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NEW CRITERIA FOR λ-KOSZUL ALGEBRAS

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The main purpose of this paper is to provide some new criteria for a standard graded algebra $A = \bigoplus_{i\geq 0} A_i$ to be a λ-Koszul algebra, which was first introduced in [12] and was another class of "Koszul-type" algebras including Koszul and d-Koszul algebras as special examples.

Key words : λ-Koszul algebras; λ-Koszul modules; Yoneda algebras.

1. Introduction

Koszul algebras, a class of quadratic algebras possessing a lot of beautiful homological properties and applications in different branches of mathematics, were first introduced by Priddy in 1970 (see [17] and [18] is a very

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good introduction book for Koszul algebras. Recently, a lot of extensions of Koszul algebras have been done, see [3, 5-7, 9-10, 12-16] etc. More precisely, in order to find periodic resolutions for the trivial extension algebras of path algebras of Dynkin quivers in bipartite orientation, the notion of *almost Koszul algebra* was introduced in [3]; it is trivial that Koszul algebras are a class of homogeneous graded algebras, as an attempt to break through the "homogeneous" condition, Cassidy and Shelton introduced the notion of $\mathcal{K}_2$-*algebra*; motivated by the cubic Artin-Schelter regular algebras (see [2]), Berger generalized the notion of Koszul algebra to non-quadratic algebras and brought in the so-called *non-quadratic Koszul algebra*, and later, many people called this class of algebras *d-Koszul algebras* (see [9], [10], etc.). In particular, it turned out that d-Koszul algebras also have many nice homological properties and applications similar to Koszul algebras. After the appearance of d-Koszul algebras, many people became interested in them. For example, Green et al generalized this class of algebras to the non-local case inspired by the quiver theory (see [9]) and He-Lu studied them by the theory of A-infinity algebras (see [10]). Further, several generalizations of d-Koszul algebras were also done in the past of 3 years: To unify the notions of Koszul algebra and d-Koszul algebra, the notion of *piecewise-Koszul algebra* was introduced in 2007 (see [13]); it is also obvious that d-Koszul algebras and piecewise-Koszul algebras both have a single "jump-degree", in order to break through this limit, the second author of the present paper introduced the notion of λ-*Koszul algebra* in 2009 (see [12]), which clearly contains Koszul algebras and d-Koszul algebras as special examples.

It is of course interesting to find conditions such that a standard graded algebra to be a λ-Koszul algebra and [12] have provided some characterizations. In this paper, we will give some new criteria for λ-Koszul algebras. Further, motivated by Eakin-Nagata Theorem: "If $R \subset S$ is a subring of the commutative ring S and S is finitely generated as an R-module, then R is noetherian if and only if S is noetherian", a natural question is for which graded algebras A and which graded subalgebras B of A, we have that A is λ-Koszul if and only if B is λ-Koszul? We will discuss this question in the last section.

Roughly speaking, the main purpose of this paper is to prove the following two results:

***Theorem* 1.1** — *Let $A = \mathbb{K}\Gamma/I$ be a standard graded algebra and*

$$\cdots \longrightarrow P_n \xrightarrow{d_n} \cdots \longrightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} A_0 \longrightarrow 0$$

a minimal graded projective resolution of the trivial A-module A_0. Then the following statements are equivalent:

