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FINITE GROUPS WITH SOME CAP-SUBGROUPS [1]

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Let G be a finite group. A subgroup H of G is called a CAP-subgroup if the following condition is satisfied: for each chief factor K/L of G either $HK = HL$ or $H \cap K = H \cap L$. Let p be a prime factor of $|G|$ and let P be a Sylow p-subgroup of G. If d is the minimum number of generators of P then there exists a family of maximal subgroups of P, denoted by $\mathcal{M}_d(P)$=$\{P_1, P_2, \ldots, P_d\}$ such that $\bigcap_{i=1}^{d} P_i = \Phi(P)$. In this paper, we investigate the group G satisfying the condition: every member of a fixed $\mathcal{M}_d(P)$ is a CAP-subgroup of G. For example, if, in addition, G is p-solvable, then G is p-supersolvable.

Key words : Cover-avoidance properties, P-supersolvable groups, p-nilpotent groups, supersolvable groups.

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1. Introduction

All groups considered in this paper are finite.

Let p be a prime dividing $|G|$ and let P be a Sylow p-subgroup of G. Consider a subset of $\mathcal{M}(P)$, the set of all maximal subgroups of P. In [8] Shirong Li introduced the following:

Definition 1.1 — Let d be the minimum number of generators of P. Then there exist maximal subgroups $P_1, \ldots, P_d$ of P such that $\bigcap_{i=1}^{d} P_i = \Phi(P)$, the Frattini subgroup of P. By $\mathcal{M}_d(P)$ denote the family of subgroups $P_1, \ldots, P_d$ with the above property.

We call $\mathcal{M}_d(P)$ a $\mathcal{M}_d$-system of P. Such $\mathcal{M}_d(P)$ is not unique in general. Fix a $\mathcal{M}_d(P)$ for a given Sylow p-subgroup P. An interesting topic is to study the influence of subgroups in $\mathcal{M}_d(P)$ on the structure of G.

The following concept is required in the paper. Let L and K be normal subgroups of G with $L < K$. A subgroup H of G covers the normal factor K/L if $HK = HL$ holds; H avoids K/L provided $H \cap K = H \cap L$. If H either covers or avoids each chief factor of G, then H is said to possess the cover-avoidance property. Such a subgroup is called a CAP-subgroup of G. This concept was introduced by W. Gaschütz in 1962 [2] and studied by many authors. For example, see [1, 3, 4, 5, 9].

The objective of this paper is to investigate the group G satisfying the following condition: each member in a fixed $\mathcal{M}_d(P)$ is a CAP-subgroup of G.

We note that if the Sylow p-subgroup P of G is of order p, then $\mathcal{M}_d(P)$ is a singleton set, its unique member is the subgroup $\{1\}$. In this case one cannot expect a detailed structure of G. So some additional condition is necessary for our research. Also, we have to be careful in proving our theorems because the condition is not inherited by subgroups and quotient groups in general.

2. Some Basic Properties

In this section we collect some results which are needed in the sequel.

Lemma 2.1 — Let S be a CAP-subgroup of a group G and $N \trianglelefteq G$. Then the following statements are true:

(i) N is a CAP-subgroup of G.

(ii) SN/N is a CAP-subgroup of G/N.

(iii) SN is a CAP-subgroup of G.

(iv) $S \cap N$ is a CAP-subgroup of G.

(v) If $N \leq S$, then N is a CAP-subgroup of S.

PROOF : (i) Let K/L be a chief factor of G. Then we can see that $L(N \cap K) \trianglelefteq G$ and so $L(N \cap K) = K$ or L. If $L(N \cap K) = K$, then $NK = NL$. If $L(N \cap K) = L$, then $N \cap K = N \cap L$. It follows that N is a CAP-subgroup of G.

By [10, Proposition 2.2.1], (ii)-(iv) follows.

The statement (v) is an immediate consequence of (i).

Lemma 2.2 — Let K/L be a chief factor of a group G. Then

(i) If H is a p-subgroup of G and H covers K/L, then K/L is a p-group.

