Indian J. Pure Appl. Math., **42**(3): 173-181, June 2011

A PROOF OF EWELL'S OCTUPLE PRODUCT IDENTITY

Zhu Cao* and Xinyun Zhu**

**Department of Mathematics, University of Mississippi,*
University, MS 38677, *USA*
e-mail: zcao3@olemiss.edu
***Department of Mathematics, University of Texas of the Permian Basin,*
Odessa, TX, 79762
e-mail: zhu_x@utpb.edu

(*Received* 2 *September* 2010; *after final revision* 27 *March* 2011;
accepted 8 *April* 2011)

We give a new proof of Ewell's octuple product identity in [7] by using a general theorem developed by the first author in [3].

Key words : Jacobi's triple product identity, octuple product identity, Ramanujan's theta functions.

1. Introduction and Main Theorem

In Theorem 1 of [7], Ewell gave the following octuple product identity. Henceforth, we call it Ewell's octuple product identity.

***Theorem* 1.1** — *For each pair of complex numbers a, x, with $a \neq 0$ and $|x| < 1$,*

$$\prod_{n=1}^{\infty}(1-x^n)^2(1-ax^n)(1-a^{-1}x^n)(1-ax^{n-1})(1-a^{-1}x^{n-1})$$
$$(1-a^2x^{2n-1})(1-a^{-2}x^{2n-1})$$
$$=2P(x)\sum_{n=-\infty}^{\infty}(-1)^n x^{2n^2}a^{4n} - Q(x)\sum_{n=-\infty}^{\infty}(-1)^n x^{2n^2}a^{4n}(ax^n+a^{-1}x^{-n}), \tag{1.1}$$

where

$$P(x) = \prod_{n=1}^{\infty}(1-x^{4n}),$$
$$Q(x) = \prod_{n=1}^{\infty}(1-x^{12n})(1-x^{12n-7})(1-x^{12n-5}) + x\prod_{n=1}^{\infty}$$
$$(1-x^{12n})(1-x^{12n-11})(1-x^{12n-1}).$$

The identity is named the octuple product identity since the left-hand side of it is an eightfold infinite product. In [4], Sin-Da Chen, Wei-Yueh Chen and Sen-Shan Huang gave two proofs of the octuple product identity as well as the proofs of some other identities.

In this paper, we shall give a new proof of Ewell's octuple product identity based on Theorem 1.4 in [3] developed by the first author.

2. PREPARATION

Before we give the proof of the octuple product identity, we introduce the customary q-product notation and some preliminary results. Define

$$(a)_0 := (a;q)_0 = 1, \qquad (a)_n := (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \qquad n \geq 1,$$

$$(a)_\infty := (a;q)_\infty = \lim_{n\to\infty}(a;q)_n, \qquad |q| < 1.$$

Ramanujan's general theta function is defined as

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \ |ab| < 1. \tag{2.1}$$

It is easy to verify that

$$f(-1,a) = 0$$

and

$$f(a,b) = a^{n(n+1)/2} b^{n(n-1)/2} f(a(ab)^n, b(ab)^{-n})$$

for any integer n.

Some special cases of $f(a,b)$

$$\varphi(q) := f(q;q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2 (q^2;q^2)_\infty\,, \tag{2.2}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}\,, \tag{2.3}$$

$$f(-q) := f(-q,-q^2) = (q;q)_\infty. \tag{2.4}$$

In Ramanujan's notation, the celebrated Jacobi's triple product identity is given by [2, p. 10]

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty.$$

Replacing x with q in Theorem 1.1 and using Ramaujan's notation in (2.1), we can rewrite Theorem 1.1 in an equivalent form

$$\begin{aligned} &f(-aq,-a^{-1})f(-a,-a^{-1}q)f(-a^2q,-a^{-2}q)/(q^2;q^2)_\infty \\ =&2(q^4;q^4)_\infty f(-a^4q^2,-a^{-4}q^2) \\ &-\left[f(-q^5,-q^7)+qf(-q,-q^{11})\right]\left[af(-a^4q^3,-a^{-4}q)+\frac{1}{a}f(-a^4q,-a^{-4}q^3)\right]. \end{aligned} \tag{2.5}$$

We find it convenient in the sequel to prove Theorem 1.1 in the form (2.5).