1. A is a λ-Koszul algebra;
2. $\mathrm{GR}_A^{\delta_\lambda}(A_0) = 0$;
3. A is a λ-Koszul module over A^e;
4. all the multiplications: $\mu : \mathrm{Ext}_A^1(A_0, A_0) \otimes \mathrm{Ext}_A^2(A_0, A_0) + \mathrm{Ext}_A^2(A_0, A_0) \otimes \mathrm{Ext}_A^1(A_0, A_0) \to \mathrm{Ext}_A^3(A_0, A_0)$, $\mu : \mathrm{Ext}_A^1(A_0, A_0) \otimes \mathrm{Ext}_A^4(A_0, A_0) + \mathrm{Ext}_A^2(A_0, A_0) \otimes \mathrm{Ext}_A^3(A_0, A_0) + \mathrm{Ext}_A^4(A_0, A_0) \otimes \mathrm{Ext}_A^1(A_0, A_0) \to \mathrm{Ext}_A^5(A_0, A_0)$, $\cdots$, $\mu : \mathrm{Ext}_A^1(A_0, A_0) \otimes \mathrm{Ext}_A^{2|\lambda|-2}(A_0, A_0) + \mathrm{Ext}_A^2(A_0, A_0) \otimes \mathrm{Ext}_A^{2|\lambda|-3}(A_0, A_0) + \mathrm{Ext}_A^4(A_0, A_0) \otimes \mathrm{Ext}_A^{2|\lambda|-5}(A_0, A_0) + \cdots + \mathrm{Ext}_A^{2|\lambda|-2}(A_0, A_0) \otimes \mathrm{Ext}_A^1(A_0, A_0) \to \mathrm{Ext}_A^{2|\lambda|-1}(A_0, A_0)$ and $\mu : \mathrm{Ext}_A^1(A_0, A_0) \otimes \mathrm{Ext}_A^{n-1}(A_0, A_0) + \mathrm{Ext}_A^2(A_0, A_0) \otimes \mathrm{Ext}_A^{n-2}(A_0, A_0) + \mathrm{Ext}_A^4(A_0, A_0) \otimes \mathrm{Ext}_A^{n-4}(A_0, A_0) + \cdots + \mathrm{Ext}_A^{2|\lambda|}(A_0, A_0) \otimes \mathrm{Ext}_A^{n-2|\lambda|}(A_0, A_0) \to \mathrm{Ext}_A^n(A_0, A_0)$ $(n \geq 2|\lambda|+1)$ are surjective, and $\mathrm{Ext}_A^{2k}(A_0, A_0) = \mathrm{Ext}_A^{2k}(A_0, A_0)_{-\delta_\lambda(2k)}$ for all $k = 1, 2, \cdots, |\lambda|$;
5. all the comultiplications: $\triangle : \mathrm{Tor}_3^A(A_0, A_0) \to \mathrm{Tor}_1^A(A_0, A_0) \otimes \mathrm{Tor}_2^A(A_0, A_0) + \mathrm{Tor}_2^A(A_0, A_0) \otimes \mathrm{Tor}_1^A(A_0, A_0)$, $\triangle : \mathrm{Tor}_5^A(A_0, A_0) \to \mathrm{Tor}_1^A(A_0, A_0) \otimes \mathrm{Tor}_4^A(A_0, A_0) + \mathrm{Tor}_2^A(A_0, A_0) \otimes \mathrm{Tor}_3^A(A_0, A_0) + \mathrm{Tor}_4^A(A_0, A_0) \otimes \mathrm{Tor}_1^A(A_0, A_0)$, $\cdots$, $\triangle : \mathrm{Tor}_{2|\lambda|-1}^A(A_0, A_0) \to \mathrm{Tor}_1^A(A_0, A_0) \otimes \mathrm{Tor}_{2|\lambda|-2}^A(A_0, A_0) + \mathrm{Tor}_2^A(A_0, A_0) \otimes \mathrm{Tor}_{2|\lambda|-3}^A(A_0, A_0) + \cdots + \mathrm{Tor}_{2|\lambda|-2}^A(A_0, A_0) \otimes \mathrm{Tor}_1^A(A_0, A_0)$ and $\triangle : \mathrm{Tor}_n^A(A_0, A_0) \to \mathrm{Tor}_1^A(A_0, A_0) \otimes \mathrm{Tor}_{n-1}^A(A_0, A_0) + \mathrm{Tor}_2^A(A_0, A_0) \otimes \mathrm{Tor}_{n-2}^A(A_0, A_0) + \cdots + \mathrm{Tor}_{2|\lambda|}^A(A_0, A_0) \otimes \mathrm{Tor}_{n-2|\lambda|}^A(A_0, A_0)$ $(n \geq 2|\lambda| + 1)$ are injective, and $\mathrm{Tor}_{2k}^A(A_0, A_0) = \mathrm{Tor}_{2k}^A(A_0, A_0)_{\delta_\lambda(2k)}$ for all $k = 1, 2, \cdots, |\lambda|$;
6. if the trivial A-module A_0 admits a pure resolution and M is a λ-Koszul module, then the Ext module $\bigoplus_{i\geq 0} \mathrm{Ext}_A^i(M, A_0)$ is generated in degree 0 as a graded $\bigoplus_{i\geq 0} \mathrm{Ext}_A^i(A_0, A_0)$-module.

***Theorem* 1.2** — *Let H be a finite dimensional semisimple and cosemisimple Hopf algebra, $A = \bigoplus_{n\geq 0} A_n$ be a graded right H-module algebra such that*

A_i is finite dimensional for all $i \geq 0$, and let $B = A^{coH}$, the coinvariant subalgebra of A. Suppose that A/B is an H-Galois graded extension. Then B is a λ-Koszul algebra if and only if A is a λ-Koszul algebra.

Throughout, $\mathbb{K}$ denotes a fixed field, $\mathbb{N} = \{0, 1, 2, \cdots\}$ denotes the set of natural numbers. A positively graded $\mathbb{K}$-algebra $A = \bigoplus_{i \geq 0} A_i$ is called standard if and only if

- $A_0 = \mathbb{K} \times \cdots \times \mathbb{K}$, a finite product of $\mathbb{K}$;
- $A_i \cdot A_j = A_{i+j}$ for all $0 \leq i, j < \infty$;
- $\dim_{\mathbb{K}} A_i < \infty$ for all $i \geq 0$.

2. Definitions and Examples

In this section, we will recall the definition of λ-Koszul algebra and give some new examples.

Definition 2.1 — ([12]) Let A be a standard graded algebra and M a finitely generated graded A-module. Let

$$\cdots \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

be a minimal graded projective resolution of M. Then M is called a *λ-Koszul module* if each P_n is generated in degree $\delta_\lambda(n)$ for all $n \geq 0$, where $\lambda : \mathbb{N}^* \to \mathbb{N}^*$ is a periodic set function and $|\lambda|$ denotes the smallest positive period of λ with $\lambda(1) \geq 1$ and λ being strictly increasing on the interval $[1, |\lambda|]$; and $\delta_\lambda : \mathbb{N} \to \mathbb{N}$ is another set function satisfying

1. $\delta_\lambda(0) = 0,\ \delta_\lambda(1) = 1,\ \delta_\lambda(2) = d$, where $d = \lambda(1) + 1$, a fixed integer;
2. $\delta_\lambda(2n+1) - \delta_\lambda(2n) = 1$ for all $n \geq 0$;
3. $\delta_\lambda(2n) - \delta_\lambda(2n-1) = \lambda(n)$ for all $n \geq 1$.

In particular, the standard graded algebra A will be called a *λ-Koszul algebra* provided that the trivial A-module A_0 is a λ-Koszul module.