(ii) If H is a maximal subgroup of a Sylow p-subgroup of G and H avoids K/L, then $|K/L|_p = 1$ or p.

PROOF : (i) By hypothesis, $HK = HL$. So $HK/L = HL/L \cong H\big/H \cap L$ and hence HK/L is a p-group. It follows that K/L is a p-group, as desired.

(ii) Suppose that (ii) is not true so that p^2 divides $|K/L|$. Let P be a Sylow p-subgroup of G containing H. Then $P \cap K$ and $P \cap L$ are Sylow p-subgroups of K and L respectively, so p^2 divides the index $|P \cap K : P \cap L|$. If $P \cap K \leq H$, then it follows from H avoids K/L that $P \cap K \leq H \cap K = H \cap L \leq P \cap L$, a contradiction. So $P = H(P \cap K)$, we have $|P/H| = |H(P \cap K)\big/H| = |P \cap K\big/H \cap K| = |P \cap K : H \cap L| = |P \cap K : P \cap L||P \cap L : H \cap L|$, thus p^2 divides $|P : H|$, in contradiction to the fact that H is a maximal subgroup of P.

Lemma 2.3 — Let p be a prime dividing the order of a group G. If the normal factor K/L is either a p'-group or a cyclic group of order p, then all maximal subgroups of Sylow p-subgroups of G either cover or avoid K/L.

PROOF : Let P be a Sylow p-subgroup of G and let P_1 be a maximal subgroup of P. It is clear that $|K/L| = |P_1K : P_1L||P_1 \cap K : P_1 \cap L|$. If K/L is a

p'-group, since $|P_1 \cap K : P_1 \cap L|$ is a number dividing the order of P_1, then $|P_1 \cap K : P_1 \cap L| = 1$ and so $P_1 \cap K = P_1 \cap L$, as desired. If $|K/L| = p$, then $|P_1K : P_1L| = 1$ or $|P_1 \cap K : P_1 \cap L| = 1$, i.e., $P_1K = P_1L$ or $P_1 \cap K = P_1 \cap L$. Hence P_1 covers or avoids K/L, the proof of the Lemma is complete.

Lemma 2.4 — If G is p-supersolvable, then every maximal subgroup of any Sylow p-subgroup of G is a CAP-subgroup of G.

PROOF : Since each chief factor of a p-supersolvable group is either a p'-group or a cyclic group of order p, the result follows from Lemma 2.3.

The following famous theorem of Gaschütz is used for Lemma 2.6 and thus in the proof of the theorems in § 3.

***Theorem* 2.5** [6, I, Hauptsatz 17.4] — *Let N be a normal abelian subgroup of a group G and let $N \leq M \leq G$ such that $(|N|, |G : M|) = 1$. If N has a complement in M, then N has a complement in G.*

Lemma 2.6 — Let P be a Sylow p-subgroup of a group G, p a prime and let $O_p(G) > 1$ be elementary abelian. Suppose that for each minimal normal subgroup N contained in $O_p(G)$ there exists a member P_i in a fixed $\mathcal{M}_d(P)$ such that P_i avoids the chief factor $N/1$. Then $O_p(G)$ is a direct product of normal subgroups of G of order p and $G = O_p(G) \rtimes M$, the semi-direct product of $O_p(G)$ with some normal subgroup M.

