Indian J. Pure Appl. Math., **41**(3): 535-553, June 2010

THE KATĚTOV-MORITA THEOREM FOR THE DIMENSION OF METRIC FRAMES

D. Brijlall and D. Baboolal

School of Science, Mathematics and Technology Education
and
School of Mathematical Sciences
University of KwaZulu-Natal, Private Bag X54001
Durban 4000, *South Africa*
e-mail : brijlalld@ukzn.ac.za; baboolald@ukzn.ac.za

(*Received* 18 *December* 2008; *after final revision* 30 *April* 2010; *accepted* 5 *May* 2010)

The results which appear here are devoted to the dimension theory of metric frames. We begin by characterizing the covering dimension dim of metric frames in terms of special sequences of covers and then prove the fundamental Katětov-Morita Theorem asserting that $\operatorname{Ind} L = \dim L$ for every metric frame L.

Next, we establish two characterizations of the dimension function Ind in metric frames, one in terms of special bases and another in terms of decompositions into subspaces of dimension zero. These characterizations yield a sum theorem.

Key words : Dimension, metric frames.

1. Introduction

In [1] the authors make a study of aspects of dimension theory in the context of frames or point-free topology. In this paper we concentrate on the class of metric frames, which are the point-free analog of metric spaces, and in this context we obtain various results on dimension, central among these being the analog of the Katětov-Morita Theorem for metric spaces X which assert that Ind $X=$ dim X.

Crucial to the proof of the above theorem is the frame analog of the following result for spaces, viz. that for every metrizable space X the following conditions are equivalent:

- The space X satisfies the inequality $\dim X \leq n$.
- For every metric ρ on the space X there exists a sequence $U_1, U_2, \ldots$ of locally finite open covers of the space X such that ord $U_i \leq n, \delta(U) < \frac{1}{i}$ for $U \in U_{i,}$ and for each $U \in U_{i+1}$ there exists a $V \in U_i$ that contains $\overline{U}$.
- There exists a metric ρ on the space X and a sequence $W_1, W_{2,} \ldots$ of open covers of the space X such that ord $W_i \leq n$, $\delta(W) < \frac{1}{i}$ for $W \in W_i$ and W_{i+1} is a refinement of W_i.

The proof of this result in the frame context is considered in Theorem 4.3. This proof follows a slight adaptation of an argument in the book by Engelking [5] which is the source we follow in developing the theory of dimension for metric frames. In this book can be found also the references to authors whose work our development depends on. We do not reference these authors individually in this paper but we give [5] as the general reference.

The theorem that for every metrizable space X we have Ind $X = \dim X$ was proved by Katětov and by Morita. Our frame counterpart theorem of this result is one of our main theorems and it appears as Theorem 5.2. It is well known also that for general metric spaces we do not have equality of the dimension functions Ind, ind and dim. However, as is also well known, we do have equality for the class of separable metric spaces. This result for spaces has its counterpart for separable metric frames, and this appears as Theorem 5.5.

2. Preliminary Concepts

A *frame* is a complete lattice L in which the infinite distributive law $a \wedge \bigvee S = \bigvee\{a \wedge s | s \in S\}$ holds for any $a \in L$ and $S \subseteq L$. The term 'complete' as used above means that all subsets of L have a meet; hence also a join. A *frame homomorphism* is a map $h : L \to M$ of frames L, M preserving all finite meets, including the unit e_L, and arbitrary joins, including the zero, 0_L. Thus we have the category **Frm** of frames and their homomorphisms. We call a frame homomorphism h *dense* if $h(x) = 0$ implies $x = 0$. Any frame homomorphism $h : L \to M$ has a *right adjoint* $h_* : M \to L$ defined by the condition

$$h(x) \leq y \text{ iff } x \leq h_*(y) \text{ for all } x \in L \text{ and } y \in M,$$

where $h_*(y) = \bigvee\{x \in L | h(x) \leq y\}$. In particular, if h is onto then $h_*(y)$ is the largest element which h maps to y. We have that h_* preserves arbitrary meets. For $B \subseteq L$, B is called a *basis* for L if $a \in L$ implies $a = \bigvee_{b \in B'} b$ for some $B' \subseteq B$. Also, for $x \in L$. The *pseudocomplement* of x (denoted by x^*) will be $x^* = \bigvee(y \in L | y \wedge x = 0_L)$. If $X \subseteq L$, X is a *cover* if $\bigvee X = e_L$. *A cover X for a frame L is called a σ-locally finite one if $X = \bigcup_{i=1}^{\infty} X_i$ with X_i locally finite for each i.* A cover X is called a *co-zero* cover if the elements of X are *co-zero* elements, that is to say elements of the form $f(\mathbb{R}\backslash\{0\})$ for some frame map $f : \mathcal{O}\mathbb{R} \longrightarrow L$. Here $\mathcal{O}\mathbb{R}$ is the frame of open sets of the real line. For $X, Y \subseteq L$ where X and Y are covers for L. X *refines* Y (we write $X \leq Y$) if for each $x \in X$ there exists $y \in Y$ such that $x \leq y$. Let L be a frame and $X \subseteq L$. L is called *compact* if $e_L \leq \bigvee X$ implies $e_L \leq \bigvee E$, for some finite $E \subseteq X$. If $a, x \in L$, we write $a \lhd x$ to mean there exists a cover A such that $Aa \leq x$, where $Aa = \bigvee\{y \in A | y \wedge a \neq 0_L\}$. For $J \subseteq L$. J is called an *ideal* if $F \subseteq J$, F finite implies $\bigvee F \in J$ and $a \leq b \in J$ implies $a \in J$. For a frame L and $a \in L$ we shall use $\uparrow a$ to denote $\{x \in L | x \geq a\}$ and $\downarrow a = \{x \in L | x \leq a\}$. A principal example of a frame is the frame $\mathcal{O}X$ of open sets of a topological space X, and the frame homomorphism $\mathcal{O}f : \mathcal{O}Y \to \mathcal{O}X$, induced by any continuous map $f : X \to Y$ between topological spaces, taking $U \in \mathcal{O}Y$ to $f^{-1}(U) \in \mathcal{O}X$. The resulting correspondence constitutes a contravariant functor $\mathcal{O} : \textbf{Top} \to \textbf{Frm}$ from the category **Top** of topological spaces and continuous maps to the category **Frm**. On the other hand, there is a contravariant functor $\Sigma : \textbf{Frm} \to \textbf{Top}$ such that for any frame L, ΣL is the space of all frame homomorphisms $\xi : L \to \mathbf{2}$ ($\mathbf{2}$ is the two-element frame $\{0, 1\}$), with open sets $\Sigma_a = \{\xi \in \Sigma L | \xi(a) = 1\}$ for $a \in L$. For any frame homomorphism $h : L \to M$, $\Sigma h : \Sigma M \to \Sigma L$ acts by composition with h, that is, $(\Sigma h)(\xi) = \xi h$. Moreover, $\mathcal{O}$ and Σ are adjoint on the right, with adjunction maps $\eta_L : L \to \mathcal{O}L$ given by

$$\eta_L(a) = \Sigma_a$$

and $\varepsilon_x : X \to \mathcal{O}X$ given by

$$\varepsilon_x(x) = \tilde{x}$$

where $\tilde{x} : \mathcal{O}X \to \mathbf{2}$ is such that $\tilde{x}(U) = 1$ iff $x \in U$ for all $U \in OX$.