Example 2.2 : The following are some trivial examples of λ-Koszul algebras:

1. Koszul algebras (see [17] and [18], etc.) are special λ-Koszul algebras in the sense of $d = 2$ and $\lambda(n) = 1$ for all $n \geq 1$.

2. d-Koszul algebras (see [5] and [9], etc.) are special λ-Koszul algebras in the sense of $|\lambda| = 1$ for all $n \geq 1$.

Example 2.3 — Let $A = \bigoplus_{i\geq 0} A_i$ be a standard graded algebra and $M = \bigoplus_{i\geq 1} M_i$ a graded A-module generated in degree 1. Let

$$B = \begin{pmatrix} A & M \\ 0 & \mathbb{K} \end{pmatrix}$$

be the matrix algebra. It is easy to see that B is also a standard graded algebra under the grading:

$$B_0 = \begin{pmatrix} A_0 & 0 \\ 0 & \mathbb{K} \end{pmatrix}, \quad \text{and} \quad B_i = \begin{pmatrix} A_i & M_i \\ 0 & 0 \end{pmatrix} \ (\forall\, i \geq 1).$$

Let

$$\cdots \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A_0 \longrightarrow 0$$

and

$$\cdots \longrightarrow Q_n \longrightarrow \cdots \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow M \longrightarrow 0$$

be the corresponding minimal graded projective resolutions. By a routine computation, we get that

$$\cdots \to \begin{pmatrix} P_n \\ 0 \end{pmatrix} \oplus \begin{pmatrix} Q_{n-1} \\ 0 \end{pmatrix} \to \cdots \to \begin{pmatrix} P_0 \\ 0 \end{pmatrix} \oplus \begin{pmatrix} M \\ \mathbb{K} \end{pmatrix} \to B_0 \to 0$$

is a minimal graded B-projective resolution of the trivial B-module B_0; moreover, for all $n \geq 1$, the graded projective module $\begin{pmatrix} P_n \\ 0 \end{pmatrix} \oplus \begin{pmatrix} Q_{n-1} \\ 0 \end{pmatrix}$ is generated in degree $s \in \mathbb{N}$ as a graded B-module if and only if both P_n and Q_{n-1} are generated in degree $s \in \mathbb{N}$ as graded A-modules. Now it is easy to see that B is a λ-Koszul algebra if and only if A is a λ-Koszul algebra, $\Omega^{2|\lambda|-1}(M)[-\delta_\lambda(2|\lambda|)]$ is a λ-Koszul module and Q_i is generated in degree $\delta_\lambda(i+1)$ for all $(i = 1,\ 2,\ \ldots,\ 2|\lambda| - 2)$, which is equivalent to that P_{i+1} and Q_i are generated in degree $\delta_\lambda(i+1)$ as graded A-modules for all $i \geq 0$.

3. Proof of Theorem 1.1

Theorem 1.1 is immediate from the following lemmas.

We begin with

Definition 3.1 — Let $M = \bigoplus_{i\geq 0} M_i$ be a finitely generated graded module over A and A a standard graded algebra. The classical *Castelnuovo-Mumford regularity* of M is defined as

$$\sup\{j - i : \mathrm{Tor}_i^A(M, A_0)_j \neq 0, \text{ for all } i \geq 0\},$$

denoted by $\mathrm{R}_A(M)$. We will call

$$\sup\{|j - f(i)| : \mathrm{Tor}_i^A(M, A_0)_j \neq 0, \text{ for all } i \geq 0\}$$

the *f-generalized Castelnuovo-Mumford regularity* of M, denoted by $\mathrm{GR}_A^f(M)$, where "$|\cdot|$" stands for the absolute value and $f : \mathbb{N} \to \mathbb{N}$ is an arbitrary set function.

Lemma 3.2 — Let A be a standard graded algebra. Then A is a δ-Koszul algebra for some set function $\delta : \mathbb{N} \to \mathbb{N}$ (i.e., the trivial A-module has a minimal graded projective resolution $\cdots \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A_0 \longrightarrow 0$ such that each P_n is generated in degree $\delta(n)$) if and only if $\mathrm{GR}_A^\delta(A_0) = 0$.

Proof : ($\Rightarrow$) Let A be a δ-Koszul algebra. Then the trivial A-module A_0 admits a minimal graded projective resolution

$$\cdots \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A_0 \longrightarrow 0$$

with each P_n generated in degree $\delta(n)$. Under a routine computation, we have $\mathrm{Tor}_i^A(A_0, A_0) = \mathrm{Tor}_i^A(A_0, A_0)_{\delta(i)}$ for all $i \geq 0$. Recall that $\mathrm{GR}_A^\delta(A_0) = \sup\{|j - \delta(i)| : \mathrm{Tor}_i^A(M, A_0)_j \neq 0, \text{ for all } i \geq 0\}$, which implies that $\mathrm{GR}_A^\delta(A_0) = 0$.

($\Leftarrow$) Assume that $\mathrm{GR}_A^\delta(A_0) = 0$, from the definition of $\mathrm{GR}_A^\delta(A_0)$, we have $\sup\{|\delta(i) - j| : \mathrm{Tor}_i^A(A_0, A_0)_j \neq 0, \text{ for all } i \geq 0\} = 0$, which implies that $\mathrm{Tor}_i^A(A_0, A_0) = \mathrm{Tor}_i^A(A_0, A_0)_{\delta(i)}$ for all $i \geq 0$. Therefore, the trivial A-module A_0 admits a minimal graded projective resolution

$$\cdots \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A_0 \longrightarrow 0$$

such that each P_i is generated by homogeneous elements of degree $\delta(i)$ for all $i \geq 0$. Thus, A is a δ-Koszul algebra. □

Lemma 3.3 — ([8]) Let A be a standard graded algebra. Then there exists a finite quiver $\Gamma = (\Gamma_0, \Gamma_1)$ and a graded ideal I in $\mathbb{K}\Gamma$ with $I \subset \sum_{n\geq 2}(\mathbb{K}\Gamma)_n$ such that $A \cong \mathbb{K}\Gamma/I$ as graded algebras, where Γ_0 denotes the set of vertices and Γ_1 denotes the set of arrows of the quiver Γ, respectively.