PROOF : Let N_1 be a minimal normal subgroup of G contained in $O_p(G)$. By hypothesis there exists a P_i in $\mathcal{M}_d(P)$ such that P_i avoids $N_1/1$, that is, $P_i \cap N_1 = P_i \cap 1 = 1$. Because P_i is a maximal subgroup of P, it follows that $|N_1| = p$. Thus the normal subgroup N_1 is complemented in P. In the light of Theorem 2.5, N_1 has a complement subgroup M in G such that $G = N_1 \rtimes M$. It is clear that $O_p(G) = N_1(O_p(G) \cap M)$ and $O_p(G) \cap M \trianglelefteq M$. Since $O_p(G)$ is an elementary abelian p-group, $O_p(G) \cap M \trianglelefteq G$. If $O_p(G) \cap M = 1$, as desired. If $O_p(G) \cap M \neq 1$, then there exists a minimal normal subgroup N_2 of G such that $N_2 \leq O_p(G) \cap M$. By hypothesis, there is a member in $\mathcal{M}_d(P)$ avoids $N_2/1$, say P_2. It follows from $P_2 \cap N_2 = P_2 \cap 1 = 1$ that $|N_2| = p$. Let $Q \in Syl_p(M)$. Since $N_i \not\leq P_2 \cap Q (i = 1, 2)$, we can see that $P = (P_2 \cap Q)N_1N_2$. By Theorem 2.5, N_1N_2 possesses a complement subgroup in G, say H. That is $G = N_1N_2 \rtimes H$. Furthermore, we may assume that there is a subgroup M of G and normal subgroups $N_i (i = 1, ..., r)$ of G of order p contained in $O_p(G)$ such that

$$G = (N_1 \times \cdots \times N_r) \rtimes M.$$

where M possesses order as small as possible. In order to finish the proof we only need to prove $O_p(G) \cap M = 1$. In fact, $O_p(G) = (N_1 \times \cdots \times N_r)(O_p(G) \cap M)$. If $O_p(G) \cap M \neq 1$, then there is a minimal normal subgroup N of G such that $N \leq O_p(G) \cap M$. By hypothesis some P_j in $\mathcal{M}_d(P)$ avoids $N/1$, i.e, $P_j \cap N = P_j \cap 1 = 1$. It follows that $|N| = p$. Thus P_j is a complement subgroup of N in P. On the other hand, we have $P = P \cap (N_1 \times \cdots \times N_r)M = (N_1 \times \cdots \times N_r)(P \cap M)$, so $P \cap M$ is a Sylow p-subgroup of M. Now, P_j is of index p in P and $P = P_j(P \cap M)$, so $P_j \cap (P \cap M)$ is a maximal subgroup of $P \cap M$. This fact means that N is complemented in the Sylow p-subgroup $P \cap M$ of M. Applying Theorem 2.5 again, we can see that N has a complement subgroup K in M. That is, $M = N \rtimes K$. Consequently, we deduce that

$$G = (N_1 \times \cdots \times N_r \times N) \rtimes K,\ K < M,$$

which contradicts to the choice of M, completing the proof.

3. Main Results

We first observe the p-supersolvability of a p-solvable group by means of CAP-subgroups.

***Theorem* 3.1** — *Let G be a p-solvable group and let P be a Sylow p-subgroup of G, where p is a fixed prime. Then the following statements are equivalent:*

(i) G is p-supersolvable.

(ii) Every member in $\mathcal{M}(P)$ is a CAP-subgroup of G.

(iii) Every member in a fixed $\mathcal{M}_d(P)$ is a CAP-subgroup of G.

PROOF : (i)⇒ (ii): It is obvious by Lemma 2.4.

(ii)⇒(iii): Trivial.

(iii)⇒(i): Suppose that the theorem is false and let G be a counterexample of minimal order. Let $\mathcal{M}_d(P)$=$\{P_1, P_2, \ldots, P_d\}$. By hypothesis, each P_i is a CAP-subgroup of G.

We prove the theorem by the following statements:

(1) $O_{p'}(G) = 1$.

Set $B = O_{p'}(G)$. Then PB/B is a Sylow p-subgroup of G/B, which is isomorphic to P, so d is also the minimum number of generators of PB/B. It is easy to check that

$$\mathcal{M}_d(PB/B) = \{P_1B/B, P_2B/B, \ldots, P_dB/B\}$$

is a $\mathcal{M}_d$-system of PB/B, and all P_iB/B are CAP-subgroups of G/B by Lemma 2.1 (ii). Hence G/B satisfies the hypothesis of theorem. If $O_{p'}(G) \neq 1$, then $G/O_{p'}(G)$ is p-supersolvable by the choice of G, and it follows that G is p-supersolvable, which is a contradiction.

(2) $\Phi(O_p(G)) = 1$, so $O_p(G)$ is elementary abelian.

