J^{(l)}(bc). \end{aligned}$$

Similarly, we have $J^{(l)}(bc) \subseteq J^{(l)}(ac)$. Now, $J^{(l)}(ac) = J^{(l)}(bc)$ and so $ac\mathcal{J}^{(l)}bc$. This shows that $\mathcal{J}^{(l)}$ is a right congruence on S. By $ac\mathcal{J}^{(l)}ca$ (by the proof above), we also see that $\mathcal{J}^{(l)}$ is a left congruence on S. Thus $\mathcal{J}^{(l)}$ is indeed a congruence on S. Together with the facts that $a\mathcal{J}^{(l)}a^2$ and $ab\mathcal{J}^{(l)}ba$, we conclude that $\mathcal{J}^{(l)}$ is a semilattice congruence on S, as required.

$(E) \Rightarrow (A)$ Suppose that $\mathcal{D}^{(l)}$ is a semilattice congruence on S. Then by Lemma 1.4, S becomes a semilattice Y of some $\mathcal{D}^{(l)}$-simple strongly rpp semigroups S_α $(\alpha \in Y)$, which are $\mathcal{D}^{(l)}$-classes of S. By Lemma 1.5, every S_α is a Rees matrix semigroup over a left cancellative monoid, in notation, $S_\alpha = \mathcal{M}(M; I_\alpha, \Lambda_\alpha; P_\alpha)$. Let $x = (a, i, \lambda)$ and $y = (b, j, \mu) \in S$. If $x\overline{\mathcal{R}}y$ then for some $\alpha \in Y$, we have $x\overline{\mathcal{R}}^{S_\alpha}y$ for $x, y \in S_\alpha$, where $\overline{\mathcal{R}}^{S_\alpha}$ denotes the relation $\overline{\mathcal{R}}$ on the semigroup S_α. It follows that $(p_{\lambda i}^{-1}, i, \lambda) = (a, i, \lambda)^\dagger\mathcal{R}(b, j, \lambda)^\dagger = (p_{\lambda j}^{-1}, j, \lambda)$ and we therefore infer that $i = j$. For $z \in S_\beta$, we have $z(p_{\lambda i}^{-1}, i, \lambda) \in S_{\beta\alpha}$, in notation, $z(p_{\lambda i}^{-1}, i, \lambda) = (c, k, \xi)$. Since

$$zx = z(p_{\lambda i}^{-1}, i, \lambda)(a, i, \lambda) = (c, k, \xi)(a, i, \lambda),$$

and

$$zy = z(p_{\lambda i}^{-1}, i, \lambda)(b, i, \mu) = (c, k, \xi)(b, i, \mu),$$

we can easily see that zx, zy are the elements of $S_{\alpha\beta}$, which can be expressed in the form $(-, k, -)$. Hence $zx\overline{\mathcal{R}}^{S_\alpha}zy$ and $zx\overline{\mathcal{R}}zy$ since $S_{\alpha\beta}$ is a $\mathcal{D}^{(l)}$-class of S. Thus $\overline{\mathcal{R}}$ is a left congruence on S. This shows that S is a super rpp semigroup.

By the proof of [3, Proposition 6.9], we observe the following result.

Lemma 2.3 — Let S be a semigroup which is a semilattice of semigroups S_α $(\alpha \in Y)$. Then the conjunction of any two of the following conditions implies the third:

(A) S is an rpp semigroup;

(B) for each $\alpha \in Y$, S_α is an rpp semigroup;

(C) for each $\alpha \in Y$ and each $a \in S_\alpha$, $L_a^*(S) = L_a^*(S_\alpha)$.

The following theorem can be regarded as a generalized version of the well-known Clifford theorem for super rpp semigroups.

***Theorem* 2.4** — *A semigroup S is super rpp if and only if S is a semilattice Y of Rees matrix semigroups over the left cancellative monoids $S_\alpha(\alpha \in Y)$ such that for $\alpha \in Y$ and $a \in S_\alpha$, $L_a^*(S) = L_a^*(S_\alpha)$.*

PROOF : The necessary part follows directly from Theorem 2.2, and Lemmas 1.4, 1.5 and 2.3. We therefore omit the details. In proving the sufficiency part, we first notice that S is an rpp semigroup by Lemma 2.3. Furthermore, by Lemma 1.5, S_α is a strongly rpp semigroup. It is clear that every S_α is a $\mathcal{D}^{(l)}$-class of S, and hence $\mathcal{D}^{(l)}$ is a semilattice congruence on S. Thus, by Theorem 2.2, S is a super rpp semigroup. □

Recall that a *superabundant semigroup* S is abundant in which each $\mathcal{H}^*$-class of S contains an idempotent. Obviously, all completely regular semigroups are superabundant. By [3, Corollary 5.2, and Theorems 4.6 and 6.8], a semigroup is superabundant if and only if it is an abundant semigroup which is a semilattice of Rees matrix semigroups over some cancellative monoids. Together with the Theorem in 2.4, we can easily show that each superabundant semigroup is a super rpp semigroup. In other words,all superabundant semigroups are abundant as well as super rpp. We now prove that the converse statement also holds.

***Theorem* 2.5** — *Let S be a super rpp semigroup. Then S is a super abundant semigroup if and only if S is an abundant semigroup.*

Proof : The necessity of the theorem is trivial and is hence omitted. We only prove the sufficiency part. Let S be a semilattice Y of Rees matrix semigroups $S_\alpha = \mathcal{M}(M_\alpha; I_\alpha, \Lambda_\alpha; P_\alpha)$ over some left cancellative monoids M_α such that for any $\alpha \in Y$ and $a \in S_\alpha$, $L_a^*(S) = L_a^*(S_\alpha)$. Let $(x, i_\alpha, \lambda_\alpha) \in S_\alpha$. If S is abundant, then there exists an idempotent $(y, i_\beta, \lambda_\beta) \in S_\beta$ such that $(x, i_\alpha, \lambda_\alpha)\mathcal{R}^*(y, i_\beta, \lambda_\beta)$. It follows that $(y, i_\beta, \lambda_\beta)(x, i_\alpha, \lambda_\alpha) = (x, i_\alpha, \lambda_\alpha)$. Since $(x, i_\alpha, \lambda_\alpha)^\dagger = (p^{-1}_{\lambda_\alpha i_\alpha}, i_\alpha, \lambda_\alpha)$, we have

$$(p^{-1}_{\lambda_\alpha i_\alpha}, i_\alpha, \lambda_\alpha)(x, i_\alpha, \lambda_\alpha) = (x, i_\alpha, \lambda_\alpha).$$

By the definition of $\mathcal{R}^*$, we derive that

$$(p^{-1}_{\lambda_\alpha i_\alpha}, i_\alpha, \lambda_\alpha)(y, i_\beta, \lambda_\beta) = (y, i_\beta, \lambda_\beta). \tag{1}$$

This shows that $S_\alpha S_\beta \subseteq S_\beta$, and hence $\alpha\beta = \beta$. On the other hand, since $(p^{-1}_{\lambda_\alpha i_\alpha}, i_\alpha, \lambda_\alpha)\overline{\mathcal{R}}(x, i_\alpha, \lambda_\alpha)$ and because $\overline{\mathcal{R}}$ is a left congruence on S, we immediately have

$$\begin{aligned}(y, i_\beta, \lambda_\beta)(p^{-1}_{\lambda_\alpha i_\alpha}, i_\alpha, \lambda_\alpha) \quad &\overline{\mathcal{R}} \quad (y, i_\beta, \lambda_\beta)(x, i_\alpha, \lambda_\alpha)\\ &= \quad (x, i_\alpha, \lambda_\alpha).\end{aligned}$$