Lemma 3.4 — Let A be a standard graded algebra and $A^e := A \otimes_{\mathbb{K}} A^{op}$ its enveloping algebra. Let $\mathbf{r}$ be the graded Jacobson radical of A^e and $f : P \to Q$ be a homomorphism of finitely generated A^e-projective modules. Then $Imf \subseteq \mathbf{r}Q$ if and only if for each simple A-module S, we have $Im(f \otimes_A 1_S) \subseteq J(Q \otimes_A S)$, where J is the graded Jacobson radical of the standard graded algebra A.

Proof : ($\Leftarrow$) For the sake of convenience, we may suppose that $Q = Av \otimes_{\mathbb{K}} wA$ is an indecomposable A^e-module, where $v, w \in \Gamma_0$ and we use the notations of Lemma 3.3. Assume f is an epimorphism, so

$$P \stackrel{f}{\longrightarrow} Av \otimes_{\mathbb{K}} wA \longrightarrow 0$$

is a splittable epimorphism, which implies the exact sequence

$$P \otimes_A M \stackrel{f\otimes_A 1_M}{\longrightarrow} Av \otimes_{\mathbb{K}} wA \otimes_A M \longrightarrow 0$$

of A-modules for any A-module M. In particular, if we choose $M = Aw/Jw := S$, a simple A-module, then we get an epimorphism

$$P \otimes_A S \stackrel{f\otimes_A 1_S}{\longrightarrow} Av \otimes_{\mathbb{K}} wA \otimes_A S \cong Av \longrightarrow 0.$$

Now by the hypothesis $Im(f \otimes_A 1_S) \subseteq J(Q \otimes_A S)$, we have that $Imf \subseteq \mathbf{r}Q$.

($\Rightarrow$) Suppose that we have the condition $Imf \subseteq \mathbf{r}Q$. Similarly, we may assume that $Q = Av \otimes_{\mathbb{K}} wA$ is an indecomposable A^e-module. Note that for each simple A-module $S \neq Aw/Jw$, we have $Q \otimes_A S = 0$. Thus it suffices to prove the case of $S = Aw/Jw$.

Consider the following commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{f} & Q \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} \\ P\otimes_A S & \xrightarrow{f\otimes_A 1_S} & Q\otimes_A S, \end{array}$$

where α and β are the splittable A-epimorphisms given by the split exact sequences in the category of finitely generated A-modules. More precisely, taking β for example, β is determined by the following split exact sequence

$$0 \longrightarrow Av\otimes_{\mathbb{k}} wJ \longrightarrow Av\otimes_{\mathbb{k}} wA \xrightarrow{\beta} Av \longrightarrow 0.$$

Note that $\beta^{-1}(v) = v\otimes w + Av\otimes wJ$, thus each element in the preimage of v is an A^e-generator for the module $Q = Av\otimes_{\mathbb{K}} wA$. If $f\otimes_A 1_S$ is an epimorphism, then βf is an epimorphism and $\beta^{-1}(v)\cap Imf \neq 0$, which implies that Imf contains an A^e-generator of the cyclic module Q, so f is an epimorphism. Therefore, we have $Im(f\otimes_A 1_S)\subseteq J(Q\otimes_A S)$, as desired.□

Lemma 3.5 — Let A be a standard graded algebra and A^e its enveloping algebra. Then A is a λ-Koszul algebra if and only if A is a λ-Koszul module over A^e.

Proof : Note that for an indecomposable A^e-projective module $P = Av\otimes_{\mathbb{K}} wA$ and an A-module M, it is easy to see that $P\otimes_A M = (Av)^{\dim wM}$ as an A-module since $Av\otimes_{\mathbb{K}} wA\otimes_A M \cong Av\otimes_{\mathbb{K}} wM$. In particular, if $M = S$ a simple A-module, then as A-modules we have $P\otimes_A S\cong Av$ if $wS\neq 0$ and $P\otimes_A S = 0$ otherwise. Hence it is enough to consider the case of $S = Aw/Jw$ here.

Let

$$\mathcal{P}_* : \cdots \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0$$

be a graded projective A^e-resolution of A. Then by Lemma 3.4 and the above observations, $\mathcal{P}_*$ is a minimal graded projective A^e-resolution of A if and only if $\mathcal{P}_*\otimes_A A_0$:

$$\cdots \longrightarrow P_n\otimes_A A_0 \longrightarrow \cdots \longrightarrow P_1\otimes_A A_0 \longrightarrow$$

$$P_0\otimes_A A_0 \longrightarrow A\otimes_A A_0 \cong A_0 \longrightarrow 0$$

is a minimal graded A-projective resolution of A_0. Further, for all $i \geq 0$, P_i is generated in degree s as a graded A^e-module if and only if $P_i \otimes_A A_0$ is generated in degree s as a graded A-module. Now we finish the proof.