Suppose that $\Phi(O_p(G)) \neq 1$. Then, by a similar argument as (1), $G\big/\Phi(O_p(G))$ satisfies the hypothesis of our theorem by Lemma 2.1(ii). We conclude that $G\big/\Phi$ $(O_p(G))$ is p-supersolvable by the minimality of G. In view of [6, III, 3.3], $\Phi(O_p(G)) \leq \Phi(G)$, it follows that $G/\Phi(G)$ is a homomorphic image of G/Φ $(O_p(G))$ and hence p-supersolvable. By a theorem of Huppert [6, VI, 8.6 a], G is p-supersolvable, a contradiction.

(3) $O_p(G) = N_1 \times N_2 \times \cdots \times N_t$, where each $N_i (i = 1, 2, \ldots, t)$ is a minimal normal subgroup of G of order p.

As G is p-solvable and $O_{p'}(G) = 1$, then $O_p(G) > 1$. Let N be a minimal normal subgroup of G such that $N \leq O_p(G)$. If every P_j in $\mathcal{M}_d(P)$ covers the chief factor $N/1$, then $P_jN = P_j$, which implies that $N \leq \bigcap_{i=1}^{d} P_i = \Phi(P)$. By [6, III, 3.3], $N \leq \Phi(G)$. We can see that G/N satisfies the hypothesis of our theorem by Lemma 2.1(ii). Thus, by the minimality of G, G/N is p-supersolvable, and so is $G/\Phi(G)$. Consequently, G is p-supersolvable, a contradiction. Therefore, there exists P_i in $\mathcal{M}_d(P)$ which avoids $N/1$. Thus (3) follows from Lemma 2.6.

(4) $G/O_p(G)$ is p-supersolvable.

In (3), each N_i is a normal subgroup of G of order p, so $Aut(N_i)$ is cyclic of order dividing $p-1$. Since $G/C_G(N_i) \lesssim Aut(N_i)$, it follows that $G/C_G(N_i)$ is cyclic of order dividing $p-1$. In particular, $G/C_G(N_i)$ is certainly p-supersolvable. As the class of all p-supersolvable groups is a saturated formation, thus we know that $G/\cap C_G(N_i)$ is p-supersolvable [6, VI, Satz 7.1, 7.2 and 8.6]. Furthermore, applying (3), we see that $G\big/\cap C_G(N_i) = G\big/C_G(O_p(G))$, so $G\big/C_G(O_p(G))$ is p-supersolvable. On the other hand, the result of [11, Theorem 9.3.1] gives $C_G(O_p(G)) \leq O_p(G)$, it follows that $G/O_p(G)$ is p-supersolvable.

Again applying (3), each chief factor of G contained in $O_p(G)$ is cyclic, so all p-chief factors of G are cyclic. It follows that G is p-supersolvable, a final contradiction, which completes our proof.

Remark 1 : A_5 and $PSL(2,7)$ show that the condition that G is p-solvable is necessary for Theorem 3.1.

A particular case of π-supersolvability, when π is the set of all primes dividing the order of G, is the usual supersolvability. Theorem 3.2 is clearly inspired by Theorem 3.1. However no hypothesis on the solvability is needed here. In fact, the solvability is deduced from the other hypothesis.

***Theorem* 3.2** — *Let G be a group. Then the following statements are equivalent:*

(i) G is supersolvable.

(ii) For each Sylow subgroup P of G, every member in $\mathcal{M}(P)$ is a CAP-subgroup of G.

(iii) For each Sylow subgroup P of G, every member in a fixed $\mathcal{M}_d(P)$ is a CAP-subgroup of G.

PROOF : (i) implies (ii) by Lemma 2.4 and (ii) implies (iii). Now, let statement (iii) hold, we want to show that G is supersolvable. Firstly, we claim that G is solvable. Suppose that there exists a non-solvable chief factor of G, say K/L. Then there exists a prime p such that p^2 divides $|K/L|$. By hypothesis, there is a $\mathcal{M}_d(P)$ such that every member M in $\mathcal{M}_d(P)$ is a CAP-subgroup of G. Apply Lemma 2.2(i) to see that M doesn't cover K/L, so M avoids K/L. Lemma 2.2(ii) indicates that $|K/L|_p = 1$ or p, which is a contradiction. Therefore we conclude that G has no non-abelian chief factor and hence G is solvable.