This yields that $(y, i_\beta, \lambda_\beta)(p^{-1}_{\lambda_\alpha i_\alpha}, i_\alpha, \lambda_\alpha) \in S_\alpha$ since $\overline{\mathcal{R}} \subseteq \mathcal{D}^{(l)}$. Thus, $S_\beta S_\alpha \subseteq S_\alpha$, and hence $\alpha\beta = \alpha$. Consequently , we have $\alpha = \beta$. Now, by Equ. (1), we deduce that $(-, i_\alpha, -) = (y, i_\beta, \lambda_\beta)$ and so $i_\alpha = i_\beta$. Note that $(y, i_\beta, \lambda_\beta) \in E(S_\alpha)$ and since $RegS_\alpha$ is a completely simple semigroup (see Lemma 1.4(B)), we observe that $(y, i_\beta, \lambda_\beta)\mathcal{R}(p^{-1}_{\lambda_\alpha i_\alpha}, i_\alpha, \lambda_\alpha)$, and hence

$$(y, i_\beta, \lambda_\beta)(p^{-1}_{\lambda_\alpha i_\alpha}, i_\alpha, \lambda_\alpha) = (p^{-1}_{\lambda_\alpha i_\alpha}, i_\alpha, \lambda_\alpha).$$

Together with Equ. (1), we obtain $(p^{-1}_{\lambda_\alpha i_\alpha}, i_\alpha, \lambda_\alpha)\mathcal{R}(y, i_\beta, \lambda_\beta)$ and so $(p^{-1}_{\lambda_\alpha i_\alpha}, i_\alpha, \lambda_\alpha)\mathcal{R}^*(x, i_\alpha, \lambda_\alpha)$. Again since $(p^{-1}_{\lambda_\alpha i_\alpha}, i_\alpha, \lambda_\alpha)\mathcal{L}^*(x, i_\alpha, \lambda_\alpha)$, we therefore obtain $(x, i_\alpha, \lambda_\alpha)\mathcal{H}^*(p^{-1}_{\lambda_\alpha i_\alpha}, i_\alpha, \lambda_\alpha)$. This shows that S is indeed a super-abundant semigroup, as required.

We remark here that the condition "super rpp" in Theorem 2.5 can not be weakened to "strongly rpp." The following example shows that there exists an abundant semigroup S which is a strongly rpp semigroup but not super abundant. Hence, S is not a super rpp semigroup.

Example 2.6 : Let N be the set of all nonnegative integers and $S = \{(m,n) \in N \times N : m \geq n\}$. Define a multiplication $*$ on S by

$$(m,n) * (p,q) = (m - n + max(n,p), q - p + max(n,p)).$$

Then, it is easy to check that $(S,*)$ is a semigroup. In fact, we can see that S is a full subsemigroup of a bicyclic semigroup (for information of bicylic semigroups, see [12]). Note that a full subsemigroup of a regular semigroup must be abundant. Thus S is an adequate semigroup (that is, an abundant semigroup whose set of idempotents forms a semilattice). Moreover, since for any $(m,n) \in S$, there exists a unique idempotent $(n,n) \in S$ such that

$$(m,n)\mathcal{L}^*(n,n) \text{ and } (n,n)(m,n) = (m,n).$$

We now see immediately that S is a strongly rpp semigroup with $(m,n)^\dagger = (n,n)$. However, S itself is not a superabundant semigroup, for if otherwise, then by [3, Theorem 6.8], S can be expressed as a strong semilattice of cancellative monoids. By [2, Proposition 2.9], $E(S)$ is central in S. But since

$$(3,3)(2,1) = (3,2) \text{ and } (2,1)(3,3) = (4,3),$$

we have $(3,3)(2,1) \neq (2,1)(3,3)$. This shows that the element $(3,3)$ is not central in S. This contradiction shows that S is not a super abundant semigroup.

3. Orthodox Super rpp Semigroups

In this section, we consider a special super rpp semigroups, namely, the orthodox super rpp semigroups. This kind of semigroups can be regarded as a generalization of orthodox semigroups which are completely regular, namely, the OS-rpp semigroups. the structure of the OS-rpp semigroups were first investigated by He, Guo and Shum in [4]. In this section, we shall give some characterizations of OS-rpp semigroups. A new construction theorem for this kind of OS-rpp semigroups will be given.

Definition 3.1 — A super rpp semigroup S is said to be an **orthodox super rpp semigroup**, in short, an **OS-rpp semigroup** if $E(S)$ constitutes a subsemigroup of S.

Let S be an OS-rpp semigroup. Define a relation on S as follows: for $a, b \in S$, define

$$a\eta b \text{ if and only if } a = ebf, \text{ for some } e, f \in E(b^\dagger),$$

where we use $E(e)$ $(e \in E(S))$ to denote the $\mathcal{D}$-class of $E(S)$ containing $e \in E(S)$. We now write $E(e) \leq E(f)$ whenever $E(e)E(f) \subseteq E(e)$. Note that for a band E, the relation $\mathcal{D}$ is a semilattice congruence on E. We observe that $E(e) = E(f)$ if and only if $E(e) \leq E(f)$ and $E(f) \leq E(e)$. It is easy to check that η is an equivalence on S . Also, since each $E(e)$ is a rectangular band , we have $e\eta f$ if and only $e \in E(f)$,for any $e, f \in E(S)$.

In what follows, a *C-rpp semigroup* means an rpp semigroup in which all idempotents are central. Obviously,the C-rpp semigroups are generalizations of the Clifford semigroups in the range of rpp semigroups. It was pointed out by Fountain in [1] that a semigroup is a C-rpp semigroup if and only if it is a semilattice of left cancellative monoids.

We now have the following results for OS-rpp semigroups.

Proposition 3.2 — Let S be an $OS-rpp$ semigroup. Then

(A) η is a congruence on S which preserves the $\mathcal{L}^{(l)}$-classes of S;

(B) S/η is a C-rpp semigroup;

(C) $\overline{\mathcal{H}} \bigcap \eta = \iota_S$ (the identity relation on S);

(D) for all $e, f \in E(S), s \in S$, if $e\mathcal{R}f$ then $(se)^\dagger\mathcal{R}(sf)^\dagger$;

(E) for all $e, f \in E(S), s \in S$, if $e\mathcal{L}f$ then $(es)^\dagger\mathcal{L}(fs)^\dagger$;

(F) for all $s, t \in S$, $(st)^\dagger = (st^\dagger)^\dagger(s^\dagger t)^\dagger$.

Proof : Since S is an OS-rpp semigroup, by Lemma 1.5 and Theorem 2.2, S is a semilattice of Rees matrix semigroups $\mathcal{M}_\alpha = \mathcal{M}_\alpha(M_\alpha; I_\alpha, \Lambda_\alpha; P_\alpha)$ over the left cancellative monoids M_α for $\alpha \in Y$. It is clear that each $\mathcal{M}_\alpha$ is an OS-rpp semigroup, thereby, $Reg\mathcal{M}_\alpha$ is an orthogroup (such a semigroup is a completely regular semigroup which is orthodox), and hence all entries of the sandwich matrix P_α can be chosen as the identity elements of M_α for all α. This implies that each $\mathcal{M}_\alpha$ is the direct product of a rectangular band and a left cancellative monoid. Write $E_\alpha = I_\alpha \times \Lambda_\alpha$. Then, we have $\mathcal{M}_\alpha = E_\alpha \times M_\alpha$. For every $(x, m) \in \mathcal{M}_\alpha$, since $(x, m)^\dagger = (x, e_\alpha)$ and $E(\mathcal{M}_\alpha) = E_\alpha \times \{e_\alpha\}$, where e_α is the identity element of the monoid M_α, we have $E((x, m)^\dagger) = E(\mathcal{M}_\alpha)$. This implies that for all $(x, m), (y, n) \in S$, we have

$$(*) \qquad (x,m)\eta(y,n) \Leftrightarrow m = n,$$

This leads to $\eta|_{\mathcal{M}_\alpha}$ is a congruence on $\mathcal{M}_\alpha$ for any $\alpha \in Y$.