For elements x and y in L, x is said to be *rather below* y, written $x \prec y$ if there exists an element $t \in L$ such that $x \wedge t = 0$ and $y \vee t = e$. L is called a *regular* frame whenever $a = \bigvee\{x \in L | x \prec a\}$ for all $a \in L$.

Anologously, for elements x and y in L, x is said to be *completely below* y, written $x \prec\prec y$ if there exists $(\mathrm{c}_{ik}), i = 0, 1, 2, ...k = 0, 1, 2, ..., 2^i$ with

$$a = c_{i0},\ b = c_{i2^i},\ c_{ik} \prec c_{i,k+1}$$

and

$$c_{ik} = c_{i+1,2k}$$

for all i, k. L is called a *completely regular* frame whenever $a = \bigvee\{x \in L | x \prec\prec a\}$ for all $a \in L$. Let L be a frame. L is called *normal* if for $a, b \in L$ with $a \vee b = e_L$ there exists $c, d \in L$ such that $c \wedge d = 0_L$ and $a \vee c = e_L = b \vee d$. Any compact regular frame is normal. For normal regular frames, $\prec$ interpolates. A *compactification* of a frame L is a dense onto homomorphism h:M$\to L$ where M is a compact regular frame.

It would be convenient in the latter half of the paper to express our results in the language of locales rather than of frames. These objects form a category **Loc** which is the opposite category of **Frm**. We refer the reader to [8] for general facts about locales.

3. The Dimension Functions to be Encountered

In this section we recall from [7] the dimension functions *Ind* and *dim*, and from [1] the dimension function *ind*. For more information and background to these functions we refer the reader to [1]. For L a regular frame and n a non-negative integer, we say that:

MU1 : ind $L = -1$ if and only if $L = 1 = \{0_L\}$

MU2 : ind $L \leq n$ if for every cover V of L there exists a cover U of L such that $U \leq V$ and for each $u \in U$, ind $(\uparrow (u^* \vee u)) \leq n - 1$.

MU3 : ind $L = n$ if ind $L \leq n$ and the inequality ind $L \leq n - 1$ does not hold.

MU4 : ind $L = \infty$ if the inequality ind $L \leq n$ does not hold for any n.

The number ind L will be called the Menger-Urysohn dimension for the frame L or the small inductive dimension of L. If L is a normal frame and n a non-negative integer, we say that

BC1: Ind $L = -1 \Leftrightarrow L = 1$

BC2: Ind $L \leq n \Leftrightarrow$ for every $a \in L$ and any $v \in L$ such that $a \vee v = e_L$ there exists $u \leq v$ such that $a \vee u = e_L$ and Ind $(\uparrow (u \vee u^*)) \leq n - 1$.

BC3: Ind $L = n$ if Ind $L \leq n$ and Ind $L \leq n - 1$ does not hold for any n.

BC4: Ind $L = \infty$ if Ind $L \leq n$ does not hold for any n.

The number Ind L is called the Brouwer-Čech dimension for frames. For a completely regular frame L and $n \geq -1$, $n \in \mathbb{Z}$ we say that

CL1: dim $L \leq n$ if every finite co-zero cover for L has a finite co-zero refinement of *order* $\leq n$.

CL2: dim $L = n$ if dim $L \leq n$ and dim $L \leq n - 1$ does not hold.

CL3: dim $L = \infty$ if dim $L \leq n$ does not hold for any n.

The number dim L is called the Čech-Lebesgue dimension for frames.

4. Introductory Results on Metric Frames

The following definitions appear in [10] and [11]. The set of all non-negative reals augmented by $+\infty$ will be represented by $\mathbb{R}^+$.

Definition 4.1 — Let L be a frame. A metric diameter on L is a map $d : L \to \mathbb{R}^+$ such that:

MD1 : $d(0_L) = 0$

MD2: If $a \leq b$ *then* $d(a) \leq d(b)$.

MD3 : If $a \wedge b \neq 0$ *then* $d(a \vee b) \leq d(a) + d(b)$.

MD4 : Each $D^d_\epsilon = \{x \in L | d(x) < \epsilon\}$ *is a cover for* L.

MD5 : For each $a \in L$ *and each* $\epsilon > 0$ *there exists* $u, v \leq a$ *such that* $d(u), d(v) < \epsilon$ *and* $d(a) < d(u \vee v) + \epsilon$.

MD6 : For each $a \in L, a = \bigvee x(x \lhd_d a)$, *where* $x \lhd_d a$ *means there exists* D^d_ϵ *such that* $D^d_\epsilon x \leq a$.

Definition 4.2 — A metric frame is a pair (L, d) with L a frame and d a metric diameter on L.

***Theorem* 4.3** — *For every metric frame* (L, d) *the following conditions are equivalent:*

(i) $\dim L \leq n$.

(ii) There exists a sequence $U_1, U_2 \dots$ *of locally finite covers of* L *such that ord* $U_i \leq n$, $d(u) < \frac{1}{i}$ *for every* $u \in U_i$, *and for each* $u \in U_{i+1}$ *there exists a* $v \in U_i$ *with* $u \prec v$.

(iii) There exists a sequence $W_1, W_2, \dots$ *of covers of* L *such that ord* $W_i \leq n$;
$d(w) < \frac{1}{i}$ *for every* $w \in W_i$, *and each* W_{i+1} *is a refinement of* W_i.

Proof : The implication (ii) implies (iii) follows from the definition of refinement.

For (i) implies (ii): The sequence $U_1, U_2, \dots$ will be defined by induction. We first define U_1. Since every metric frame is paracompact, the cover D^d_1 has a locally finite refinement V, say. By [3] V has a locally finite shrinking U_1 which has order $\leq n$. This defines U_1. Now assume that $k > 1$ and the covers U_i are already defined for $i < k$. Since L is regular, $u \in U_{k-1}$ implies $u = \bigvee t(t \prec u)$. For each t, consider the cover $D^d_{\frac{1}{k}}$, $t = \bigvee(t \wedge x)(x \in D^d_{\frac{1}{k}})$. Now call

$z = t \wedge x$, then $z = t \wedge x \leq t \prec u$, so $z \prec u$ and $d_L(z) = d_L(t \wedge x) \leq d_L(x) < \frac{1}{k}$. Let $U'_{\frac{i}{k}} = \{z\}$. But then, since L is metric and hence paracompact, $U'_{\frac{1}{k}}$ has a locally finite refinement and by [3], we get a locally finite shrinking U_k which has order $\leq n$. The sequence $U_1, U_2, \ldots$ obtained in this way satisfies (ii).