Recall [1, pp. 48-49, Entry 31]

$$f(a,b)=\sum_{r=0}^{k-1} a^{r(r+1)/2}b^{r(r-1)/2}f(a^{k(k+1)/2+kr}b^{k(k-1)/2+kr},\\ a^{k(k-1)/2-kr}b^{k(k+1)/2-kr}). \tag{2.6}$$

Letting $k=2$ in (2.6), we have

$$f(a,b)=f(a^3b,ab^3)+af(b/a,a^5b^3), \tag{2.7}$$

which is used in this paper.

We need to cite the quintuple product identity [2, p. 19].

***Theorem* 2.1** — *(The quintuple product identity).*

$$\frac{f(-x^2,-\lambda x)f(-\lambda x^3)}{f(-x,-\lambda x^2)}=f(-\lambda^2x^3,-\lambda x^6)+xf(-\lambda,-\lambda^2x^9). \tag{2.8}$$

For the history of the quintuple product identity, the reader can refer to Cooper's survey [5].

A system of congruences $a_i \pmod{n_i}$ with $1\le i\le k$ is called a covering system (or complete residue system) if every integer y satisfies $y\equiv a_i \pmod{n_i}$ for at least one value of i.

A covering system in which each integer is covered by just one congruence is called an exact covering system (ECS). In other words, an exact covering system is a partition of the integers into a finite set of arithmetic sequences.

It is easy to see that (2.6) corresponds to an exact covering system $\{r \pmod{k}\}_{r=0}^{k-1}$ of $\mathbb{Z}$. A natural question is: for a product of n $(n\ge 2)$ theta functions, do we have similar results to (2.6)? In other words, can we write a product of n theta functions as the linear combination of other products of theta functions? In [3], we prove that there is a natural correspondence between product identities for product of n theta functions and exact covering systems of $\mathbb{Z}^n$. First we give a general theorem to write a product of n theta functions as a linear combination of other products of theta functions.

***Theorem* 2.2** — [3, Theorem 1.4] *Let* $l_i \in \mathbb{Z}^+$, $a_i b_i = q^{l_i}$, $(i = 1, 2, \ldots, n)$. *Let* $B = (b_{ij})_{n\times n}$ *be an invertible integer matrix satisfying*

$$\begin{cases} l_1 b_{11} b_{12} + l_2 b_{21} b_{22} + \cdots + l_n b_{n1} b_{n2} = 0, \\ l_1 b_{11} b_{13} + l_2 b_{21} b_{23} + \cdots + l_n b_{n1} b_{n3} = 0, \\ \vdots \\ l_1 b_{1(n-1)} b_{1n} + l_2 b_{2(n-1)} b_{2n} + \cdots + l_n b_{n(n-1)} b_{nn} = 0. \end{cases} \tag{2.9}$$

Let $k = |\det B|$. *Let* B^* *be the adjoint of* B. *Assume at least one of the entries of* B^* *has no common factor with* k. *We suppose an entry in the* jth *column of* B^* *is coprime to* k. *Then*