(A) If η is a congruence on S, then by the implication $(*)$, S/η is a semilattice of the semigroups $\mathcal{M}_\alpha/\eta|_{\mathcal{M}_\alpha}$ with $\alpha \in Y$. But $\mathcal{M}_\alpha/\eta|_{\mathcal{M}_\alpha}$ is isomorphic to M_α for each $\alpha \in Y$, we see that S/η is a semilattice of left cancellative monoids. This shows that S/η is rpp (see [1]). Now, by Lemma 2.3, $\mathcal{L}^*(\mathcal{M}_\alpha/\eta|_{\mathcal{M}_\alpha}) = \mathcal{L}^* \cap \mathcal{M}_\alpha/\eta|_{\mathcal{M}_\alpha}$ and whence η preserves the $\mathcal{L}^{(l)}$-classes of S. Thus, in order to prove (A), it suffices to prove that η is a congruence on S. In fact, we only need to prove that η is compatible with the semigroup multiplication since η is an equivalence on S. We first verify that η is left compatible with semigroup multiplication and dually we can show that η is right compatible with semigroup multiplication. Now let $(x,m),(y,n)$, $(z,p) \in S$ and $(x,m)\eta(y,n)$. Then $m = n$. If $m,n \in M_\alpha$ and $p \in M_\beta$, then $x,y \in E_\alpha$. We now divide the proof into two steps:

Step 1 : Here we consider the case $\beta = \alpha\beta = \beta\alpha$ and $p = e_\beta$. Denote $(z,e_\beta)(x,m) = (u,q)$. It is obvious that $(u,q) \in \mathcal{M}_\beta$. Hence, we derive that

$$\begin{aligned}(z,e_\beta)(x,m) &= (u,e_\beta)(u,q)(u,e_\beta)\\ &= (u,e_\beta)(z,e_\beta)(x,m)(u,e_\beta)\\ &= (uz,e_\beta)(x,m)(u,e_\beta)\\ &\quad \text{(because } (u,e_\beta),(z,e_\beta \in \mathcal{M}_\beta \text{ and}\\ &\quad (u,e_\beta)(z,e_\beta) = (uz,e_\beta))\\ &= (uz,e_\beta)(x,e_\alpha)(y,n)(x,e_\alpha)(u,e_\beta)\\ &\quad \text{(because } (x,e_\alpha),(y,n) \in \mathcal{M}_\alpha \text{ and}\\ &\quad (x,e_\alpha)(y,n)(x,e_\alpha) = (xyx,n) = (x,n).)\end{aligned} \tag{2}$$

Observe that $(v,t) := (y,m)(x,e_\alpha)(u,e_\beta), (w,e_\beta) := (uz,e_\beta)(x,e_\alpha) \in \mathcal{M}_\beta$ and since $E(\mathcal{M}_\beta)$ is a rectangular band, we have

$$\begin{aligned}(w,e_\beta)(v,t) &= (w,e_\beta)(v,e_\beta)(v,t)\\ &\quad \text{(because } (v,e_\beta),(v,t) \in E_\beta \times M_\beta \text{ and}\\ &\qquad (v,e_\beta)(v,t) = (vv,t) = (v,t))\\ &= (uz,e_\beta)(w,e_\beta)(v,e_\beta)(v,t)\\ &\quad \text{(because } (uz,e_\beta),(w,e_\beta),(v,e_\beta) \in \mathcal{M}_\beta \text{ and}\\ &\quad (w,e_\beta) = (uz,e_\beta)(w,e_\beta))\\ &= (uz,e_\beta)(v,e_\beta)(v,t)\\ &= (uz,e_\beta)(v,t).\end{aligned}$$

Thus, by Eq. (2), we obtain

$$\begin{aligned}(z,e_\beta)(x,m) &= (uz,e_\beta)(y,m)(x,e_\alpha)(u,e_\beta)\\ &= (u,e_\beta)(z,e_\beta)(y,n)(x,e_\alpha)(u,e_\beta),\\ &\text{because } (u,e_\beta),(z,e_\beta)\in E(\mathcal{M}_\beta) \text{ and } (u,e_\beta)(z,e_\beta)=(uz,e_\beta)\end{aligned}$$

This shows that

$$(**)\quad (z,e_\beta)(x,m)\eta|_{\mathcal{M}_\beta}(z,e_\beta)(y,m)$$

because $(u,e_\beta),(x,e_\alpha)(u,e_\beta)\in E(\mathcal{M}_\beta)$ and $(z,e_\beta)(y,m)\in\mathcal{M}_\beta$.

Step 2: We now consider the general case. For this, take $(z,p)(x,m)=(u,q)$ and $(z,p)(y,m)=(v,t)$. Now, it is clear that $(u,q),(v,t)\in\mathcal{M}_{\alpha\beta}$, and hence, we have

$$(u,q)=(u,e_{\alpha\beta})(u,q)(u,e_{\alpha\beta})=[(u,e_{\alpha\beta})(z,p)](x,m)(u,e_{\alpha\beta}) \tag{3}$$

and

$$\begin{aligned}(v,t) &= (v,e_{\alpha\beta})(v,t)(v,e_{\alpha\beta})\\ &= (v,e_{\alpha\beta})(z,p)(y,n)(v,e_{\alpha\beta})\\ &= (v,e_{\alpha\beta})(u,e_{\alpha\beta})(v,e_{\alpha\beta})(v,t)(v,e_{\alpha\beta})(u,e_{\alpha\beta})(v,e_{\alpha\beta}),\\ &\quad\text{because } (u,e_{\alpha\beta}),(v,e_{\alpha\beta})\in E(\mathcal{M}_{\alpha\beta})\\ &= (v,e_{\alpha\beta})(u,e_{\alpha\beta})(v,t)(u,e_{\alpha\beta})(v,e_{\alpha\beta})\\ &= [(v,e_{\alpha\beta})(u,e_{\alpha\beta})](z,p)(y,n)[(u,e_{\alpha\beta})(v,e_{\alpha\beta})]\end{aligned}$$

and $(u,e_{\alpha\beta})\eta|_{\mathcal{M}_{\alpha\beta}}(u,e_{\alpha\beta})(v,e_{\alpha\beta})$. Since $(u,e_{\alpha\beta}),(v,e_{\alpha\beta})\in E(\mathcal{M}_{\alpha\beta})$ and $E(\mathcal{M}_{\alpha\beta})$ is a rectangular band, we have $(u,e_{\alpha\beta})\ \eta|_{\mathcal{M}_{\alpha\beta}}\ (v,e_{\alpha\beta})$. Note that $(u,e_{\alpha\beta})(z,p)\in\mathcal{M}_{\alpha\beta}$ and since $\eta|_{\mathcal{M}_{\alpha\beta}}$ is a congruence on $\mathcal{M}_{\alpha\beta}$, we therefore deduce that

$$(u,e_{\alpha\beta})(z,p)=(u,e_{\alpha\beta})(u,e_{\alpha\beta})(z,p)\ \eta|_{\mathcal{M}_{\alpha\beta}}\ (v,e_{\alpha\beta})(u,e_{\alpha\beta})(z,p). \tag{4}$$

On the other hand, since $\alpha(\alpha\beta)=\alpha\beta=(\alpha\beta)\alpha, m=n\in\mathcal{M}_\alpha$ and by the dual of Formula $(**)$, we have $(x,m)(u,e_{\alpha\beta})\ \eta|_{\mathcal{M}_{\alpha\beta}}\ (y,n)(u,e_{\alpha\beta})$, so that

$$\begin{aligned}(x,m)(u,e_{\alpha\beta}) &= (x,m)(u,e_{\alpha\beta})(u,e_{\alpha\beta})\\ &\eta|_{\mathcal{M}_{\alpha\beta}}\ (y,n)(u,e_{\alpha\beta})(u,e_{\alpha\beta})(v,e_{\alpha\beta})=(y,n)(u,e_{\alpha\beta})(v,e_{\alpha\beta}).\end{aligned}$$

By applying Eq. (4) again, we can show that

$$\begin{aligned}(z,p)(x,m) &= [(u,e_{\alpha\beta})(z,p)][(x,m)(u,e_{\alpha\beta})]\\ &\eta|_{\mathcal{M}_{\alpha\beta}}\ [(v,e_{\alpha\beta})(u,e_{\alpha\beta})(z,p)][(y,n)(u,e_{\alpha\beta})](v,e_{\alpha\beta})=(z,p)(y,n).\end{aligned}$$

Therefore, η is left compatible, as required.