For (iii) implies (i). Take for every i a mapping $f_i^{i+1} : W_{i+1} \to W_i$ with $w \leq f_i^{i+1}(w)$, for all, $w \in W_{i+1}$.

Call $f_i^k = f_i^{i+1} \circ f_{i+1}^{i+2} \circ \ldots \circ f_{k-1}^k$, for $i < k$ and let $f_i^i : W_i \to W_i$ be defined as $f_i^i(w) = w$ for all $w \in W_i$.

For every $w \in W_k$ and $i \leq k, w \leq f_k^k(w)$ follows from $f_i^k = f_i^{i+1} \circ f_{i+1}^{i+2} \circ \ldots \circ f_{k-1}^k$. Let $\{h_j\}_{j=1}^l$ be any finite cover for L. Consider the collection $\{x_i\}_{i=1,}^{\infty}$ where $x_k = \bigvee_k \{w \in W_k |$ there exists $j \leq l$ such that $W_k w \leq h_j\}$.

We claim that this collection is a cover for L. To prove this for each $h_j, (1 \leq j \leq l), h_j = \bigvee x(x \lhd h_j)$. Take any $x \lhd h_j$. This implies there exists $\epsilon > 0$ such that $D_\epsilon x \leq h_j$. We can choose $i \in \mathbb{N}$ with $\frac{1}{i} < \epsilon$. Since W_{2i} is a cover for L there exists $w \in W_{2i}$, such that $x \wedge w \neq 0_L$. Take $w^1 \in W_{2i}$, with $w^1 \wedge w \neq 0_L$. Hence, $d(w^1 \vee w) \leq d(w^1) + d(w) < \frac{1}{2i} + \frac{1}{2i} = \frac{1}{i} < \epsilon$ implying $w^1 \vee w \in D_\epsilon$ and $(w^1 \vee w) \wedge x \neq 0_L$ so that $w^1 \vee w \leq h_j$ and thus $w^1 \leq h_j$ whence $W_{2i} w \leq h_j$. But,

$$\begin{aligned} x &= \bigvee (x \wedge w)(x \wedge w \neq 0_L, w \in W_{2i}) \\ &\leq \bigvee w(x \wedge w \neq 0_L, w \in W_{2i}) \\ &\leq x_{2i}. \end{aligned}$$

Also,

$$\begin{aligned} e_L &= h_1 \vee h_2 \vee \ldots \vee h_l \\ &= \bigvee x(x \lhd h_1) \vee \bigvee x(x \lhd h_2) \vee \ldots \bigvee x(x \lhd h_l) \end{aligned}$$

Therefore,

$$\begin{aligned} e_L &= \bigvee x(x \lhd h_j, \text{ for some } j \leq l) \\ &\leq \bigvee_i x_{2i} \end{aligned}$$

and the claim is proved by renaming of subscripts.

Consider the subfamilies $U_k = \{u \in W_k | u \wedge x_k \neq 0_L\}$ and $V_k = \{v \in U_k | v \wedge \left(\bigvee_{j<k} x_j \right) = 0_L\}$ of the cover W_k, where $k \in \mathbb{N}$. For every $u \in U_k$,

we have $f_k^k(u) \wedge x_k = u \wedge x_k \neq 0_L$; denote by $i(u)$ the smallest integer $i \leq k$ such that $f_i^k(u) \wedge x_i \neq 0_L$. Since,

$$\begin{aligned} V_1 &= \{v \in U_1 | v \wedge 0_L = 0_L\} \\ &= U_1 \end{aligned}$$

and $f_{i(u)}^k(u) \leq f_{i(u)-1}^k(u) \leq \ldots \leq f_1^k(u)$ whenever $i(u) > 1$, we have $f_{i(u)}^k(u) \wedge x_j = 0_L$ for $j < i(u)$ and so $f_{i(u)}^k(u) \in V_{i(u)}$. For every $v \in V_i$, call

$$v^1 = \bigvee_{k=1}^{\infty} (\bigvee \{u \wedge x_k | u \in U_k, f_i^k(u) = v \text{ and } i(u) = i\}$$

As $f_i^k(u) \wedge x_i \neq 0_L$, from the definition of V_i, $v \wedge x_i \neq 0_L$. Also, there exists $w \in W_i$ such that $v \wedge w \neq 0_L$ and a $j(v) \leq l$ satisfying $v \leq W_i w \leq h_{j(v)}$ from the definition of x_i. Now, for each $k, u \wedge x_k \leq u \leq f_i^k(u) = v$ and so, $v^1 \leq v \leq h_{j(v)}$. Whenever $i \neq j$ $V_i \bigcap V_j = \phi$ for we may assume $i < j$ and for every $s \in V_i, s \wedge x_i \neq 0_L$ and so $s \wedge \bigvee\limits_{p<j} x_p \neq 0_L$ implying $s \notin V_j$ and vice versa.

Thus, for every $v \in V = \bigcup\limits_{i=1}^{\infty} V_i$, the element v^1 and the integer $j(v)$ are well defined.

To conclude the proof it suffices to show that the family $\{v_j\}_{j=1}^l$ where $v_j = \bigvee \{v^1 | v \in V \text{ and } j(v) = j) \leq h_j$, or equivalently that $V^1 = \{v^1 | v \in V\}$ is a cover for L has order $\leq n$. But for each $k, x_k = \bigvee \{u \wedge x_k | u \in W_k\}$ so $x_k \leq \bigvee \{u \wedge x_k | u \wedge x_k \neq 0_L\} = \bigvee \{u | u \in U_k\}$. Hence, $\bigvee\limits_{k=1}^{\infty} (\bigvee U_k) = e_L$ and so $U = \bigcup\limits_{k=1}^{\infty} U_k$ is a cover for L. So let $u \in U_k$, then $u \wedge x_k \neq 0_L$ implying $f_i^k(u) \wedge x_i \neq 0_L$ for some $i \leq k$. Call $f_i^k(u) = v$ so, $u \wedge x_k \leq v^1$ and so $\bigvee (u \wedge x_k) \leq \bigvee v^1$, but $x_k \leq \bigvee (u \wedge x_k | u \wedge x_k \neq 0_L)$ implies $x_k \leq \bigvee v^1$ for all k so that $\bigvee\limits_{k=1} x_k \leq \bigvee v^1$ and V^1 is a cover.