$$\begin{aligned} &\prod_{i=1}^{n} f(a_i, b_i) \\ &= \sum_{r=0}^{k-1} a_j^{\frac{r^2+r}{2}} b_j^{\frac{r^2-r}{2}} f\Big(a_1^{\frac{b_{11}^2+b_{11}}{2}} b_1^{\frac{b_{11}^2-b_{11}}{2}} \cdots a_j^{\frac{b_{j1}^2+b_{j1}}{2}+b_{j1}r} b_j^{\frac{b_{j1}^2-b_{j1}}{2}+b_{j1}r} \cdots \\ &\quad a_n^{\frac{b_{n1}^2+b_{n1}}{2}} b_n^{\frac{b_{n1}^2-b_{n1}}{2}}, \\ &\quad a_1^{\frac{b_{11}^2-b_{11}}{2}} b_1^{\frac{b_{11}^2+b_{11}}{2}} \cdots a_j^{\frac{b_{j1}^2-b_{j1}}{2}-b_{j1}r} b_j^{\frac{b_{j1}^2+b_{j1}}{2}-b_{j1}r} \cdots a_n^{\frac{b_{n1}^2-b_{n1}}{2}} b_n^{\frac{b_{n1}^2+b_{n1}}{2}}\Big) \cdots \\ &\quad \times f\Big(a_1^{\frac{b_{1n}^2+b_{1n}}{2}} b_1^{\frac{b_{1n}^2-b_{1n}}{2}} \cdots a_j^{\frac{b_{jn}^2+b_{jn}}{2}+b_{jn}r} b_j^{\frac{b_{jn}^2-b_{jn}}{2}+b_{jn}r} \cdots a_n^{\frac{b_{nn}^2+b_{nn}}{2}} b_n^{\frac{b_{nn}^2-b_{nn}}{2}}, \\ &\quad a_1^{\frac{b_{1n}^2-b_{1n}}{2}} b_1^{\frac{b_{1n}^2+b_{1n}}{2}} \cdots a_j^{\frac{b_{jn}^2-b_{jn}}{2}-b_{jn}r} b_j^{\frac{b_{jn}^2+b_{jn}}{2}-b_{jn}r} \cdots a_n^{\frac{b_{nn}^2-b_{nn}}{2}} b_n^{\frac{b_{nn}^2+b_{nn}}{2}}\Big). \end{aligned} \tag{2.10}$$

Here we need to cite a corollary in [3], which is a special case of the general theorem for products of two theta functions.

Corollary 2.1 — [3, Corollary 2.2] If $|ab| < 1$ and $(cd) = (ab)^{k_1 k_2}$, where both k_1 and k_2 are positive integers, then

$$\begin{aligned} f(a,b)f(c,d) = \sum_{r=0}^{k_1+k_2-1} & a^{\frac{r^2+r}{2}} b^{\frac{r^2-r}{2}} f\Big(a^{\frac{k_1^2+k_1}{2}+k_1 r} b^{\frac{k_1^2-k_1}{2}+k_1 r} c, a^{\frac{k_1^2-k_1}{2}-k_1 r} b^{\frac{k_1^2+k_1}{2}-k_1 r} d\Big) \\ & \times f\Big(a^{\frac{k_2^2+k_2}{2}+k_2 r} b^{\frac{k_2^2-k_2}{2}+k_2 r} d, a^{\frac{k_2^2-k_2}{2}-k_2 r} b^{\frac{k_2^2+k_2}{2}-k_2 r} c\Big). \end{aligned} \tag{2.11}$$

It is easy too see that (2.11) is symmetric with respect to a and b, c and d, and k_1 and k_2.

3. Proof of Ewell's Octuple Product Identity

Because Corollary 2.1 gives the structure of many identities for products of two theta functions, we can apply it to prove (2.5), which is equivalent to Theorem 1.1.

PROOF : Let $k_1 = 1$ and $k_2 = 2$ in Corollary 2.1, we have the following generalization of the quintuple product identity. For $ab = q$ and $cd = q^2$,

$$\begin{aligned} f(a,b)f(c,d) =& f(ac,bd)f(a^2dq,b^2cq) + af(acq,bd/q)f(a^2dq^3,b^2c/q) \\ &+ bf(ac/q,bdq)f(a^2d/q,b^2cq^3). \end{aligned} \tag{3.1}$$

We replace a with $-a$, b with $-a^{-1}q$, c with $-a^2q$ and d with $-a^{-2}q$ in (3.1). Since $f(-1,a) = 0$ for any complex number a with $|a| < 1$, we obtain

$$\begin{aligned} & f(-a,-a^{-1}q)f(-a^2q,-a^{-2}q) \\ =& (q^2;q^2)_\infty[f(a^3q,a^{-3}q^2) - af(a^3q^2,a^{-3}q)]. \end{aligned} \tag{3.2}$$

It follows from (3.2) that the left-hand side of (2.5) is

$$\begin{aligned} & f(-aq,-a^{-1})f(-a,-a^{-1}q)f(-a^2q,-a^{-2}q)/(q^2;q^2)_\infty \\ =& f(-aq,-a^{-1})[f(a^3q,a^{-3}q^2) - af(a^3q^2,a^{-3}q)]. \end{aligned} \tag{3.3}$$