(B) By (A), we can see immediately that S/η is an rpp semigroup and a semilattice of $\mathcal{M}_\alpha/\eta|_{\mathcal{M}_\alpha}$ for every $\alpha \in Y$. Because $\mathcal{M}_\alpha/\eta|_{\mathcal{M}_\alpha}$ is isomorphic to M_α, it follows immediately that S/η is a semilattice of left cancellative monoids, that is, S/η is a C-rpp semigroup.

(C) Let $s, t \in S$ and $(s,t) \in \overline{\mathcal{H}} \bigcap \eta$. Then $s^\dagger = t^\dagger$ and $s\eta = t\eta$. By the definition of η, there exist $e, f \in E(t^\dagger)$ such that $s = etf$. Thus, we can easily deduce that $(s^\dagger etfs^\dagger) = (s^\dagger ss^\dagger = s)$, and hence,

$$\overline{\mathcal{H}} \bigcap \eta = \iota_S.$$

(D) Let $e, f \in E(S)$ and $s \in S$. If $e\mathcal{R}f$, then $e = fe$, $f = ef$ and so $se = sfe\ sf = sef$. This implies that

$$(sf)^\dagger se = (sf)^\dagger sfe = se = (se)^\dagger se \text{ and } (se)^\dagger (sf) = (se)^\dagger (se)f = (se)f = sf \tag{5}$$

These imply that there exists $\beta \in Y$ such that $(sf)^\dagger, se \in \mathcal{M}_\beta$, because S is a semilattice of $\mathcal{M}_\alpha$ with $\alpha \in Y$. Now we may let $(sf)^\dagger = (g, e_\beta), se = (h, m), (se)^\dagger = (k, e_\beta) \in E_\beta \times M_\beta$. By Eq. (5) ,

$$\begin{aligned} (gh, m) &= (g, e_\beta)(h, m) = (sf)^\dagger se = (se)^\dagger se \\ &= (k, e_\beta)(h, m) = (kh, m) = se \\ &= (h, m) \end{aligned}$$

By comparing the corresponding components, we have $gh = h = kh$. This shows that $g\mathcal{R}k$ in $E_{\alpha\beta}$, since $g, h \in E_{\alpha\beta}$ and $E_{\alpha\beta}$ is a rectangular band. Consequently, $(se)^\dagger \mathcal{R} (sf)^\dagger$.

(E) This part is the dual of part (D).

(F) Let $s, t \in S$. Since S is a strongly rpp semigroup, $(st)^\dagger st = st = (st^\dagger)t = (st^\dagger)^\dagger (st^\dagger)t = (st^\dagger)^\dagger st$. Note that $st, (st)^\dagger, (st^\dagger)^\dagger$ are in some $\mathcal{M}_\alpha$. By using same arguments as in the proof of (D), we obtain $(st)^\dagger \mathcal{R} (st^\dagger)^\dagger$. On the other hand, since $(st)(st)^\dagger = st = s(s^\dagger t)(s^\dagger t)^\dagger = (st)(s^\dagger t)^\dagger$, by using similar arguments as in the proof of (D), we have $(st)^\dagger \mathcal{L} (s^\dagger t)^\dagger$. Thus $(st)^\dagger = (st)^\dagger (st)^\dagger = (st^\dagger)^\dagger (st)^\dagger (s^\dagger t)^\dagger = (st^\dagger)^\dagger (s^\dagger t)^\dagger$ because $E(\mathcal{M}_\alpha)$ is a rectangular band.

We now proceed to establish a construction theorem for OS-rpp semigroups. We first introduce the concept of weak semi-spined product.

Let $T = \bigcup_{\alpha \in Y} T_\alpha$ and $I = \bigcup_{\alpha \in Y} I_\alpha$ be the semilattice decomposition of the semigroup T and the band I into the left cancellative submonoids T_α and the rectangular bands I_α on the semilattice Y, respectively. For each $\alpha \in Y$, construct the direct product $S_\alpha = T_\alpha \times I_\alpha$ and denote $\bigcup_{\alpha \in Y} S_\alpha$ by S. Define a mapping φ by

$$\varphi : S \to \mathcal{T}_l(I),\ (a, i) \mapsto \varphi_{a,i}$$

with

$$\varphi_{a,i} : I \to I,\ j \mapsto \varphi_{a,i} j,$$

where $\mathcal{T}_l$ is the right transformation semigroup on I . Similarly , we define a mapping ψ by

$$\psi : S \to \mathcal{T}_r(I),\ (a, i) \mapsto \psi_{a,i}$$

with

$$\psi_{a,i} : I \to I,\ j \mapsto j\psi_{a,i},$$

where $\mathcal{T}_r(I)$ is the left transformation semigroup on I. Assume that the following conditions are satisfied:

(SS1) If $(a, i) \in S_\alpha, j \in I_\beta$, then $\varphi_{a,i} j, j\psi_{a,i} \in I_{\alpha\beta}$;

(SS2) If $(a, i) \in S_\alpha, j \in I_\alpha$ then $j\psi_{a,i} = j \bullet i, \varphi_{a,i} j = i \bullet j$, where $j \bullet i$ $[i \bullet j]$ is the product of j $[i]$ and i $[j]$ in the semigroup I;

(SS3) If $(a, i) \in S_\alpha, (b, j) \in S_\beta, k \in I$, then $\varphi_{a,i}\varphi_{b,j} k \mathcal{R} \varphi_{ab, \varphi_{a,i} j \bullet i\psi_{b,j}} k$ and $k\psi_{a,i}\psi_{b,j} \mathcal{L} k\psi_{ab, \varphi_{a,i} j \bullet i\psi_{b,j}}$, where ab is the product of a and b in the semigroup T ;

(SS4) If $(a, i) \in S, j, k \in I$ and $j\mathcal{R}k$, then $\varphi_{a,i} j \mathcal{R} \varphi_{a,i} k$;

(SS5) If $(a, i) \in S, j, k \in I$ and $j\mathcal{L}k$, then $j\psi_{a,i} \mathcal{L} k\psi_{a,i}$.

We call $(T, I; \varphi, \psi)$ satisfying all the above conditions an $OS-quadruple$. Define a binary operation " $\circ$ " on the set $S = \bigcup_{\alpha \in Y}(T_\alpha \times I_\alpha)$ by

$$(a, i) \circ (b, j) = (ab, \varphi_{a,i} j \bullet i\psi_{b,j}).$$

Obviously, " $\circ$ " is well defined.

We now formulate the following lemma.

Lemma 3.3 — With respect to the multiplication " $\circ$ " defined above, $(S, \circ)$ forms a semigroup.

PROOF : Let $(a,i),(b,j),(c,k) \in S$. Then

$$\begin{aligned}[(a,i)\circ(b,j)]\circ(c,k) &= (ab, \varphi_{a,i}j \bullet i\psi_{b,j})(c,k)\\ &= ((ab)c, \varphi_{ab,\varphi_{a,i}j\bullet i\psi_{b,j}}k \bullet (\varphi_{a,i}j \bullet i\psi_{b,j})\psi_{c,k})\\ &= (abc, \varphi_{a,i}\varphi_{b,j}k \bullet i\psi_{b,j}\psi_{c,k})\\ &\quad (by\ (SS3)\ and\ (SS5)).\end{aligned}$$

Similarly, $(a,i)\circ[(b,j)\circ(c,k)] = (abc, \varphi_{a,i}\varphi_{b,j}k \bullet i\psi_{b,j}\psi_{c,k})$. Thus $[(a,i)\circ(b,j)]\circ(c,k) = (a,i)\circ[(b,j)\circ(c,k)]$. Hence the associative law holds on S and so $(S,\circ)$ forms a semigroup. □

Definition 3.4 — The above semigroup $(S,\circ)$ is called the **weak semi-spined product** of the semigroups T and I with respect to the structure homomorphisms φ and ψ, respectively.