Lemma 3.6 — Let A be a standard graded algebra with the trivial A-module A_0 possessing a pure resolution and M a λ-Koszul module. Then the Ext module $\bigoplus_{i\geq 0} \mathrm{Ext}_A^i(M, A_0)$ is generated in degree 0 as a graded $\bigoplus_{i\geq 0} \mathrm{Ext}_A^i(A_0, A_0)$-module if and only if A is a λ-Koszul algebra.

PROOF : Let

$$\mathcal{P}_* : \cdots \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A_0 \longrightarrow 0$$

and

$$\mathcal{Q}_* : \cdots \longrightarrow Q_n \longrightarrow \cdots \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow M \longrightarrow 0$$

be the minimal graded projective resolutions of A_0 and M, respectively. By hypothesis, M is a λ-Koszul module. Then for all $n \geq 0$, Q_n is generated in degree $\delta_\lambda(n)$.

($\Rightarrow$) For all $n \geq 1$, then $\mathrm{Ext}_A^n(M, A_0) = \mathrm{Ext}_A^n(A_0, A_0) \cdot \mathrm{Ext}_A^0(M, A_0)$ by hypothesis. Note that $\mathrm{Ext}_A^0(M, A_0) = \mathrm{Ext}_A^0(M, A_0)_0$ and A is a positively graded algebra with the trivial A-module A_0 having a pure resolution, which implies that $\mathrm{Ext}_A^n(A_0, A_0) = \mathrm{Ext}_A^n(A_0, A_0)_{-s}$ for some natural number s. Now observe that $\mathrm{Ext}_A^n(M, A_0) = \mathrm{Ext}_A^n(M, A_0)_{-\delta_\lambda(n)}$ since M is a λ-Koszul module, so we have $\mathrm{Ext}_A^n(A_0, A_0) = \mathrm{Ext}_A^n(A_0, A_0)_{-\delta_\lambda(n)}$ for all $n \geq 0$, which implies easily that A is a λ-Koszul algebra.

($\Leftarrow$) Suppose that A is a λ-Koszul algebra. Then as a trivial A-module, A_0 admits a minimal graded projective resolution

$$\cdots \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A_0 \longrightarrow 0$$

such that each projective module P_n is generated in degree $\delta_\lambda(n)$ for all $n \geq 0$. Note that M is also a λ-Koszul module. Thus M has a minimal graded projective resolution

$$\cdots \longrightarrow Q_n \longrightarrow \cdots \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow M \longrightarrow 0$$

such that each projective module Q_n is generated in degree $\delta_\lambda(n)$ for all $n \geq 0$. Then by Proposition 3.5 of [9], we have $\mathrm{Ext}_A^i(M, A_0) = \mathrm{Ext}_A^i(A_0, A_0)\cdot$

$\mathrm{Ext}_A^0(M, A_0)$ for all $i \geq 0$. That is, $\bigoplus_{i\geq 0} \mathrm{Ext}_A^i(M, A_0)$ is generated by $\mathrm{Ext}_A^0(M, A_0)$. □

Now let us recall the so-called normalized bar and cobar resolution.

Let $A = \bigoplus_{i\geq 0} A_i$ be a standard graded algebra and $J = \bigoplus_{i\geq 1} A_i$ be the graded Jacobson radical of A. Then the trivial A-module A_0 possesses a canonical graded projective resolution:

$$\cdots \longrightarrow Bar^n(A) \xrightarrow{\partial'_n} \cdots \longrightarrow Bar^1(A) \xrightarrow{\partial'_1} Bar^0(A) \xrightarrow{\partial'_0} A_0 \longrightarrow 0,$$

where for all $n \geq 0$, $Bar^n(A) := A \otimes_{A_0} J^{\otimes n}$ and the differential $\partial'_n : A \otimes_{A_0} J^{\otimes n} \longrightarrow A \otimes_{A_0} J^{\otimes n-1}$ is defined by

$$\partial'_n(a_0 \otimes a_1 \otimes \cdots \otimes a_n) := \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n,\ (a_0 \in A, a_1, \cdots, a_n \in J).$$

Note that $A_0 \otimes_A Bar^n(A) = A_0 \otimes_A A \otimes_{A_0} J^{\otimes n} \cong J^{\otimes n}$ for all $n \geq 0$, we get the following complex

$$\cdots \longrightarrow J^{\otimes n} \xrightarrow{\partial_n} \cdots \longrightarrow J^{\otimes 2} \xrightarrow{\partial_2} J^{\otimes 1} \xrightarrow{\partial_1} J^0 \xrightarrow{\partial_0} A_0$$

with

$$\partial_n(a_1 \otimes a_2 \otimes \cdots \otimes a_n) := \sum_{i=0}^{n-1} (-1)^i a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n,\ (a_1, \cdots, a_n \in J).$$

Now it is obvious that

$$\mathrm{Tor}_n^A(A_0, A_0) = \ker \partial_n / Im \partial_{n+1}.$$

The cobar complex is the cochain complex $Cob^\bullet(A)$ defined by $Cob^n(A) := \mathrm{Hom}_A(J^{\otimes n}, A_0)$ for all $n \geq 0$, where the differential $\partial^*_{n+1} : Cob^n(A) \to Cob^{n+1}(A)$ is the pullback of ∂. Clearly, for all $n \geq 0$, we have

$$\mathrm{Ext}_A^n(A_0, A_0) = \ker \partial^*_{n+1} / Im \partial^*_n.$$

We usually call $T(A)$ the *Yoneda coalgebra* of A, and $E(A)$ the *Yoneda algebra* of A.

Lemma 3.7 — Using the above notations, we have the following statements.