Next, observing that $-aq \cdot -a^{-1} = q$ and $a^3q \cdot a^{-3}q^2 = a^3q^2 \cdot a^{-3}q = q^3$, we set $k_1 = 1$ and $k_2 = 3$ in (2.11) to obtain for $ab = q$ and $cd = q^3$,

$$\begin{aligned} f(a,b)f(c,d) =& f(ac,bd)f(a^3dq^3,b^3cq^3) + af(acq,bd/q)f(a^3dq^6,b^3c) \\ &+ bf(ac/q,bdq)f(a^3d,b^3cq^6) + a^3bf(acq^2,bd/q^2)f(a^3dq^9,b^3c/q^3). \end{aligned} \tag{3.4}$$

Next, we replace a with $-aq$, b with $-a^{-1}$, c with a^3q, and d with $a^{-3}q^2$ in (3.4) to find that

$$\begin{aligned} & f(-aq,-a^{-1})f(a^3q,a^{-3}q^2) \\ =& f(-a^4q^2,-a^{-4}q^2)f(-q^4,-q^8) - aqf(-a^4q^3,-a^{-4}q)f(-q,-q^{11}) \\ &- a^{-1}f(-a^4q,-a^{-4}q^3)f(-q^7,-q^5) + a^{-2}qf(-a^4,-a^{-4}q^4)f(-q^{10},-q^2). \end{aligned} \tag{3.5}$$

Similarly, by replacing a with $-aq$, b with $-a^{-1}$, c with a^3q^2, and d with $a^{-3}q$ in (3.4), we obtain

$$\begin{aligned} &af(-aq,-a^{-1})f(a^3q^2,a^{-3}q)\\ =&af(-a^4q^3,-a^{-4}q)f(-q^5,-q^7)+a^{-2}qf(-a^4,-a^{-4}q^4)f(-q^2,-q^{10})\\ &-f(-a^4q^2,-a^{-4}q^2)f(-q^8,-q^4)+a^{-1}qf(-a^4q,-a^{-4}q^3)f(-q^{11},-q). \end{aligned} \tag{3.6}$$

By taking the difference of (3.5) and (3.6), we deduce (2.5). We have finished the proof of Theorem 1.1. □

4. Further Discussions

If we set $a=q$ and $b=-q^2$ in (2.7), we can find that

$$f(-q^5,-q^7)+qf(-q,-q^{11})=f(q,-q^2). \tag{4.1}$$

Or we can set $\lambda=x=q$ in the quintuple product identity (2.8) to find that

$$f(-q^5,-q^7)+qf(-q,-q^{11})=\frac{f(-q^2,-q^2)f(-q^3)}{f(-q,-q^3)}=(q;q^2)_\infty(-q;q^2)^2_\infty(q^3;q^3)_\infty. \tag{4.2}$$

Applying (4.1) and (4.2) to (2.5), we can slightly simplify the right-hand side of (2.5).

Chen, Chen, and Huang [4] developed the following corollary as a by-product in the process of proving the octuple product identity. We shall give a different proof.

Corollary 4.1 — [4, Corollary 2.2] For $|x|<1$, we have

$$\prod_{n=-\infty}^{\infty}\frac{(1-x^{2n})(1+x^{2n})}{(1-x^{2n-1})(1+x^{2n-1})}=\sum_{n=-\infty}^{\infty}x^{16n^2+4n}(1+x^{8n+2}) \tag{4.3}$$

and

$$\prod_{n=-\infty}^{\infty}\frac{(1-x^{2n})(1+x^{2n-1})}{(1+x^{2n})(1-x^{2n-1})}=\sum_{n=-\infty}^{\infty}x^{16n^2}(1+2x^{8n+1}+x^{16n+4}). \tag{4.4}$$

PROOF : It is easy to see that (4.3) is equivalent to $\psi(q^2) = f(q^{12}, q^{20}) + q^2 f(q^4, q^{28})$, which can be derived from (2.7). And (4.4) is equivalent to $\varphi(q) = \varphi(q^{16}) + 2q\psi(q^8) + 2q^4\psi(q^{32})$. By setting $a = b = q$ in (2.7), we have $\varphi(q) = \varphi(q^4) + 2q\psi(q^8)$. Expanding $\varphi(q^4)$ using (2.7) again, we can obtain (4.4).