For $v_i \in V_{m_i}$ consider $v_1^1 \wedge v_2^1 \bigwedge \ldots \bigwedge v_q^1 \neq 0_L$, where $v_i \in V_{m_i}$ and $v_i \neq v_j$ whenever $i \neq j$. But, for each v_i^1 there exists $u_i, i \in \{1, 2, \ldots q\}$ such that $u_i \in U_{k_i}, f_{m_i}^{k_i}(u_i) = v_i, i(u_i) = m_i$. The elements $w_1, w_2, \ldots w_q$ where $w_i = f_k^{k_i}(u_i) \in W_k$.

Also, $0_L \neq v_1^1 \wedge v_i^1 \wedge \ldots \wedge v_q^1 \leq v_i^1 \leq v_i = f_k^{k_i}(u_i)$. So, $w_1 \wedge w_2 \wedge \ldots w_q \neq 0_L$. As ord $W_k \leq n$, we need to show that $w_i \neq w_j$ for $i \neq j$. Now, $w_i \in W_k$

and $i(w_i) = i(u_i) = m_i$ by the definition of $i(u_i)$; whenever $i \neq j, m_i \neq m_j$ and $v_i \neq v_j$ or $f_k^{k_i}(u_i) \neq f_k^{k_i}(u_j)$ so $w_i \neq w_j$. When $m_i = m_j, f_{m_i}^k(w_i) = f_{m_i}^{k_i}(u_i) = v_i \neq v_j = f_{m_j}^{k_j}(u_j) = f_{m_j}^{k_j}(w_j)$.

Hence, $\{v_i^1\}_{i-1}^a$has order $\leq n$ and refines $\{h_j\}_{j=1}^l$ and $\dim L \leq n$.

Lemma 4.4 — [1] Let (L, d) be a metric frame and let $S \subseteq D_\epsilon^d$ be such that $a \leq \bigvee S$. Then there exists $x, y \in S$ such that $d(x \vee y) > d(a) - \epsilon$.

PROOF : Now, for $x, y \in S$, $x \vee y \leq \bigvee S$ so that $d(x \vee y) \leq d(\bigvee S)$ by MD2. Thus, $d(\bigvee S) - \epsilon < d(x \vee y)$, for $\epsilon > 0$. But, $a \leq \bigvee S$ implies $d(a) - \epsilon \leq d(\bigvee S) - \epsilon < d(x \vee y)$.

5. The Main Result

Remark 5.1 : Let ρ be the usual diameter on $\mathcal{O}\mathbb{R}$ and $f : \mathcal{O}\mathbb{R} \to L$ a frame homomorphism. Define $\bar{d} : L \to \mathbb{R}^+$ by $\bar{d}(a) = \inf\{\rho(u) | a \leq f(u)\}$. Pultr [10], has proved that $\bar{d}$ is a metric diameter on L.

***Theorem* 5.1** — *For every metric frame (L, d) we have that $Ind\, L = \dim L$.*

PROOF : By [1] it suffices to show that $\operatorname{Ind} L \leq \dim L$; obviously, we can suppose that $\dim L < \infty$. We shall apply induction with respect to $\dim L$. If $\dim L = -1$, $L = \mathbf{1}$ and Ind $L = -1$ so that Ind $L = \dim L$. Assume that our inequality holds for all metric frames with $\dim L \leq n - 1$ and consider a metric frame L such that $\dim L = n$. Let $b \in L$ with $a \vee b = e_L$, we may assume $a \neq e_L$. By Urysohn's lemma we have $f : \mathcal{O}\mathbb{R} \to L$ a frame map with $f(\mathbb{R}\backslash\{0\}) \leq a$ and $f(\mathbb{R}\backslash\{1\}) \leq b$.

By Theorem 4.3 there exists a sequence $U_1, U_2, \ldots$ of locally finite covers of the frame L such that ord $U_i \leq n, d(u) < \frac{1}{i}$, for $u \in U_i$ and for any $u \in U_{i+1}$ there exists a $v \in U_i$ with $u \prec v$.

Let

$$\begin{aligned} h_0 &= a, g_0 = b, \\ g_i &= \bigvee\{u \in U_i | u^* \vee g_{i-1} = e_L\} \qquad \text{and} \\ h_i &= \bigvee\{u \in U_i | u^* \vee g_{i-1} \neq e_L\}. \end{aligned}$$

In this way two sequences $h_0, h_1, h_2, \ldots$ and $g_0, g_1, g_2,$ are defined.

Let us observe that if $u \in U_i$ and $u^* \vee g_{i-1} \neq e_L$ then $u^* \vee h_{i-1} = e_L$.

The validity of this follows by induction : For $i = 1$, consider $d^1 = d + \bar{d}$ where $\bar{d}$ is the previously defined metric diameter from the above remark. We have thus $u \in L, u^* \vee b \neq e_L$ and $d^1(u) < 1$ since $u \in U_1$. Following from this we deduce that $\bar{d}(u) < 1$ implying there exists $u_1 \in \mathcal{O}\mathbb{R}$ such that $\bar{d}(u) \leq \rho(u_1) < 1,\ u \leq f(u_1)$. Since $\rho(u_1) < 1$ either the distance from the real number 0 to the set u_1 is greater than 0 or the distance from the real number 1 to u_1 is greater than 0. If the former is the case then $0 \notin \bar{u_1}$, i.e. $\bar{u_1} \subseteq \mathbb{R}\backslash\{0\}$ and hence $u_1 \prec \mathbb{R}\backslash\{0\}$. If the latter is the case then $u_1 \prec \mathbb{R}\backslash\{1\}$. This then implies $f(u_1) \prec f(\mathbb{R}\backslash\{0\}) \leq a$ or $f(u_1) \prec f(\mathbb{R}\backslash\{1\}) \leq b$. Therefore, $u \prec a$ or $u \prec b$. But $u^* \vee b \neq e_L$ implies then that $u^* \vee a = e_L$ or $u^* \vee h_0 = e_L$.

Suppose that for $i < j$, if $u \in U_i$ and $u^* \vee g_{i-1}, \neq e_L$ then $u^* \vee h_{i-1} = e_L$. Let $u \in U_j$ and $u^* \vee g_{j-1} \neq e_L$. Then for any $v \in U_{j-1}$ with $u \prec v, v^* \vee g_{j-1} \neq e_L$ (otherwise $v \prec g_{j-1}$ and so $u \prec g_{i-1}$ and $u^* \vee g_{j-1} = e_L$). By the inductive assumption then $v^* \vee h_{j-1} = e_L$ and hence $u \prec h_{j-1}$ and $u^* \vee h_{j-1} = e_L$.