Now we are in the hypothesis of Theorem 3.1 for all primes p. Consequently G is p-supersolvable for all primes p. That is, G is supersolvable.

The following two theorems study the p-nilpotency of a group.

***Theorem* 3.3** — *Let p be the smallest prime dividing the order of a group G and let P be a Sylow p-subgroup of G. Then the following statements are equivalent:*

(i) G is p-nilpotent.

(ii) Every member in $\mathcal{M}(P)$ is a CAP-subgroup of G.

(iii) Every member in a fixed $\mathcal{M}_d(P)$ is a CAP-subgroup of G.

PROOF : By Lemma 2.4, statement (i) implies statement (ii). And it is clear that statement (ii) implies statement (iii). Now we assume that statement (iii) is true, that is, every member in $\mathcal{M}_d(P)$ is a CAP-subgroup of G. We want to show that G is p-nilpotent. Firstly, we claim that G is p-solvable. In fact, let K/L be a chief factor of G and the order of K/L is divisible by p. If $M \in \mathcal{M}_d(P)$, then M is a CAP-subgroup of G by hypothesis. If M covers K/L, then K/L is a p-group by Lemma 2.2(i), as desired. If M avoids K/L, then $|K/L|_p = p$ by Lemma 2.2(ii). As p is the smallest prime dividing the order of G, by Burnside's theorem [6, IV, 2.6 and 2.7], K/L is p-nilpotent. Hence K/L is a p-group. We can see that G is p-solvable. It follows from Theorem 3.1 that G is a p-supersolvable group.

Let

$$1 = G_0 < G_1 < \cdots < G_n = G$$

be a chief series of G. Then any p-chief factor G_i/G_{i-1} of G is cyclic of order p, so

$$(G/G_{i-1})/C_{G/G_{i-1}}(G_i/G_{i-1})$$

is cyclic of order dividing $p-1$. On the other hand, the minimality of p implies that $G/G_{i-1} = C_{G/G_{i-1}}(G_i/G_{i-1})$, i.e., $G_i/G_{i-1} \leq Z(G/G_{i-1})$. By [6, IV, 4.4], it follows that G is p-nilpotent.

If we remove the hypothesis of Theorem 3.3 that p is the smallest prime, then we have the following theorem:

***Theorem* 3.4** — *Let p be a prime dividing the order of a group G and let P be a Sylow p-subgroup of G. If $N_G(P)$ is p-nilpotent, then the following statements are equivalent:*

(i) G is p-nilpotent.

(ii) Every member in $\mathcal{M}(P)$ is a CAP-subgroup of G.

(iii) Every member in a fixed $\mathcal{M}_d(P)$ is a CAP-subgroup of G.

PROOF : (i) $\Rightarrow$ (ii): A p-nilpotent group is p-supersolvable, conclusion (ii) follows from Lemma 2.4.

(ii)$\Rightarrow$(iii): Clear.

(iii)$\Rightarrow$(i): Suppose that the result is false and let G be a counterexample of minimal order. Write

$$\mathcal{M}_d(P) = \{P_1, \ldots, P_d\}.$$

(1) $O_{p'}(G) = 1$.

In fact, if $O_{p'}(G) \neq 1$, then we can consider the quotient group $G/O_{p'}(G)$. Set $N = O_{p'}(G)$. Then every Sylow p-subgroup of G/N possesses the form PN/N, where P is a Sylow p-subgroup of G. Since $PN/N \cong P$, it follows that PN/N is generated by d elements. Thus we can take $\mathcal{M}_d(PN/N) = \{P_iN/N | P_i \in \mathcal{M}_d(P)\}$. By Lemma 2.1(ii), we can see that P_iN/N is a CAP-subgroup of G/N for all i. Also, $N_{G/N}(PN/N) = N_G(P)N/N$, hence it is p-nilpotent because $N_G(P)$ is p-nilpotent. Thus $G/O_{p'}(G)$ satisfies the hypothesis of our theorem. By the choice of G, $G/O_{p'}(G)$ is p-nilpotent and it follows that G is p-nilpotent, a contradiction.