1. $T(A) := \bigoplus_{n\geq 0} \mathrm{Tor}_n^A(A_0, A_0)$ is a bigraded coalgebra with the comultiplication $\overline{\triangle} = \sum_{n,i} \overline{\triangle}_{n,i}$, where $\overline{\triangle}_{n,i}$ is induced by $\triangle_{n,i} : J^{\otimes n} \to J^{\otimes i} \otimes J^{\otimes n-i}$ via $\triangle_{n,i}(a_1 \otimes \cdots \otimes a_n) = (a_1 \otimes \cdots \otimes a_i) \otimes (a_{i+1} \otimes \cdots \otimes a_n)$.

2. $E(A) := \bigoplus_{n\geq 0} \mathrm{Ext}_A^n(A_0, A_0)$ is a bigraded algebra with the multiplication $\widetilde{\mu} = \sum_{i,n} \widetilde{\mu}_{i,n-i}$, where $\widetilde{\mu}_{i,n-i}$ is induced by $\mu_{i,n-i} : Cob^i(A) \otimes Cob^{n-i}(A) \to Cob^n(A)$ via $\mu_{i,n-i}(f \otimes g)(a_1 \otimes a_2) := f(a_1) \otimes g(a_2)$.

Proof : (1) It is easy to check that $\triangle = \sum_{n,i} \triangle_{n,i}$ provides a comultiplicative structure for the complex $J^{\otimes \bullet}$ and preserves kernels and images. Thus $(J^{\otimes \bullet}, \partial, \triangle)$ is a differential graded coalgebra and $T(A)$ a graded coalgebra. Note that now A is a standard graded algebra, which implies that $T(A)$ a bigraded coalgebra.

(2) It is also easy to check that $\mu = \sum_{n,i} \mu_{i,n-i}$ provides a multiplicative structure for the complex $Cob^\bullet(A)$ and preserves kernels and images. Thus $(Cob^\bullet(A), \partial^*, \mu)$ is a differential graded algebra and $E(A)$ a graded algebra. Note that now A is a standard graded algebra, which implies that $E(A)$ a bigraded algebra.

Lemma 3.8 — The map $\mu_{n-i,i} : Cob^{n-i}(A) \otimes Cob^i(A) \to Cob^n(A)$ and $\triangle_{n,i} : J^{\otimes n} \to J^{\otimes n-i} \otimes J^{\otimes i}$ are dual to one another.

Proof : Let $f_1 \otimes \cdots \otimes f_i \in Cob^i(A)$, $g_1 \otimes \cdots \otimes g_{n-i} \in Cob^{n-i}(A)$ and $a_1 \otimes \cdots \otimes a_n \in J^{\otimes n}$. Then

$$\begin{aligned}\triangle^*((f_1 \otimes \cdots \otimes f_i) \quad \otimes \quad & (g_1 \otimes \cdots \otimes g_{n-i}))(a_1 \otimes \cdots \otimes a_n) \\ = \quad & ((f_1 \otimes \cdots \otimes f_i) \otimes (g_1 \otimes \cdots \otimes g_{n-i}))\triangle(a_1 \otimes \cdots \otimes a_n) \\ = \quad & (f_1 \otimes \cdots \otimes f_i)(a_1 \otimes \cdots \otimes a_i)(g_1 \otimes \cdots \otimes g_{n-i}) \\ & (a_{i+1} \otimes \cdots \otimes a_n) \\ = \quad & \mu((f_1 \otimes \cdots \otimes f_i) \otimes (g_1 \otimes \cdots \otimes g_{n-i}))(a_1 \otimes \cdots \otimes a_n).\end{aligned}$$

Therefore, we are done.

Lemma 3.9 — Let A be a standard graded algebra. Then the following statements are equivalent:

1. A is a λ-Koszul algebra;

2. all the multiplications: $\mu : \mathrm{Ext}_A^1(A_0, A_0) \otimes \mathrm{Ext}_A^2(A_0, A_0) + \mathrm{Ext}_A^2(A_0, A_0) \otimes \mathrm{Ext}_A^1(A_0, A_0) \to \mathrm{Ext}_A^3(A_0, A_0)$, $\mu : \mathrm{Ext}_A^1(A_0, A_0) \otimes \mathrm{Ext}_A^4(A_0, A_0) + \mathrm{Ext}_A^2(A_0, A_0) \otimes \mathrm{Ext}_A^3(A_0, A_0) + \mathrm{Ext}_A^4(A_0, A_0) \otimes \mathrm{Ext}_A^1(A_0, A_0) \to \mathrm{Ext}_A^5(A_0, A_0), \cdots, \mu : \mathrm{Ext}_A^1(A_0, A_0) \otimes \mathrm{Ext}_A^{2|\lambda|-2}(A_0, A_0) + \mathrm{Ext}_A^2(A_0, A_0) \otimes \mathrm{Ext}_A^{2|\lambda|-3}(A_0, A_0) + \mathrm{Ext}_A^4(A_0, A_0) \otimes \mathrm{Ext}_A^{2|\lambda|-5}(A_0, A_0) + \cdots + \mathrm{Ext}_A^{2|\lambda|-2}(A_0, A_0) \otimes \mathrm{Ext}_A^1(A_0, A_0) \to \mathrm{Ext}_A^{2|\lambda|-1}(A_0, A_0)$ and $\mu : \mathrm{Ext}_A^1(A_0, A_0) \otimes \mathrm{Ext}_A^{n-1}(A_0, A_0) + \mathrm{Ext}_A^2(A_0, A_0) \otimes \mathrm{Ext}_A^{n-2}(A_0, A_0) + \mathrm{Ext}_A^4(A_0, A_0) \otimes \mathrm{Ext}_A^{n-4}(A_0, A_0) + \cdots + \mathrm{Ext}_A^{2|\lambda|}(A_0, A_0) \otimes \mathrm{Ext}_A^{n-2|\lambda|}(A_0, A_0) \to \mathrm{Ext}_A^n(A_0, A_0)$ $(n \geq 2|\lambda|+1)$ are surjective, and $\mathrm{Ext}_A^{2k}(A_0, A_0) = \mathrm{Ext}_A^{2k}(A_0, A_0)_{-\delta_\lambda(2k)}$ for all $k = 1, 2, \cdots, |\lambda|$;