At this stage we recall a result by Sun [12]: Let $\{x_i, i \in J\}$ be locally finite and let $x_i \prec y_i$, for all $i \in J$. Then we have $\bigvee\{x_i, i \in J\} \prec \bigvee\{y_i, i \in J\}$.

Hence, from the local finiteness of U_i together with the above result, the definitions of g_i and h_i, we have

$$\begin{aligned} g_i &= \bigvee\{u \in U_i | u^* \vee g_{i-1} = e_L\} \\ &= \bigvee\{u \in U_i | u \prec g_{i-1}\} \\ &\prec g_{i-1} \end{aligned}$$

Also ,

$$\begin{aligned} h_i &= \bigvee\{u \in U_i | u^* \vee g_{i-1} \neq e_L\} \\ &= \bigvee\{u \in U_i | u^* \vee h_{i-1} = e_L\} \\ &= \bigvee\{u \in U_i | u \prec h_{i-1}\} \\ &\prec h_{i-1} \end{aligned}$$

Call $g = \bigvee_{i=1}^{\infty} g_i$. Now, $g_i \prec b = g_0$, for all $i \in \mathbb{N}$, so that $g = \bigvee_{i=1}^{\infty} g_i \leq b$ and $e_L = \bigvee_{i=1}^{\infty} h_i \vee \bigvee_{i=1}^{\infty} g_i \leq a \vee \bigvee_{i=1}^{\infty} g_i = a \vee g$. Let for each i, $l_i = g_i \wedge h_i$ and $l = \bigwedge_{i=1}^{\infty} l_i$. Now, for each i, $\bigwedge_{i=1}^{\infty} l_i \leq l_i = g_i \wedge h_i \leq g_i \leq g \leq g \vee g^*$ and therefore $\uparrow (g \vee g^*) \subseteq \uparrow l$. To conclude the proof it suffices to show that $\dim(\uparrow l) \leq n-1$

since by Theorem 4.9 of [1] and the inductive assumption Ind $\uparrow (g \vee g^*) \leq$ Ind $\uparrow l \leq \dim \uparrow l \leq n-1$.

Consider the family $W_i = \{u \wedge \uparrow l | u \in U_i, u^* \vee g_{i-1} \neq e_L\}$, where $u \vee \uparrow l = \{u \vee x | l \leq x\}$. For each i, and each $u \in U_i$, $l \leq u \vee l$ and so $\uparrow (u \vee l) \subseteq \uparrow l$ and $\bigvee W_i = e_L$ implying W_i is a cover for $\uparrow l$. Further for any $x \in \uparrow l$, there is at least one $u \in U_i$, with , $u^* \vee g_{i-1} = e_L$. Otherwise $u^* \vee g_{i-1} \neq e_L$ for all $u \in U_i$ and so $u^* \vee h_{i-1} = e_L$ for all $u \in U_i$ implying $e_L = \bigvee_{u \in U_i} u \leq \bigvee_i h_{i-1} \leq a$ but we have assumed $a \neq e_L$. Also, ord $U_i \leq n$ means that ord $W_i \leq n-1$, for each i. Now let $y \in W_{i+1}$. Then $y = u \vee x$ for $u \in U_{i+1}, l \leq x$ and $u^* \vee g_i \neq e_L$. For any $v \in U_i$, with $u \prec v$ we have $v^* \vee g_i \neq e_L$. Also, if $v^* \vee g_{i-1} = e_L$, then $v \leq g_i$ by definition of g_i implying $u \prec v \leq g_i$ which is not true.

Therefore, $v^* \vee g_{i-1} \neq e_L$ and so $x \vee u \leq x \vee v$ and $x \wedge v \in W_{i+1}$. Hence W_{i+1} is a refinement of W_i and $d^1(w) < \frac{1}{i}$ for $w \in W_i \subseteq U_i$. Thus by Theorem 4.3, dim $\uparrow l \leq n-1$ and we are done.

Definition 5.3 — ([4]) Let L be a frame and $S \subseteq L \backslash \{e_L\}, S$ countable. L is said to be *separable* if for each $x \in L,\ x \neq 0_L$, there exists $s \in S$ such that $x \vee s = e_L$.

Remark 5.6 : We call S in the above definition a separator for L. It is shown by Dube [4] that Urysohn's Metrization Theorem for Locales holds, namely that a metric frame L is separable if and only if it has a countable base.

***Theorem* 5.5** — *For every separable metric frame* (L, d) *we have ind* $L =$ *Ind* $L = \dim L$.

PROOF : The proof follows from the above remark, Proposition 4.8 of [1] and Theorem 5.2.

Remark 5.6 : For a separable metric frame L, we observe from the above theorem, [1] and [10] that ind $L = \dim \beta L$.

The following result, The Countable Sum Theorem, was proved in [1]: If a normal frame L has a countable collection $\{a_j\}_{j=1}^n$ of elements with $\bigcap_{j=1}^{\infty} \nabla_{a_j} = \triangle$ and dim $\uparrow a_j \leq n$ for all j, then $\dim L \leq n$.

***Theorem* 5.7** — *If a metric frame* (L, d) *has a* $\sigma-$*locally finite cover*

$\{a_s\}_{s\in S}$ *such that Ind* $(\uparrow a_s) \leq n$ *for every* $s \in S$, *then Ind* $L \leq n$.

PROOF : Since by [10], a metric frame is normal, the proof follows from The Countable Sum Theorem and the Katětov-Morita theorem for frames.

Remark 5.8 : We shall now prove a theorem characterizing the Brouwer-Čech dimension Ind of metric frames in terms of special bases. Before that, let us prove the following result which is required for the proof of Lemma 5.10.

Lemma 5.9 — Let L be a frame and $U = \{u_i\}_{i\in I} \subseteq L$ a locally finite collection with u_i complemented for all i. Then $u = \bigvee_{i\in I} u_i$ is complemented.

PROOF : Since U is a locally finite collection, let A be a cover for L such that for each $a \in A$ there exists finite $F_a \subseteq U$ such that $a \wedge u_i \neq 0_L$ for $u_i \in F_a$ and $a \wedge u_i = 0_L$ for $u_i \notin F_a$. Call $x_a = \bigvee F_a$. Then by the finiteness of F_a, x_a is complemented. Let $u = \bigvee_i u_i$.