We denote this semigroup $(S,\circ)$ by $OS(T,I;\varphi,\psi)$, where the homomorphisms φ,ψ are called the weak semi-spined homomorphisms of S.

We remark here that the concept of "weak semi-spined product of semigroups" is a generalized concept of the so called " spined product of semigroups" given in [11].

The following proposition gives some properties of the weak semi-spined product of a C-rpp semigroup T and a band I.

Proposition 3.5 — Let $(T,I;\varphi,\psi)$ be an OS-quadruple. Then the following statements hold for the semigroup $S = OS(T,I;\varphi,\psi)$:

(A) $E(S) = \{(x,i) \in S : x \in E(T)\}$;

(B) Let the weak semi-spined homomorphism φ satisfies the following condition:

(OS) If $\varphi_{a,i}j\mathcal{R}\varphi_{a,i}k$, then $\varphi_{b,i}j\mathcal{R}\varphi_{b,i}k$, for all $(a,i),(b,i) \in S_\alpha = T_\alpha \times I_\alpha$, and $j,k \in I_\beta$,

Hence, S is an OS-rpp semigroup.

PROOF : (A) By Condition (SS2), it is easy to see that $E(S) = \{(x,i) \in S : x \in E(T)\}$.

(B) By the definition of "$\circ$", we can easily see that $(E(S),\circ)$ is a band. Since T is a C-rpp semigroup, the semigroup operation of S restricted to the set S_α is precisely the operation defined on the direct product $T_\alpha \times I_\alpha$, for each $\alpha \in Y$, and hence every S_α is a $\mathcal{D}^{(l)}$-simple strongly rpp semigroup. By the definition of "$\circ$" again, S is a semilattice of S_α, that is, S is a semilattice of Rees matrix semigroups S_α for $\alpha \in Y$.

Let $(a, i) \in S_\alpha$. Denote by e_α the identity element of T_α. We now prove that $(a, i)\mathcal{L}^{(l)}(e_\alpha, i)$. If $(x, j), (y, k) \in S$ and $(a, i) \circ (x, j) = (a, i) \circ (y, k)$, then $(ax, \varphi_{a,i}j \bullet i\psi_{x,j}) = (ay, \varphi_{a,i}k \bullet i\psi_{y,k})$ and by comparing the corresponding components, we have $ax = ay$ and

$$\varphi_{a,i}j \bullet i\psi_{x,j} = \varphi_{a,i}k \bullet i\psi_{y,k}. \tag{6}$$

In fact, the above equality implies that $e_\alpha x = e_\alpha y$ since $e_\alpha \mathcal{L}^{(l)} a$ in T. By Condition (SS1), there exist $\beta, \gamma \in Y$ such that $\varphi_{a,i}j, i\psi_{x,j} \in I_\beta$ and $\varphi_{a,i}k, i\psi_{y,k} \in I_\gamma$. Thus, by Eq. (6), we have $\beta = \gamma$ and by applying Eq. (6) again and since I_β is a rectangular band, we have $\varphi_{a,i}j\mathcal{R}\varphi_{a,i}k$ and $i\psi_{x,j}\mathcal{L}i\psi_{y,k}$. This shows that $\varphi_{e_\alpha,i}j\mathcal{R}\varphi_{e_\alpha,i}k$ by Condition (OS). Also, by Condition (SS1), we have $\varphi_{e_\alpha,i}j, \varphi_{e_\alpha,i}k \in I_\beta$. Thus by $\varphi_{e_\alpha,i}j\mathcal{R}\varphi_{e_\alpha,i}k$ and $i\psi_{x,j}\mathcal{L}i\psi_{y,k}$ and because I_β is a rectangular band, we have $\varphi_{e_\alpha,i}j \bullet i\psi_{x,j} = \varphi_{e_\alpha,i}k \bullet i\psi_{y,k}$, and therefore

$$\begin{aligned}(e_\alpha, i) \circ (x, j) &= (e_\alpha x, \varphi_{e_\alpha,i}j \bullet i\psi_{x,j}) \\ &= (e_\alpha y, \varphi_{e_\alpha,i}k \bullet i\psi_{y,k}) = (e_\alpha, i) \circ (y, k).\end{aligned}$$

Together with $(a, i) \circ (e_\alpha, i) = (a, i)$, we have $(a, i)\mathcal{L}^{(l)}(e_\alpha, i)$. This proves that S is an rpp semigroup. Now, by Lemma 2.3, $L^*(S_\alpha)_a = L^*_a \cap S_\alpha$, for every $\alpha \in Y$ and $a \in S_\alpha$, where $L^*(S_\alpha)_a$ is the $\mathcal{L}^*$-class of S_α containing a. Consequently, by Theorem 2.4, S is a super rpp semigroup, and thus S is indeed an OS-rpp semigroup.

Finally, we proceed to prove that every OS-rpp semigroup is isomorphic to some semigroup $OS(T, I; \varphi, \psi)$. For the sake of brevity, in the rest of this section, we assume that S is an $OS - rpp$ semigroup with a band E of idempotents. Let $E = \bigcup_{\alpha\in Y} E_\alpha$ be the semilattice decomposition of E into rectangular bands E_α. For $\alpha \in Y$. By Proposition 3.2, S/η is a C-rpp semigroup. We observe that

- $(x\eta)^2 = x\eta$ implies $x^2 = x$ for $x \in S$, and
- $\eta \subseteq \mathcal{D}$.

It is easy to see that $E(S/\eta) = E(S)/\eta$ and is isomorphic to the semilattice Y. For the sake of simplicity, we first identify $E(S/\eta)$ by the semilattice Y. Thus, we can let $S/\eta = \bigcup_{\alpha\in Y} T_\alpha$ be the semilattice decomposition of the semigroup S/η into some left cancellative monoids M_α, where $\alpha \in Y$. Write $S_\alpha = T_\alpha \times E_\alpha$ and $T = \bigcup_{\alpha\in Y} T_\alpha$.

Now, we have the following lemma.

Lemma 3.6 — For all $(x, i) \in T$, there exists a unique element s of S such that $s^\dagger = i$ and $x = s\eta$.

Proof : Since $x \in S/\eta$, we have $t \in S$ such that $t\eta = x$. It is clear that $t^\dagger \in E(i)$. Now, for any $u, v \in S^1$ with $(iti)u = (iti)v$, we have

$$\begin{aligned} (ti)u &= (t^\dagger ti)u = (t^\dagger it^\dagger)(ti)u \\ &= t^\dagger(iti)u = t^\dagger(iti)v \\ &= (t^\dagger it^\dagger)tiv = t^\dagger tiv = tiv. \end{aligned}$$

This implies that $t^\dagger iu = t^\dagger iv$ since $t^\dagger\mathcal{L}^*t$. Thus,

$$iu = (it^\dagger i)u = i(t^\dagger iu) = i(t^\dagger iv) = iv.$$

By the above equality and the fact that $(iti)i = iti$, we obtain $iti\mathcal{L}^*i$. Because $i(iti) = iti$ and S is a strongly rpp semigroup, we have $(iti)^\dagger = i$. On the other hand, if $x \in T_\alpha$ for some $\alpha \in Y$, then

$$(iti)\eta = (i\eta)t\eta(i\eta) = e_\alpha x e_\alpha = x,$$

where e_α is the identity element of T_α. Now let s be an element of S such that $s^\dagger = i$ and $s\eta = x$. Since $s\eta = (iti)\eta$, there exist $j, k \in E(i)$ such that $s = jitik$. Thus $s = isi = (iji)t(iki) = iti$, and hence there exists a unique element s of S such that $s^\dagger = i$ and $s\eta = x$. The proof is completed. □

Consider the following mappings:

$$\lambda : T \to \mathcal{T}_l(E),\ (x, i) \mapsto \lambda_{x,i},$$

with $\lambda_{x,i} : E \to E,\ j \mapsto \lambda_{x,i}j = (sj)^\dagger$; and

$$\rho : T \to \mathcal{T}_r(E),\ (x, i) \mapsto \rho_{x,i},$$

with $\rho_{x,i} : E \to E,\ j \mapsto \rho_{x,i} = (js)^\dagger$, where s is the unique element of the semigroup S such that $s^\dagger = i$ and $s\eta = x$.