3. all the comultiplications: $\triangle : \mathrm{Tor}_3^A(A_0, A_0) \to \mathrm{Tor}_1^A(A_0, A_0) \otimes \mathrm{Tor}_2^A(A_0, A_0) + \mathrm{Tor}_2^A(A_0, A_0) \otimes \mathrm{Tor}_1^A(A_0, A_0)$, $\triangle : \mathrm{Tor}_5^A(A_0, A_0) \to \mathrm{Tor}_1^A(A_0, A_0) \otimes \mathrm{Tor}_4^A(A_0, A_0) + \mathrm{Tor}_2^A(A_0, A_0) \otimes \mathrm{Tor}_3^A(A_0, A_0) + \mathrm{Tor}_4^A(A_0, A_0) \otimes \mathrm{Tor}_1^A(A_0, A_0), \cdots, \triangle : \mathrm{Tor}_{2|\lambda|-1}^A(A_0, A_0) \to \mathrm{Tor}_1^A(A_0, A_0) \otimes \mathrm{Tor}_{2|\lambda|-2}^A(A_0, A_0) + \mathrm{Tor}_2^A(A_0, A_0) \otimes \mathrm{Tor}_{2|\lambda|-3}^A(A_0, A_0) + \cdots + \mathrm{Tor}_{2|\lambda|-2}^A(A_0, A_0) \otimes \mathrm{Tor}_1^A(A_0, A_0)$ and $\triangle : \mathrm{Tor}_n^A(A_0, A_0) \to \mathrm{Tor}_1^A(A_0, A_0) \otimes \mathrm{Tor}_{n-1}^A(A_0, A_0) + \mathrm{Tor}_2^A(A_0, A_0) \otimes \mathrm{Tor}_{n-2}^A(A_0, A_0) + \cdots + \mathrm{Tor}_{2|\lambda|}^A(A_0, A_0) \otimes \mathrm{Tor}_{n-2|\lambda|}^A(A_0, A_0)$ $(n \geq 2|\lambda| + 1)$ are injective, and $\mathrm{Tor}_{2k}^A(A_0, A_0) = \mathrm{Tor}_{2k}^A(A_0, A_0)_{\delta_\lambda(2k)}$ for all $k = 1, 2, \cdots, |\lambda|$.

PROOF : For the sake of simplicity, let us recall one of the main results of [12]–Theorem 2.1 first: The standard graded algebra $A = \bigoplus_{i \geq 0} A_i$ is λ-Koszul if and only if the Yoneda algebra $E(A)$ is minimally generated in degrees $1, 2, 4, \cdots, 2|\lambda|$ and $\mathrm{Ext}_A^{2k}(A_0, A_0) = \mathrm{Ext}_A^{2k}(A_0, A_0)_{-\delta_\lambda(2k)}$ for all $k = 1, 2, \cdots, |\lambda|$. Now by Lemma 3.8, we only need to prove (1) $\Leftrightarrow$ (2) since (3) is the dual version of (2).

(1) $\Rightarrow$ (2) Suppose that A is a λ-Koszul algebra, then by Theorem 2.1 of [12], we have that the Yoneda algebra $E(A)$ is minimally generated in degrees $1, 2, 4, \cdots, 2|\lambda|$ and $\mathrm{Ext}_A^{2k}(A_0, A_0) = \mathrm{Ext}_A^{2k}(A_0, A_0)_{-\delta_\lambda(2k)}$ for all $k = 1, 2, \cdots, |\lambda|$, which implies (2) clearly.

(2) $\Rightarrow$ (1) Suppose we have the condition (2), obviously we have that the Yoneda algebra $E(A)$ is minimally generated in degrees $1, 2, 4, \cdots, 2|\lambda|$ and $\mathrm{Ext}_A^{2k}(A_0, A_0) = \mathrm{Ext}_A^{2k}(A_0, A_0)_{-\delta_\lambda(2k)}$ for all $k = 1, 2, \cdots, |\lambda|$. Now by Theorem 2.1 of [12], we obtain that A is a λ-Koszul algebra. $\square$

Now putting Lemmas 3.2, 3.5, 3.6 and 3.9 together, we have proved Theorem 1.1.

4. Proof of Theorem 1.2

The following lemma is obvious and we omit the proof.

Lemma 4.1 — Let A be a standard graded algebra and $\mathrm{Ext}_A^*(A_0, A_0)$ be its Yoneda algebra. Then A is an λ-Koszul algebra if and only if $\mathrm{Ext}_A^i(A_0, A_0) = \mathrm{Ext}_A^i(A_0, A_0)_{-\delta_\lambda(i)}$ for all $i \geq 0$.

Lemma 4.2 — ([11]) Let H be a finite dimensional semisimple and cosemisimple Hopf algebra and A/B be an H-Galois graded extension. If $A = \bigoplus_{i\geq 0} A_i$ is a positively graded algebra, then A_0/B_0 is an H-Galois extension.