Now, to show that $u \vee u^* = e_L$ or $u \vee (\bigvee u_i)^* = e_L$. Since L is a frame this is good as proving $u \vee \bigwedge_i u_i^* = e_L$. The meet $\bigwedge_i u_{i^*}$ can be broken down as

$$\bigwedge_i u_i^* \quad = \quad \bigwedge u_i^*(a \wedge u_i \neq 0_L) \wedge \bigwedge u_i^*(a \wedge u_i = 0_L).$$

Here, for all i with $a \wedge u_i = 0_L$, $a \leq u_i^*$ and so $a \leq \bigwedge u_i^*$. Also, $x_a^* = \bigwedge u_i^*(a \wedge u_i \neq 0_L)$ and so,

$$\begin{aligned}
a \wedge x_a^* \quad & \leq \quad \bigwedge_i u_i^*(a \wedge u_i = 0_L) \wedge \bigwedge_i u_i^*(a \wedge u_i \neq 0_L) \\
\text{or} \quad a \wedge x_a^* \quad & \leq \quad u \vee \bigwedge_i u_i^* \qquad \text{implying} \\
a \wedge x_a^* \quad & \leq \quad x_a^* \wedge (u \vee \bigwedge u_i^*) \\
& = \quad (u \wedge x_a^*) \vee (\bigwedge_i u_i^* \wedge x_a^*) \\
& \leq \quad (u \wedge x_a^*) \vee (\bigwedge_i u_i^* \wedge x_a) \vee (\bigwedge_i u_i^* \wedge x_a^*)
\end{aligned}$$

Hence,

$$\begin{aligned}
a = a \wedge e_L \quad & = \quad a \wedge (x_a \vee x_a^*) \\
& = \quad (a \wedge x_a) \vee (a \wedge x_a^*) \\
& \leq \quad x_a \vee (u \vee x_a^*) \vee (\bigwedge_i u_i^* \wedge x_a) \vee (\bigwedge_i u_i^* \wedge x_a^*)
\end{aligned}$$

Since $x_a \leq u$, we have then that

$$\begin{aligned} a &\leq [(u \wedge x_a) \vee (u \vee x_a^*)] \vee (\bigwedge_i u_i^* \wedge x_a) \vee (\bigwedge_i u_i^* \wedge x_a^*) \\ &= (u \wedge (x_a \vee x_a^*)) \vee (\bigwedge_i u_i^* \wedge (x_i \vee x_a^*)) \\ &= u \vee \bigwedge_i u_i^*, \text{ for all } a \in A. \end{aligned}$$

Hence, $\bigvee_{a \in A} a \leq u \vee \bigwedge_i u_i^*$ and so $e_L = u \vee \bigwedge_i u_i^* = u \vee (\bigvee u_i)^* = u \vee u^*$ and u is complemented.

Lemma 5.10 — If a metric frame (L, d) has a σ-locally finite base B consisting of complemented elements, then Ind $L \leq 0$.

Proof : Let $B = \bigcup_{i=1}^{\infty} B_i$, where the families B_i are locally finite. Consider $a \in L$ and $v \in L$ with $a \vee v = e_L$. For $i = 1, 2, 3, \ldots$ let $v_{2i} = \bigvee(u \in B_i | u \leq v)$ and $v_{2i+1} = \bigvee(u \in B_i | u \leq a)$.

Now, B a basis would imply $v = \bigvee_{b \in B} b$, and for such b, $b \leq v$ and so, $\bigvee_{i=1}^{\infty} v_{2i} = v$ and by a similar argument $\bigvee_{i=1} v_{2i+1} = a$ and so, $\{v_i\}_{i=1}^{\infty}$ is a cover for L. Call $u_{2i} = v_{2_i} \wedge \bigvee_{j<2i} v_j^*$. Now, $u_{2i} \leq v$ and so $u = \bigvee_i u_{2i} \leq v$ and $a \vee u = a \vee \bigvee_{i=1}^{\infty} u_{2i} = a \vee \bigvee_i (v_{2i} \wedge \bigvee_{j<2i} v_{j^*}) = (a \vee \bigvee v_{2i}) \wedge (a \vee \bigvee_i \bigvee_{j<2i} vj^*) = (a \vee v) \wedge (a \vee \bigvee_i \bigvee_{j<2i} vj^*)$ but $a \vee v = e_L$, So $a \vee u = (a \vee \bigvee_i \bigvee_{j<2i} v_j^*)$.

But, for any odd $j, v_j \leq a$ and so $e_L = v_j^* \vee v_j \leq v_j^* \vee a$ meaning $a \vee u = e_L$. Further, u complemented means that $\uparrow (u \vee u^*) = 1$ and hence Ind $(\uparrow (u \vee u^*)) = 0$ and by definition of the Brouwer-Čech dimension Ind $L \leq 0$.

Before proving a result on a hereditary property of the Brouwer-Čech dimension we prove the following result for a general frame L. In fact the formulation is best done in the category of locales. For this purpose, and for what is to follow, we recall some of the fundamentals on locales.

The following is fairly standard in the literature on locales (see e.g. [8], [9], and we recall some basic facts. As far as notation is concerned we shall denote locales by the letters X, Y or Z. We also write $S(X)$ for the lattice of sublocales of X, these being the regular subojects of X in **Loc**. The lattice

$S(X)$ is complete and satisfies the coframe distributive law: $S \vee \bigwedge T_i = \bigwedge(S \vee T_i)$. For a locale X, and $S, R \in S(X)$ the *relative complement* of S *with respect to* R written $R\backslash S$) is $R\backslash S = \bigvee\{T \in S(X) | T \leq R \text{ and } T \wedge S = 0\}$. The *supplement* of S written $\sup(S)$ is $\sup(S) = \bigwedge\{R \in S(X) | R \vee S = X\}$. Since $S(X)$ is a coframe $S \vee \sup(S) = X$. Hence $\sup(S)$ is the smallest sublocale whose join with S is X. It is known that $X\backslash S = \sup(S)$. A sublocale C is *complemented* if it is a complemented element in $S(X)$. It can be characterized in terms of its supplement by $C \wedge \sup(C) = 0$. A sublocale S is said to be *open* if it is isomorphic to $\downarrow a$ for some $a \in X$. Open sublocales have complements in $S(X)$. These are the *closed* sublocales of X. The closure of a sublocale S is the smallest closed sublocale that contains S. We denote the closure of a sublocale S of X by $\overline{S}$ or $cl_X S$.

Definition 5.11 — Let X be a locale and $A, B \in S(X)$. A and B are called *separated* if $A \wedge \overline{B} = 0 = \overline{A} \wedge B$.