By using the above mappings, we have the following lemma.

Lemma 3.7 — The quadruple $(S/\eta, E; \lambda, \rho)$ is an OS-rpp quadruple.

Proof: If $(x, i) \in T_\alpha$ and $j \in E_\beta$, then by Lemma 3.6, there exists $s \in S$ such that $s\eta = x$ and $s^\dagger = i$. By Proposition 3.2, S/η is a C-rpp semigroup so that $E(S/\eta)$ is central in the semigroup S/η, and hence $(js)^\dagger \eta \mathcal{L}^*(js)\eta = (sj)\eta\mathcal{L}^*(s^\dagger j)\eta$ since $i\eta \in E(S/\eta)$. This shows that $(js)^\dagger\eta = (js^\dagger)\eta$. On the other hand, it is clear that $(sj)^\dagger\eta = (s^\dagger j)\eta$. Because $\eta|_{E(S)} = \mathcal{D}^{E(S)}$, we have $(js)^\dagger \mathcal{D}^{E(S)} js^\dagger$ and $(sj)^\dagger \mathcal{D}^E s^\dagger j$. By using the arguments before Lemma 3.6, we have $E((sj)^\dagger) = E(s^\dagger j) \supseteq E(s^\dagger)E(j)$. On the other hand, by using the same reasonings, we have $E(j) = E_\beta$ and $E_\alpha = E(i) = E(s^\dagger)$. Thus $E((sj)^\dagger) \subseteq E_{\alpha\beta}$. These mean that $\lambda_{x,i} j \in E_{\alpha\beta}$; and similarly, $j\rho_{x,i} \in E_{\alpha\beta}$. Hence condition (SS1) is satisfied.

Let $(a, i) \in T_\alpha$ and $j \in E_\alpha$. Then there exists $s \in S$ such that $s^\dagger = i$ and $s\eta = a$. Since $ij, ji \in E_\alpha$, by the proof of Lemma 3.6, we have $(ijsij)^\dagger = ij$ and $(jisji)^\dagger = ji$. Since E_α is a rectangular band and $i, j \in E_\alpha$, we derive that

$$(sj)^\dagger = (s^\dagger s s^\dagger j)^\dagger = (isij)^\dagger = (ijisij)^\dagger = (ijsij)^\dagger = ij$$

and

$$(js)^\dagger = (jisi)^\dagger = (jisiji)^\dagger = (jisji)^\dagger = ji.$$

These lead to $\lambda_{x,i} j = ij$ and $j\rho_{x,i} = ji$. Consequently, condition (SS2) is a satisfied.

If $(a, i) \in T_\alpha, (b, j) \in T_\beta$, then there exist $s, t \in S$ such that $s^\dagger = i, t^\dagger = j, s\eta = a$ and $t\eta = b$. By Proposition 3.2, we have $(st)\eta = ab$ and consequently, $(st)^\dagger = (st^\dagger)^\dagger (s^\dagger t)^\dagger = \lambda_{a,i} j \bullet i\rho_{b,j}$. On the other hand, if S is a semilattice of Rees matrix semigroups $\mathcal{M}_\alpha$ for $\alpha \in Y$, then because S is an OS-rpp semigroup, each $\mathcal{M}_\alpha$ is the direct product of a rectangular band and a left cancellative monoid which, together with the fact that for all $k \in E$, we see that

$$(stk)^\dagger stk = stk = s(tk)^\dagger tk = (s(tk)^\dagger)^\dagger s(tk)^\dagger tk = (s(tk)^\dagger)^\dagger stk,$$

This implies $(stk)^\dagger \mathcal{R} (s(tk)^\dagger)^\dagger$ (for more details, see the proof of Lemma 3.2(D)). By Lemma 3.6, we have $\lambda_{ab, \lambda_{a,i} j \bullet i\rho_{b,j}} k = (stk)^\dagger$ and $\lambda_{a,i}\lambda_{b,j} k = \lambda_{a,i}(tk)^\dagger = (s(tk)^\dagger)^\dagger$. Thus, we have $\lambda_{ab, \lambda_{a,i} j \bullet i\rho_{b,j}} k \mathcal{R} \lambda_{a,i}\lambda_{b,j} k$. Similarly, we can prove that $k\rho_{ab, \lambda_{a,i} j \bullet i\rho_{b,j}} \mathcal{L} k\rho_{a,i}\rho_{b,j}$. Thus, the condition (SS3) is satisfied.

The conditions (SS4) and (SS5) clearly hold by the results (D) and (E) in Proposition 3.2, respectively. Consequently, $(S/\eta, E; \lambda, \rho)$ is an OS-quadruple. □

We now characterize the OS-rpp semigroups. Our theorem clearly extends and generalizes the theorems related to orthogroups in the classes of regular semigroups and completely regular semigroups, as described in [15] and [16].

***Theorem* 3.8** — *The semigroup S is isomorphic to the constructed semigroup $OS(S/\eta, E; \lambda, \rho)$.*

Proof : We only need to prove that the mapping

$$\theta : S \to OS(S/\eta, E; \lambda, \rho), \ s \mapsto (s\eta, s^{\dagger})$$

is a semigroup isomorphism. By Lemma 3.6, θ is both surjective and injective. Now let $s, t \in S$. Then, by Proposition 3.2 and the definition of the multiplication $\circ$, we have

$$\begin{aligned} \theta(st) &= ((st)\eta, (st)^{\dagger}) = ((s\eta)(t\eta), (st^{\dagger})^{\dagger}(s^{\dagger}t)^{\dagger}) = (s\eta, s^{\dagger}) \circ (t\eta, t^{\dagger}) \\ &= \theta(s)\theta(t). \end{aligned}$$

This shows that θ is a homomorphism, and whence θ is a semigroup isomorphism, as required.

4. Left C-rpp Semigroups and Right C-rpp Semigroups

In the following section , we consider some special $OS - rpp$ semigroups. Recall that a strongly rpp semigroup S is a *left C-rpp semigroup* if $\mathcal{L}^{(l)}$ is a semilattice congruence on S. In the literature, the left C-rpp semigroups were first investigated by Guo, Shum and Zhu in [11].

We now describe the left C-rpp semigroup S by using the relation $\mathcal{L}^*$ on S. Our result is an improved result.

***Theorem* 4.1** — *Let S be a strongly rpp semigroup. Then S is a left C-rpp semigroup if and only if $\mathcal{J}^{(l)} = \mathcal{L}^{(l)}$.*

Proof : $\Rightarrow$) Suppose that S is a left C-rpp semigroup. Then, by Lemma 1.1, it suffices to prove that for every $a \in S$, $J^{(l)}(a) = L^*(a)$. Obviously,

$L^*(a) \subseteq J^{(l)}(a)$. Conversely, if $b \in J^{(l)}(a)$, then by Lemma 1.3, there exist $a_0, a_1, \ldots, a_n \in S$ with $a_0 = a, a_n = b$ and $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in S^1$ such that $a_i \mathcal{L}^{(l)}(x_i a_{i-1} y_i)$ for all $i = 1, 2, \ldots, n$. Because S is a left C-rpp semigroup, $\mathcal{L}^{(l)}$ is a semilattice congruence on S. Hence,it follows that $x_i a_{i-1} y_i \mathcal{L}^{(l)} y_i x_i a_{i-1}$, that is, $a_i \mathcal{L}^{(l)} y_i x_i a_{i-1}$. This shows that $b \in L^*(a)$, and $J^{(l)}(a) \subseteq L^*(a)$. Therefore, $J^{(l)}(a) = L^*(a)$, as required.