Remark 4.1 : Chen, Chen, and Huang sketched ideas similar to the above proof of Corollary 4.1 in Remark 1 of their paper [4]. They use the same method as in their proof of the octuple product identity, writing the theta function as infinite series, and splitting the infinite sum according to congruence classes. We only need to apply (2.6) to decompose a theta function. □

The following Theorem 4.1 in [4] was first given by Gauss and later proved by Ewell in [6].

***Theorem* 4.1** – *[4, Theorem 4.1] For $a \neq 0$ and $|x| < 1$, we have*

$$\prod_{n=1}^{\infty} \left(1 + ax^{2n-1}\right)^2 \left(1 + a^{-1}x^{2n-1}\right)^2$$
$$= A_0(x) \sum_{m=-\infty}^{\infty} a^{2m} x^{2m^2} + A_1(x) \sum_{m=0}^{\infty} x^{2m(m+1)} \left(a^{2m+1} + a^{-2m-1}\right),$$

where

$$A_0(x) = \left(\prod_{n=1}^{\infty} \left(1 - x^{2n}\right)^{-2}\right) \sum_{n=-\infty}^{\infty} x^{2n^2}$$

and

$$A_1(x) = \left(\prod_{n=1}^{\infty} \left(1 - x^{2n}\right)^{-2}\right) x \sum_{n=-\infty}^{\infty} x^{2n(n+1)}.$$

Remark 4.2 : The proof of Theorem 4.1 in 4.1 involved changing series indices and separating them into congruence classes. In [6], Ewell deduced several corollaries from Theorem 4.1, including a new representation of $(q;q)_\infty^6$ and a new proof of Ramanujan's congruence $p(7n+5) \equiv 0 \pmod 7$, where $p(n)$ denotes the partition function.

As showed in [3], Theorem 2.1 is a special case of Ewell's sextuple product identity which first appeared in [8].

***Theorem* 4.2** — *[8, The Sextuple Product Identity]*

$$(-xyq;q^2)_\infty(-q/xy;q^2)_\infty(-qx/y;q^2)_\infty(-qy/x;q^2)_\infty(q^2;q^2)^2_\infty$$
$$= \sum_{i=-\infty}^{\infty} q^{2i^2}x^{2i} \sum_{j=-\infty}^{\infty} q^{2j^2}y^{2j} + q \sum_{i=-\infty}^{\infty} q^{2i(i+1)}x^{2i+1} \sum_{j=-\infty}^{\infty} q^{2j(j+1)}y^{2j+1}. \quad (4.5)$$

It is has been shown in [3] that the sextuple product identity is a special case of Corollary 2.1 corresponding to the case $k_1 = k_2 = 1$. Hence we can give a new proof of Theorem 4.1. We omit the proof here.

REFERENCES

1. B. C. Berndt, *Ramanujan's notebooks. Part III*, Springer-Verlag, New York, 1991.
2. B. C. Berndt, *Number theory in the spirit of Ramanujan*, Student Mathematical Library, vol. 34, American Mathematical Society, Providence, RI, 2006.
3. Z. Cao, Integer matrix exact covering systems and product identities for theta functions, *Int. Math. Res. Not. IMRN*, 2010; doi: 10.1093/imrn/rnq253.
4. Sin-Da Chen, Wei-Yueh Chen and Sen-Shan Huang, A new construcation of Ewell's octuple product identity, *Indian J. Pure Appl. Math*., **35** (2004), 1241–1253.
5. S. Cooper, The quintuple product identity, *Int. J. Number Theory*, **2**(1) (2006), 115–161.
6. J. A. Ewell, Completion of a Gaussian derivation, *Proc. Amer. Math. Soc*., **84**(2) (1982), 311–314.
7. J. A. Ewell, On an octuple-product identity, *Rocky Mountain J. Math*., **12** (1982), 279–282.
8. J. A. Ewell, Arithmetic consequence of a sextuple product identity, *Rocky Mountain J. Math*., **25** (1995), 1287–1293.