***Theorem* 5.12** — *Let X be a locale such that every open sublocale is normal. Then whenever the sublocales S and T are separated, there exists open sublocales U and V such that $S \leq U$, $T \leq V$ and $U \wedge V = 0$.*

PROOF : Consider the sublocale $K = (X\backslash\overline{S}) \vee (X\backslash\overline{T})$. We have $S, T \leq K$, this following from the fact that $S \wedge \overline{T} = 0$ implies $S \leq X\backslash\overline{T}$ and $T \wedge \overline{S} = 0$ implies $T \leq X\backslash\overline{S}$. Furthermore $cl_K S = \overline{S} \wedge K = \overline{S} \wedge ((X\backslash\overline{S}) \vee (X\backslash\overline{T})) = [\overline{S} \wedge (X\backslash\overline{S})] \vee [\overline{S} \wedge (X\backslash\overline{T})] = 0 \vee (\overline{S} \wedge (X\backslash\overline{T})) = \overline{S} \wedge (X\backslash\overline{T})$. Similarly $cl_K T = \overline{T} \wedge (X\backslash\overline{S})$, so $cl_K S \wedge cl_K T = \overline{S} \wedge \overline{T} \wedge (X\backslash\overline{T}) \wedge (X\backslash\overline{S}) = 0$. Since K is an open sublocale there exists open sublocales U and V of K (and hence of X) such that $cl_K S \leq U$, $cl_K T \leq V$ and $U \wedge V = 0$.

Definition 5.13 — Let X be a locale and $U \in S(X)$. We define the *frontier* of U (denoted by FrU) by $FrU = \overline{U}\backslash U$.

Remark 5.14 : Note that for the Definition of the Brouwer - Čech dimension for frames the condition of $BC1$, $BC3$ and $BC4$ readily apply. However, for $BC2$ we observe the following :

$$\begin{aligned} \mathcal{O}(FrU) &\cong \uparrow (X\backslash FrU) \\ \mathcal{O}(FrU) &\cong \uparrow ((X\backslash\overline{U}) \cup U) \text{ in } \mathcal{O}X \\ &= \uparrow (U \vee U^*). \end{aligned}$$

Thus the formulation for $BC2$ in the language of locales is:

Ind $X \le n$ if for every closed sublocale $A \subseteq X$ and any open sublocale $V \subset X$ that contains A there exists an open sublocale $U \subset V$ such that $A \subset U \subset V$ and Ind $FrU \le n-1$.

Remark 5.15 : The localic result in the following lemma is a generalization of the space version which appears in [5], (p405).

Lemma 5.16 — Let X be a metric locale, Z a sublocale of X. If Ind $Z \le 0$, then for every closed sublocale A of X and any open sublocale V of X such that $A \subseteq V$, there exists an open sublocale U of X such that

$$A \subseteq U \subseteq \overline{U} \subseteq V \quad \text{and} \quad Z \wedge FrU = 0.$$

PROOF : Since $A \subseteq V$ and X is a normal locale we can take open sublocales W_1 and W_2 of X satisfying the inclusions

$$A \subseteq W_1 \subseteq \overline{W}_1 \subseteq W_2 \subseteq \overline{W}_2 \subseteq V.$$

We can take Ind $Z = 0$ (for if Ind $Z = -1$, we could take $U = W_1$ and $Z \wedge F_r W_1 = 0$). For ind $Z = 0$ there exists a complemented sublocale U_0 in Z such that $Z \wedge \overline{W}_1 \subseteq U_0 \subseteq Z \wedge W_2$. Clearly, we have $Z \backslash U_0 \subseteq Z \backslash (Z \wedge W_1) = Z \backslash W_1 \subseteq X \backslash W_1$ and $\overline{U}_0 \subseteq \overline{W}_2$. Now,

$$\begin{aligned} & (A \vee U_0) \cap [(X \backslash V) \vee (Z \backslash U_0)] \\ = \ & [(A \vee U_0) \wedge (X \backslash V)] \vee [(A \vee U_0) \wedge (Z \backslash U_0)] \end{aligned}$$

But, $A \subseteq V$, implies $A \wedge (X \backslash V) = 0; U_0 \subseteq Z \wedge W_2 \subseteq \overline{W}_2 \subseteq V$ so that $(A \bigvee U_0) \wedge (X \backslash V) = 0$. Also, $Z \backslash U_0 \subseteq Z \backslash W_1$ and $A \wedge (Z \backslash U_0) \subseteq A \wedge (Z \backslash W_1) = 0$ and thence $(A \vee U_0)$ and $((X \backslash V) \vee (Z \backslash U_0))$ are separated. By virtue of Theorem 5.12 there exists an open sublocale $V \subseteq X$ such that $A \bigvee U_0 \subseteq V$ and $\overline{V} \wedge [(X \backslash V) \bigvee (Z \backslash U_0)] = 0$ implying $\overline{U} \wedge (X \backslash V) = 0$ and $\overline{U} \subseteq V$. Also, $Z \wedge F_r U = Z \wedge (\overline{U} \backslash U) = (U_0 \cup (Z \backslash U_0)) \wedge (\overline{U} \backslash U) = (U_0 \wedge (\overline{U} \backslash U)) \bigvee ((Z \backslash U_0) \wedge (\overline{U} \backslash U)) = 0 \vee ((Z \backslash U_0) \wedge (\overline{U} \backslash U)) \subseteq (Z \bigvee U_0) \wedge \overline{U} = 0$.

The following definitions were introduced by Dowker and Strauss in [3]. Let L be a frame. A family $(a_\alpha)_{\alpha \in A}$ is called a *closed covering* of L if for each $u \in L$,

$$u = \bigwedge_{\alpha \in A} (u \vee a_\alpha).$$

A family $(a_\alpha)_{a \in A}$ of elements of L is called *point-finite* if $(c_\beta)_{\beta \in B}$ (where B is the set of all finite subsets of A and $c_\beta = \bigvee_{a \notin \beta} a_\alpha$) is a closed covering.

Remark 5.17 : Notice in the definition of a closed covering $u \leq \bigwedge u \vee a_\alpha$ is always true. The condition $u = \bigwedge_{\alpha \in A} (u \vee a_\alpha)$ can be stated as $\bigwedge_{\alpha \in A} (u \vee a_\alpha) \leq u$. Dowker and Strauss [3] showed that a frame L is normal if and only if each point-finite cover is shrinkable. This will certainly apply for metric frames. This result is needed in the following theorem, as is the result that "metric locales are paracompact" which was proved by Sun in [12]. It should be further remarked that the point-finite notion of Dowker and Strauss has some shortcomings mainly because it does not behave so well as its classical counterpart. In a recent article by Ferreira and Picado (see [6]), this deficiency was corrected. These authors proposed a stronger version of point-finiteness, still weaker than local finiteness however. They propose the following: Let $\mathcal{S} = \{S_i | i \in I\}$ be a family of sublocales of X. For every finite subset F of I let $\mathcal{S}_F = \bigvee\{S_i | i \in I \backslash F\}$. Then the family $\mathcal{S}$ is said to be *point finite* whenever

$$T = \bigwedge\nolimits_{F \in \mathcal{P}_f(I)} (T \vee \mathcal{S}_F) \quad \text{for every } T \in S(X),$$

where $\mathcal{P}_f(I)$ stands for all the finite subsets of I. They show also that the pointfree version of the theorem of Lefschetz that a frame L is normal if and only if each point-finite cover is shrinkable still holds.