$\Leftarrow$) Assume that $\mathcal{J}^{(l)} = \mathcal{L}^{(l)}$ on S. Since $\mathcal{L}^{(l)} \subseteq \mathcal{D}^{(l)} \subseteq \mathcal{J}^{(l)}$, $\mathcal{D}^{(l)} = \mathcal{J}^{(l)}$. Now, applying Theorem 2.2, we can easily see that S is a super rpp semigroup, and so $\mathcal{D}^{(l)}$ is a semilattice congruence on S. This shows that $\mathcal{L}^{(l)}$ is a semilattice congruence on S, and whence S is a left C-rpp semigroup.

The following corollary follows immediately from Theorems 2.2 and 4.1.

Corollary 4.2 — Any left C-rpp semigroup is a super rpp semigroup.

It is known in [11] that a left C-rpp semigroup S satisfies the condition $eS \subseteq Se$ for all $e \in E(S)$, hence $E(S)$ is a left regular band, that is, a band satisfying the identity $xy = xyx$. This shows that the left C-rpp semigroup is a special OS-rpp semigroup.

The following theorem gives a description for the left C-rpp semigroups.

***Theorem* 4.3** — *Let S be an OS-rpp semigroup S. Then S is a left C-rpp semigroup if and only if $E(S)$ is a left regular band.*

PROOF : We only prove the sufficiency because the necessity is trivial. By Theorem 3.8, S is isomorphic to some OS-rpp semigroup $OS(S/\eta, E; \lambda, \rho)$. By the definition of multiplication of weak semi-spined product, S is a semilattice of $T_\alpha \times I_\alpha$ for any $\alpha \in Y$. Since $E = E(S)$ is a left regular band, I_α is a left zero band. Thus S is a semilattice of direct products of a left zero band and a left cancellative monoid. By applying [11, Theorem 2.4], S is a left C-rpp semigroup. □

We now study the right C-rpp semigroups. In fact, right C-rpp semigroups were first studied by Guo in [10]. A *right C-rpp semigroup* is defined as an rpp semigroup S on which $\mathcal{D}^{(l)}$ is a congruence and for every $e^2 = e \in S$, $Se \subseteq eS$. Equivalently , an rpp semigroup S is a right C-rpp semigroup if and only if S is a strongly rpp semigroup on which $\mathcal{D}^{(l)}$ is a semilattice congruence and $\mathcal{D}|_{RegS} = \mathcal{R}|_{RegS}$, where $RegS$ is the set of regular elements of S. The structure of right C-rpp semigroups was later investigated by Shum and Ren in [20]. It is easy to see that any right C-rpp semigroup is a super rpp smigroup. On the other hand, by $Se \subseteq eS$ for all $e \in E(S)$, we can easily see that $E(S)$ is a right regular band, that is, a band satisfying the identity $xy = yxy$. Hence, a right C-rpp semigroup is an $OS - rpp$ semigroup whose band of idempotents is a right regular band.

In comparing with Theorem 4.1, the following theorem shows that a right C-rpp semigroup is not the dual of a left C-rpp semigroup, although it is also a special OS-rpp semigroup. However, we have the following characterization theorem for right C-rpp semigroups.

***Theorem* 4.4** — *Let S be a strongly rpp semigroup. Then S is a right C-rpp semigroup if and only if $\mathcal{J}^{(l)} = \bar{\mathcal{R}}$.*

PROOF : $\Rightarrow$) Suppose that S is a right C-rpp semigroup. Then S is a super rpp semigroup and that $\mathcal{D}|_{RegS} = \mathcal{R}|_{RegS}$. Hence each $\mathcal{D}^{(l)}$-class of S contains exactly one regular $\mathcal{R}$-class. Thus $\mathcal{D}^{(l)} = \overline{\mathcal{R}}$. Now, it follows from Theorem 2.2 that $\mathcal{J}^{(l)} = \overline{\mathcal{R}}$ on S.

$\Leftarrow$) Assume that $\mathcal{J}^{(l)} = \overline{\mathcal{R}}$ on S. Then, since $\overline{\mathcal{R}} \subseteq \mathcal{D}^{(l)} \subseteq \mathcal{J}^{(l)}$, we have $\mathcal{D}^{(l)} = \mathcal{J}^{(l)}$, and so S is a super rpp semigroup. By Theorem 2.2, $\mathcal{D}^{(l)}$ is a semilattice congruence on S. In order to show that S is a right C-rpp semigroup, we still need to prove that $\mathcal{D}|_{RegS} = \mathcal{R}|_{RegS}$. Since $\overline{\mathcal{R}} = \mathcal{J}^{(l)}$, we have $e\overline{\mathcal{R}}f$, for any idempotents e, f in the same $\mathcal{D}^{(l)}$-class, and so $e\mathcal{R}f$ since $e = e^{\dagger}$ and $f = f^{\dagger}$. Hence each $\mathcal{D}^{(l)}$-class contains exactly one regular $\mathcal{R}$-class. Because $\mathcal{D} \subseteq \mathcal{D}^{(l)}$, we have $\mathcal{D}|_{RegS} = \mathcal{R}|_{RegS}$, as required.

***Theorem* 4.5** — *Let S be an OS-rpp semigroup. Then S is a right C-rpp semigroup if and only if $E(S)$ is a right regular band.*

PROOF : We only prove the sufficiency. Let $E(S)$ be a right regular band. Then , by Theorem 2.4, S is a semilattice of Rees matrix semigroups S_α over a left cancellative monoid , for every $\alpha \in Y$. Since S is an OS-rpp semigroup, each S_α is the direct product of a rectangular band and a left cancellative monoid. When $E(S)$ is a right regular band, each $E(S_\alpha)$ is a right zero band and hence $\mathcal{D}^{(l)}|_{RegS_\alpha} = \mathcal{R}|_{RegS_\alpha}$ on S. Since each S_α is precisely one of the $\mathcal{D}^{(l)}$-classes of S, we have $\mathcal{D}^{(l)}|_{RegS} = \mathcal{R}|_{RegS}$. Thus S is a right C-rpp semigroup.

It was noticed by Guo in [10] that an rpp semigroup is a right C-rpp semigroup if and only if it is a semilattice of direct products of a left cancellative monoid and a right zero band. It was also pointed out by Shum and Ren in [20] that right C-rpp semigroups are not the dual of left C-rpp semigroups. Despite of this fact, we still have the following theorem of right C-rpp semigroups which is analogous to left C-rpp semigroups.

***Theorem* 4.6** — *Let S be a strongly rpp semigroup. Then S is a right C-rpp semigroup if and only if $\overline{\mathcal{R}}$ is a semilattice congruence on S.*

Proof : The necessity follows from Theorem 4.4. To prove the sufficiency, we assume that $\overline{\mathcal{R}}$ is a semilattice congruence on S. Then it is clear that $\overline{\mathcal{R}}$ is a left congruence on S, and S is a super rpp semigroup. Now, let S be a semilattice of Rees matrix semigroups $S_\alpha = \mathcal{M}(M_\alpha; I_\alpha, \Lambda_\alpha; P_\alpha)$ for $\alpha \in Y$. Then, by our hypothesis, $\overline{\mathcal{R}}$ is a semilattice congruence on S_α for all $\alpha \in Y$. On the other hand, it is easy to check that $(a, i, \lambda)\overline{\mathcal{R}}(b, j, \mu)$ if and only if $i = j$, for all $(a, i, \lambda), (b, j, \mu) \in S$. Hence, by an easy computation, we can show that $|I_\alpha| = 1$ for all $\alpha \in Y$, since $\overline{\mathcal{R}}$ is a semilattice congruence on S_α. Also, it can be easily checked that every semigroup component S_α is the direct product of a left cancellative monoid and a right zero band, and thereby S is a semilattice of direct products of a left cancellative monoid and a right zero band. Consequently, S is a right C-rpp semigroup. □

5. Primitive Strongly rpp Semigroups

In this section, we investigate the properties of a semigroup which is primitive and is strongly rpp. In what follows, the phrase "primitive strongly rpp semigroup" always means a strongly rpp semigroup being primitive. In fact, a primitive strongly rpp semigroup is always super rpp and has a 0-direct union structure which is similar to a primitive regular semigroup. For the sake of brevity, we simply call a semigroup S which is both primitive and strongly rpp the primitive strongly rpp semigroup. As usual, we let E be the set of all idempotents of S and assume that $0 \in E$ unless it is stated otherwise.