(2) $Core_G(\Phi(P)) = 1$, particularly $\Phi(U) = 1$ where $U = O_p(G)$.

It is clear that $\Phi(U)$ is normal in G and $\Phi(U) \leq \Phi(P)$. If $Core_G(\Phi(P)) \neq 1$, then there exists a nontrivial normal subgroup N of G contained in $Core_G(\Phi(P))$. As in the proof of (1), it is easy to check that G/N satisfies the hypothesis of our theorem. By the minimal choice of G, G/N is p-nilpotent. Since N is a subgroup of $\Phi(P)$, $N \leq \Phi(G)$ by [6, III, 3.3]. Let H/N be a normal p-complement of G/N. By Schur-Zassenhaus theorem [11, Theorem 9.1.2]Robinson, there exists a complement K of N in H, this implies that K^g is also a complement of N in H for any g in G. It follows from Schur-Zassenhaus theorem that there is an element $h \in H$ such that $K^{gh} = K$ and hence $g \in h^{-1}N_G(K) \subseteq HN_G(K)$. Thus $G = HN_G(K)$. Since N is a subgroup of $\Phi(G)$, $G = HN_G(K) = NKN_G(K) = N_G(K)$ and K is normal in G. Therefore G is p-nilpotent.

(3) If N is a minimal normal subgroup of G and N is a p-subgroup, then $|N| = p$.

If all $P_i \in \mathcal{M}_d(P)$ cover the chief factor $N/1$, then $P_iN = P_i$, so N is contained in each P_i and hence $N \leq \bigcap P_i = \Phi(P)$, contrary to (2). Thus there exists some P_i avoids $N/1$, then $P_i \cap N = P_i \cap 1 = 1$, which gives $P = P_i \times N$ and hence $|N| = p$.

(4) All minimal normal subgroups of G are in $O_p(G)$.

Suppose that (4) is not true and there is a minimal normal subgroup H satisfying $H \nsubseteq O_p(G)$. As $O_{p'}(G) = 1$, we have $p \big| |H|$ and H is a non-abelian characteristically simple group. We shall finish the proof of (4) by claiming the following statements:

(4-1) All P_i avoid the chief factor $H/1$, H is a non-abelian simple group with $|H|_p = p$.

As H is not a p-group, by lemma 2.2(i), all P_i avoid $H/1$. Also, because P_i is a maximal subgroup of P, Lemma 2.2(ii) shows that $|H|_p = 1$ or p. Consequently, $|H|_p = p$. Since H is a direct product of isomorphic simple subgroups of H [6, I, Satz 9.12], H is a non-abelian simple group.

(4-2) $C_G(H) = 1$.

Let $C_G(H) > 1$. As H is a non-abelian simple group, we have $H \cap C_G(H) = 1$. Then we can find a minimal normal subgroup H^* of G which is contained in $C_G(H)$ such that $H \cap H^* = 1$. Consider the chief factor HH^*/H. If some P_i avoids HH^*/H, then $P_i \cap (HH^*) = P_i \cap H$=1 (as P_i avoids $H/1$, see (4-1)), which implies $p^2 \dagger |HH^*|$, and thus H^* would be a p'-group, contrary to (1). Thus all P_i cover HH^*/H. By definition, $P_i(HH^*) = P_iH$, so $H^* \leq P_iH$. Moreover, by Lemma 2.2(i), HH^*/H is a p-group. Since $H \cap H^* = 1$, we see that H^* must be a p-subgroup. Hence $|H^*| = p$ by (3).