Definition 5.18 — Let X be a locale. B is called a *basis* for X if B consists of open sublocales and each open sublocale U of X is a join of members from B.

***Theorem* 5.19** — *For every metric locale (X, d) and any integer $n \geq 0$ the following conditions are equivalent :*

(i) Ind $X \leq n$.

(ii) X has a $\sigma-$ locally finite base B such that Ind $FrU \leq n-1$ for every $U \in B$.

(iii) $X = Y \vee Z$, where Ind $Y \leq n-1$ and Ind $Z \leq 0$.

Proof : We shall show that (i)$\Rightarrow$(ii) . Let (X, d) be a metric locale with Ind $X \leq n$. Since every metric locale is paracompact, it follows that for $i = 1, 2, \ldots$ the locale X has a locally finite cover $V_i = \{V\}_{s \in S}$ of $\{V_s\}_{s \in S}$ consisting of diameter less than $\frac{1}{i}$. By the remarks preceding Definition 5.18, for every i there exists a shrinkage $\{F_s\}_{s\in S}$ of $\{V_s\}_{s\in S}$ with $\overline{F}_s \leq V_s$, for each s. As Ind $X \leq n$, for every $s \in S_i$ there exists an open sublocale $U_s \in S(X)$ such that $F_s \subseteq U_s \subseteq V_s$ and Ind $(FrU_s) \leq n-1$. The family $B_i = \{U_s\}_{s\in S_i}$ is a locally finite open cover of the locale X consisting of members of diameter less that $\frac{1}{i}$. Hence $B = \bigcup_{i=1}^{\infty} B_i$ is a $\sigma-$locally finite basis for X.

To prove (ii)⇒(iii) Observe that by virtue of Theorem 5.17 the sublocale $Y = \bigvee\{FrU|U \in B\} \subseteq X$ satisfies the inequality Ind $Y \leq n-1$ and that the sublocale $Z = X\backslash Y$ has a $\sigma-$locally finite basis $\{Z \wedge U|U \in B\}$ consisting of complemented sublocales so that Ind $Z \leq 0$ by Lemma 5.9.
The implication (iii)⇒(i) follows readily from the Lemma 5.16 and [1].

The above theorem yields:

***Theorem* 5.20** — *A metric locale X satisfies the inequality Ind $X \leq n \geq 0$ if and only if $X = Z_1 \vee Z_2 \vee \ldots \vee Z_{n+1}$, where Ind $Z_i \leq 0$ for $i = 1, 2, \ldots, n+1$.*

The above theorem in turn yields.

***Theorem* 5.21** — *Let X be a metric locale. For every pair of open sublocales $A, B \in S(X)$ we have*

$$\mathrm{Ind}(A \vee B) \leq \mathrm{Ind}\, A + \mathrm{Ind}\, B + 1.$$

We now set to out to generalize Lemma 5.16.

***Theorem* 5.22** — *Let X be a metric locale and M a sublocale of X. If Ind $M \leq n \geq 0$, then for every closed sublocale A of X and any open sublocale $V \subseteq X$ such that $A \subseteq V$ there exists an open sublocale U of X such that $A \subseteq U \subseteq \overline{U} \subseteq V$ and Ind $(M \wedge FrU) \leq n-1$.*

Proof : By Theorem 5.19, we have $M = Y \vee Z$ where Ind $Z \leq 0$. Now, the open sublocale U satisfying Lemma 5.16 also satisfies this theorem, because $M \wedge FrU \subseteq M\backslash Z \subseteq Y$ and thus Ind $(M \wedge FrU) \leq n-1$ by [1].

From the above theorem we obtain the following

Corollary 5.23 — Let X be a metric locale. For every sequence

$$\langle (A_1, B_1), (A_2, B_2), \ldots, (A_{n+1}, B_{n+1})\rangle$$

of $(n+1)$ pairs of disjoint closed sublocales of X with Ind $X \leq n \geq 0$ there exists open sublocales $U_1, U_2, \ldots U_{n+1}$ such that $A_i \subseteq U_i \subseteq \overline{U}_i \subseteq X \backslash B_i$, for $i = 1, 2, \ldots, n+1$ and $FrU_1 \wedge FrU_2 \wedge \ldots \wedge FrU_{n+1} = 0$.

From [10], [1] and Theorem 5.2 we have the following theorem, where $L \oplus M$ represents the coproduct of L and M.

***Theorem* 5.24** — *Let L and M be metric frames. Then Ind* $(L \oplus M) =$ *Ind* $\beta(L \oplus M)$.

Remark 5.25 : Banaschewski and Gilmour [2] have shown that for regular $\sigma-$frames L and M with a quotient map $h : L \to M, \dim(L \oplus M) \leq \dim L + \dim M$. For metric frames L and M of which at least one is non-zero, the result
Ind $(L \oplus M) \leq$ Ind L + Ind M is at present an open problem.

Acknowledgement

We are indebted to the referee for valuable comments and suggestions which helped to improve the quality of the paper. In particular we thank the referee for alerting us to the paper by Ferreira and Picado (see [6]), which to our mind as well, has the right pointfree definition of point-finiteness.

References

1. D. Brijlall and D. Baboolal, Some aspects of dimension theory of frames, *Indian Journal of Pure and Applied Mathematics*, **39**(5) (2008), 375-402.

2. B. Banaschewski and C. Gilmour, Stone-Čech compactification and dimension theory for regular σ-frames, *J. London Math. Soc.*, **39**(2) (1989), 1-8.

3. C. H. Dowker and D. P. Strauss, Paracompact frames and closed maps, *Sympos. Math.*, **16** (1975), 93-116.

4. T. Dube, Separability in locales, *Questiones Mathematicae*, **17** (1994), 333-338.

5. R. Engelking, *General Topology*, Helderman Verlag, Berlin, 1989.

6. M. J. Ferreira and J. Picado, On Point-finiteness in Pointfree Topology, *Applied Categorical Structures*, **15** (2007), 185-198.

7. J. R. Isbell, *Graduation and dimension in locales*, L.M.S. Lecture Note Series, Vo. 93, pp. 195-210, Cambridge University Press, 1985.

8. P. T. Johnstone, *Stone Spaces*, Cambridge University Oress, Cambridge, 1982.

9. T. Plewe, Quotient Maps of Locales, *Applied Categorical Structures*, **8** (2000), 17-24.

10. A. Pultr, *Remarks on Metrizable Locales*, Proceedings of the 12th Winter School on Abstract Analysis, 1984, pp. 105-120.

11. A. Pultr, *Categories of Diametric Frames*, Math. Proc. Camb. Phil. Soc., (1989), 285-297.

12. S. Sun, Paracompact Locales and Metric Locales, *Comment. Math. Univ. Carolinae*, **30**(1) (1989), 101-107.