It was noted by [2, Lemma 3.1] that for any idempotents e, f of a primitive semigroup T, $eT \subseteq fT(Te \subseteq Tf)$ implies that $eT = fT(Te = Tf)$ or $e = 0$ if it exists.

We now establish the following lemmas.

Lemma 5.1 — Let $a, b \in S$. Then $ab = 0$ or $ab\mathcal{L}^{(l)}b$.

PROOF : Let $a, b \in S$. If $ab \neq 0$, then by $L_0^* = \{0\}$, $(ab)^\dagger \neq 0$ while $b \neq 0$, so $b^\dagger \neq 0$. Observe that $bb^\dagger = b$. Then $abb^\dagger = ab$. Since $ab\mathcal{L}^{(l)}(ab)^\dagger$, we have $(ab)^\dagger b^\dagger = (ab)^\dagger$, and hence $S(ab)^\dagger \subseteq Sb^\dagger$. Because S is primitive, by using the arguments before Lemma 5.1, we can easily deduce that $S(ab)^\dagger = Sb^\dagger$ and so $(ab)^\dagger\mathcal{L}b^\dagger$. This proves that $ab\mathcal{L}^{(l)}b$.

Lemma 5.2 — Let a, b be elements of S. Then $ab \neq 0$ if and only if for every $x \in D_a^{(l)}$ and $y \in D_b^{(l)}, xy \neq 0$.

PROOF : We only prove the necessity because the sufficiency is trivial. Suppose that $ab \neq 0$. Then by Lemma 5.1, we have $ab\mathcal{L}^{(l)}b$. Now, we claim that $ya \neq 0$ for every $y \in D_b^{(l)}$. In fact, if $ya = 0$ then $yab = 0$. Since $ab\mathcal{L}^{(l)}b$ and $ab \in D_b^{(l)}$, we have $yab \in \overline{R}_y \bigcap L_b^{(l)}$ by Lemma 1.6. This leads to $0 \in L_b^{(l)}$. Hence, by $L_0^{(l)} = \{0\}$, we have $b = 0$, which is a contradiction. Thus $ya \neq 0$ and our claim is established. By applying a similar argument to $ya \neq 0$, we also obtain $xy \neq 0$ for every $x \in D_a^{(l)}$. Thus our lemma is proved.

Lemma 5.3 — S is a super rpp semigroup.

PROOF : It suffices to prove that $\overline{\mathcal{R}}$ is a left congruence . For this purpose, we let $a, x, y \in S$ such that $x\overline{\mathcal{R}}y$. Then, by Lemma 5.2 , $ax = 0$ if and only if $ay = 0$. Consider the following cases:

(i) If $ax = 0$, then $ay = 0$, and hence $ax\overline{\mathcal{R}}ay$.

(ii) If $ax \neq 0$, then $ay \neq 0$. This implies that $ax^\dagger \neq 0$ and $ay^\dagger \neq 0$, by Lemma 5.2. Now, by Lemma 5.1, we have $ax^\dagger\mathcal{L}^{(l)}x^\dagger$ and $ay^\dagger\mathcal{L}^{(l)}y^\dagger$ and so $ax^\dagger\mathcal{D}^{(l)}x\mathcal{D}^{(l)}y\mathcal{D}^{(l)}ay^\dagger$. Applying Lemma 1.6, we have

$$ax = a(x^\dagger x) = (ax^\dagger)x \in \overline{R}_{ax^\dagger}.$$

Because $x\overline{\mathcal{R}}y$, we have $x^\dagger\mathcal{R}y^\dagger$. This leads to

$$ay = a(y^\dagger y) = a(x^\dagger y^\dagger)y = (ax^\dagger)y \in \overline{R}_{ax^\dagger},$$

and hence $ax\overline{\mathcal{R}}ay$. This shows that $\overline{\mathcal{R}}$ is a left congruence on S. Thus S is a super rpp semigroup. □

Lemma 5.4 — Let $a, b \in S$. If $(a, b) \notin \mathcal{D}^{(l)}$, then $ab = 0$.

PROOF : By Lemma 5.3, S is a super rpp semigroup.

Now, by Theorem 2.2, S can be expressed by a semilattice Y of $\mathcal{D}^{(l)}$-simple and strongly rpp semigroups S_α for $\alpha \in Y$, where each S_α is a $\mathcal{D}^{(l)}$-class of S. Let $a, b \in S$ and suppose that $(a, b) \notin \mathcal{D}^{(l)}$. Then for some $\alpha, \beta \in Y$ and $\alpha \neq \beta$, we have $a \in S_\alpha$ and $b \in S_\beta$. Hence $ab \in S_{\alpha\beta}$. If $ab \neq 0$, then, by Lemma 5.1, $ab\mathcal{L}^{(l)}b$ and so $ab \in S_\beta$. This shows that $S_{\alpha\beta} = S_\beta$ and therefore $\alpha\beta = \beta$. On the other hand, by the proof of Lemma 5.2, $ab \neq 0$ implies that $ba \neq 0$. Since $ba \neq 0$, we have $ba\mathcal{L}^*a$ and so $\alpha\beta = \alpha$. Thus $a\mathcal{D}^{(l)}b$, a contradiction. Therefore, $ab = 0$ and the proof is completed.

We are now ready to give a description for the structure of primitive strongly rpp semigroups.

***Theorem* 5.5** — *A semigroup S is a primitive strongly rpp demigroup if and only if S is a 0-direct union of some semigroups S_α^0, where each S_α is either a Rees matrix semigroup over a left cancellative monoid or is just* $\{0\}$.

Proof : The necessity follows from Theorem 2.4, Lemma 5.2, Lemma 5.4 and the fact that $D_0^{(l)} = \{0\}$.

For the sufficiency, it is easy to see that every S_α is a strongly rpp semigroup. Hence, we can easily see that the whole semigroup S is a strongly rpp semigroup. On the other hand, $RegS_\alpha$ forms a completely simple semigroup (Lemma 1.4). Because $RegS = \bigcup RegS_\alpha$, $RegS$ is the 0-direct union of $(RegS_\alpha)^0$. It follows immediately that S is a primitive semigroup (see, [12, pp. 71]howie). Therefore S is a primitive semigroup as well as a strongly rpp semigroup. □

Finally, the following corollary is a direct consequence of Theorem 5.5.

Corollary 5.6 — A semigroup S without zero is a primitive strongly rpp semigroup if and only if it is isomorphic to some Rees matrix semigroup over a left cancellative monoid.

It has already been mentioned by M. Petrich in [16] that a congruence on a completely regular semigroup can be completely determined by its kernel and its trace instead of using its idempotent classes, this is because the structure of a completely regular semigroup has been described explicitly and its structure has been known transparently. It is well known that a semigroup is a completely regular semigroup if and only if it is a semilattice of Rees matrix semigroups over groups. By Theorem 2.4, The super rpp semigroups have similar structure. Hence, It is natural to ask : Can we extend the results of Petrich for congruences on a completely regular semigroup (see [16]) to a super rpp semigroup by using the kernel-trace approaches?

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