Consider the chief factor HH^*/H^*. As H is not a p-group, but H^* is a p-group, we have $P_i(HH^*) \neq P_iH^*(= P)$. It follows that all P_i avoid HH^*/H^*, i.e., $P_i \cap H^* = P_i \cap (HH^*)$. As each P_i is of index p in P and $p^2 \big| |HH^*|$, it follows that $P_i \cap (HH^*) \neq 1$ for all i. Consequently, $P_i \cap H^* \neq 1$ for all P_i and hence, as H^* is of order p, $H^* \leq \bigcap_{i=1}^{d} P_i = \Phi(P)$, contrary to (2). Thus (4-2) holds.

(4-3) $G = PH$.

By $C_G(H) = 1$ (see (4-2)), we know that the non-abelian simple group H is a unique minimal normal subgroup of the subgroup PH. So all chief factors of PH are $H/1$ and some cyclic groups of order p. By (4-1) and Lemma 2.3, every member P_i in $\mathcal{M}_d(P)$ covers or avoids every chief factor of PH. Hence every member in $\mathcal{M}_d(P)$ is a CAP-subgroup of PH. Also, $N_{PH}(P)(\leq N_G(P))$ is certainly p-nilpotent since $N_G(P)$ is p-nilpotent. That is, the subgroup PH satisfies the hypothesis of the theorem. As H is not p-nilpotent, we conclude that $G = PH$ by the choice of G.

(4-4) Finishing the proof of (4).

Set $H_p = P \cap H$. It is clear that $H_p \neq 1$ is a Sylow p-subgroup of H and $P \leq N_G(H_p)$. By Frattini argument, we have $G = N_G(H_p)H$. Then $G/H \cong N_G(H_p)/N_H(H_p)$ is a p-group. Thus $N_G(H_p)$ has a normal series $N_G(H_p) \geq N_H(H_p) \geq H_p > 1$, where $N_H(H_p)/H_p$ is a p'-group. Therefore all chief factors of $N_G(H_p)$ either are p'-group or cyclic group of order p. By Lemma 2.4, all P_i are CAP-subgroups of $N_G(H_p)$. Noting that $P \leq N_G(H_p)$, we can see that $N_G(H_p)$ satisfies the hypothesis of our theorem and hence p-nilpotent. Thereby $N_H(H_p)$ is p-nilpotent. But then, H would be p-nilpotent by a theorem of Burnside [7, Theorem 7.2.1], in contradiction to the fact that H is a non-abelian simple group.

(5) $G = N_G(P)$.

By (2) and (4), we see that $O_p(G)$ is an elementary abelian p-group of order p^f, $f \geq 1$. Let N be any minimal normal subgroup of G contained in $O_p(G)$. From the proof of (3) we know that there exists a $P_i \in \mathcal{M}_d(P)$ such that P_i avoids the chief factor $N/1$. Applying Lemma 2.6, there exist normal subgroups $N_i(i = 1, ..., r)$ of G of order p and a subgroup M such that $O_p(G) = N_1 \times \cdots \times N_r$ and $G = O_p(G) \rtimes M$. Consider the intersection $T = \bigcap_{i=1}^{r} C_G(N_i)$. Since any normal subgroup $N_i(i = 1, 2, \ldots, r)$ of G of order p is normal in P, $N_i \leq Z(P)$ and so $P \leq T$. We claim that $T \cap M = 1$. Otherwise there exists a minimal normal subgroup N of G such that $N \leq T \cap M$, and N is not in $O_p(G)$, contrary to (4). Thus $T \cap M = 1$ holds and it follows that $P = O_p(G)$. Consequently, $G = N_G(P)$, as desired.

(6) Final contradiction.

Now, applying the condition that $N_G(P)$ is p-nilpotent, we can conclude that G is p-nilpotent, which is a final contradiction. The proof of the theorem is now complete.

Remark 2 : In proving Theorem 3.4, the assumption that $N_G(P)$ is p-nilpotent is essential. To illustrate the situation, we consider $G = A_5$ and $p = 5$. In this case, since the maximal subgroup of Sylow 5-subgroup G_5 of G is 1. We see that every member of $\mathcal{M}_d(G_5)$ is a CAP-subgroup of G, but G is not 5-nilpotent.

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