Lemma 4.3 — ([11]) Let H be a finite dimensional semisimple and cosemisimple Hopf algebra, $A = \bigoplus_{n\geq 0} A_n$ be a graded right H-module algebra and $B = A^{coH}$, the coinvariant subalgebra. Suppose that A/B is an H-Galois graded extension. Then we have an isomorphism of bigraded algebras

$$\mathrm{Ext}_B^*(A_0, A_0) \cong \mathrm{Ext}_A^*(A_0, A_0)\#H,$$

where the bigradeding of $\mathrm{Ext}_A^*(A_0, A_0)\#H$ is induced from that of $\mathrm{Ext}_A^*(A_0, A_0)$.

***Theorem* 4.4** — *Let H be a finite dimensional semisimple and cosemisimple Hopf algebra, $A = \bigoplus_{n\geq 0} A_n$ be a graded right H-module algebra such that A_i is finite dimensional for all $i \geq 0$, and let $B = A^{coH}$, the coinvariant subalgebra of A. Suppose that A/B is an H-Galois graded extension. Then B is a λ-Koszul algebra if and only if A is a λ-Koszul algebra.*

Proof : ($\Rightarrow$) Suppose that B is a λ-Koszul algebra, then by the definition, B_0 is a finite dimensional semisimple algebra. By Lemma 4.2, A_0/B_0 is an H-Galois extension since A/B is an H-Galois graded extension. Now note that $A_0\#H$ and B_0, A_0 and $(A_0\#H)\#H^*$ are both Morita equivalent, and H is a finite dimensional semisimple and cosemisimple Hopf algebra, we have that A_0 is a semisimple algebra. Further, as a right B_0-module, $A_0 = B_0 \oplus S$ for some finite dimensional B_0-module S. Now recall that B is a λ-Koszul algebra, by Lemma 4.1, which is equivalent to that $\mathrm{Ext}_B^i(B_0, B_0) = \mathrm{Ext}_B^i(B_0, B_0)_{-\delta_\lambda(i)}$ for all $i \geq 0$. Note that S is a direct summand of a finite sum of B_0, which implies that $\mathrm{Ext}_B^i(B_0, S) = \mathrm{Ext}_B^i(B_0, S)_{-\delta_\lambda(i)}$,

$\mathrm{Ext}_B^i(S, B_0) = \mathrm{Ext}_B^i(S, B_0)_{-\delta_\lambda(i)}$ and $\mathrm{Ext}_B^i(S, S) = \mathrm{Ext}_B^i(S, S)_{-\delta_\lambda(i)}$ for all $i \geq 0$. Also observe that we have the following isomorphism

$$\mathrm{Ext}_B^i(A_0, A_0) = \mathrm{Ext}_B^i(B_0, B_0) \oplus \mathrm{Ext}_B^i(B_0, S) \oplus \mathrm{Ext}_B^i(S, B_0) \oplus \mathrm{Ext}_B^i(S, S)$$

for all $i \geq 0$, which implies that $\mathrm{Ext}_B^i(A_0, A_0) = \mathrm{Ext}_B^i(A_0, A_0)_{-\delta_\lambda(i)}$ for all $i \geq 0$. By Lemma 4.3, we have

$$\mathrm{Ext}_A^i(A_0, A_0)\#H = (\mathrm{Ext}_A^i(A_0, A_0)\#H)_{-\delta_\lambda(i)}$$

for all $i \geq 0$. Now by the definition of the bigrading of $\mathrm{Ext}_A^i(A_0, A_0)\#H$, we obtain

$$\mathrm{Ext}_A^i(A_0, A_0) = \mathrm{Ext}_A^i(A_0, A_0)_{-\delta_\lambda(i)}$$

for all $i \geq 0$. By Lemma 4.1 again, we get that A is a λ-Koszul algebra.

($\Leftarrow$) Suppose that A is a λ-Koszul algebra, then A_0 is a semisimple algebra. Similar to the proof of necessity, we have that A_0 is a semisimple algebra and as a right B_0-module, $A_0 = B_0 \oplus S$ for some finite dimensional B_0-module S. Now recall that A is a λ-Koszul algebra, by Lemma 4.1, which is equivalent to

$$\mathrm{Ext}_A^i(A_0, A_0) = \mathrm{Ext}_A^i(A_0, A_0)_{-\delta_\lambda(i)}$$

for all $i \geq 0$. By Lemma 4.3, we have

$$\mathrm{Ext}_B^i(A_0, A_0) = \mathrm{Ext}_B^i(A_0, A_0)_{-\delta_\lambda(i)}$$

for all $i \geq 0$. Note that $A_0 = B_0 \oplus S$ and S is a direct summand of a finite sum of B_0, which imply that

$$\mathrm{Ext}_B^i(A_0, A_0) = \mathrm{Ext}_B^i(B_0, B_0) \oplus \mathrm{Ext}_B^i(B_0, S) \oplus \mathrm{Ext}_B^i(S, B_0) \oplus \mathrm{Ext}_B^i(S, S)$$

for all $i \geq 0$. Now it is easy to see that $\mathrm{Ext}_B^i(B_0, B_0) = \mathrm{Ext}_B^i(B_0, B_0)_{-\delta_\lambda(i)}$ for all $i \geq 0$. By Lemma 4.1, we get that B is a λ-Koszul algebra. □

Remark 4.5 : Theorem 4.4 perfects Theorem 2.11 of [11] since d-Koszul algebras are special λ-Koszul algebras.

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