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A NECESSARY CONDITION FOR NON-ABELIAN FINITE p-GROUPS WITH SECOND CENTRE OF ORDER p^2

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Let G be a non-abelian finite p-group such that $|Z_2(G)| = p^2$. In this paper we prove that each maximal subgroup $M \neq C_G(Z_2(G))$ is non-abelian and has cyclic centre.

Key words : Finite p-groups; maximal subgroups.

1. Introduction and Result

Throughout p denotes a prime number. Let G be a group and H be a subgroup of G. We denote by $Z(G)$, $C_G(H)$ and Aut(G), respectively the centre, the centralizer of H in G and the automorphism group of G. We write C_n for the cyclic group of order n. If $x, y \in G$, then $[x, y]$ denotes the commutator $x^{-1}y^{-1}xy$. For subsets A, B of G , $[A, B]$ denotes the group generated by all commutators $[a, b]$, where $a \in A$ and $b \in B$. Let $\gamma_1(G) = G$ and $\gamma_i(G) = [\gamma_{i-1}(G), G]$ for each $i \geq 2$. Then the sequence $G = \gamma_1(G) \geq \gamma_2(G) = G' \geq \gamma_3(G) \geq \cdots$ is called the lower central series of G. The upper central series $1 = Z_0(G) \leq Z_1(G) = Z(G) \leq Z_2(G) \leq \cdots$ of G is defined inductively by the rules $Z_0(G) = 1$ and $Z_i(G)/Z_{i-1}(G) =$

$Z(G/Z_{i-1}(G))$ for each $i \geq 1$. Therefore $x \in Z_2(G)$ if and only if $[g, x] \in Z(G)$ for all $g \in G$. If m is the smallest integer such that $Z_m(G) = G$ or $\gamma_{m+1}(G) = 1$, then G is called nilpotent of class m. The nilpotency class of G is denoted by $\mathrm{cl}(G)$. Set $\Omega_1(G) = \langle x \in G \mid x^p = 1 \rangle$. In this article we give a necessary condition for non-abelian finite p-groups G with $|Z_2(G)| = p^2$.

Theorem — *Let G be a non-abelian finite p-group such that $|Z_2(G)| = p^2$. Then*
(a) $C_G(Z_2(G))$ is a maximal subgroup of G;
(b) Each maximal subgroup $M \neq C_G(Z_2(G))$ is non-abelian and has cyclic centre.

A p-group G of order p^n with $n \geq 3$ and nilpotency class $n-1$ is said to be of maximal class. The cornerstone in the theory of p-groups of maximal class is the paper by Blackburn [1] (see also Huppert's book [2, III.14]).

As a consequence of Theorem, every finite p-group of maximal class satisfies in above Theorem.

2. Proof of the Theorem

We split the proof into a number of steps.

Step 1 — $C_G(Z_2(G))$ is a maximal subgroup of G.

PROOF : Since $|Z_2(G)| = p^2$, we have $|Z(G)| = p$. If $Z_2(G)$ is cyclic, then by [2, Satz 7.7] $p = 2$. Therefore $|\mathrm{Aut}(Z_2(G))| = 2$. If $Z_2(G)$ is not cyclic, then $|\mathrm{Aut}(Z_2(G))| = (p^2-1)(p^2-p)$. Since $G/C_G(Z_2(G)) \hookrightarrow \mathrm{Aut}(Z_2(G))$, it follows that $[G : C_G(Z_2(G))] = p$ and so $C_G(Z_2(G))$ is a maximal subgroup of G.

Step 2 — If M is a maximal subgroup of G such that $M \neq C_G(Z_2(G))$, then M is non-abelian.

PROOF : Suppose, for a contradiction, that M is abelian. It follows from $Z(G) \subseteq Z_2(G) \cap M$, that $p \leq |Z_2(G) \cap M| \leq p^2$. If $|Z_2(G) \cap M| = p$, then $G = Z_2(G)M$ and hence G is nilpotent of class at most 2 which is a contradiction. Therefore $|Z_2(G) \cap M| = p^2$ whence $Z_2(G) \subseteq M$. Hence $M = Z(M) = C_G(M) \subseteq C_G(Z_2(G))$ and so $M = C_G(Z_2(G))$ which is impossible.

Step 3 — If M is a maximal subgroup of G such that $M \neq C_G(Z_2(G))$, then $\Omega_1(Z(M)) \leq Z_2(G)$.

PROOF : Suppose that $y \in \Omega_1(Z(M))$. If $y \in Z(G)$, then $y \in Z_2(G)$. Let $y \notin Z(G)$. Choose an element x in G such that $x \notin M$. Therefore $G = \langle x \rangle M$. Now we have $[y, G] = [\langle y \rangle, \langle x \rangle M] = [\langle y \rangle, \langle x \rangle] = [\langle y \rangle, x] = \langle [y, x] \rangle$. Also $[y, x]^p = [y^p, x] = 1$. Since $1 \neq [y, G] \trianglelefteq G$, we have $Z(G) = [y, G]$. Therefore $y \in Z_2(G)$.

Step 4 — If M is a maximal subgroup of G such that $M \neq C_G(Z_2(G))$, then $Z(M)$ is cyclic.

PROOF : By Step 3, we have $\Omega_1(Z(M)) \leq Z_2(G)$. Therefore $C_G(Z_2(G)) \leq C_G(\Omega_1(Z(M)))$. Thus $C_G(\Omega_1(Z(M))) = C_G(Z_2(G))$ or $C_G(\Omega_1(Z(M))) = G$ by Step 1.

If $C_G(\Omega_1(Z(M))) = C_G(Z_2(G))$, then $M \leq C_G(\Omega_1(Z(M))) = C_G(Z_2(G))$, whence $M = C_G(Z_2(G))$ which is a contradiction. Thus $C_G(\Omega_1(Z(M))) = G$, whence $|\Omega_1(Z(M))| = p$. Therefore $Z(M)$ is cyclic. □

We finish with some examples of p-groups G that satisfy the condition $|Z_2(G)| = p^2$.

Example 2.1 : The following p-groups G satisfy the condition $|Z_2(G)| = p^2$.

1. Let $G = \langle x, x_1, x_2, \ldots, x_n \mid x_1^p = x_2^p = \cdots = x_n^p = 1, x^p = x_n, [x_i, x] = x_{i+1}, 1 \leq i \leq n-1, \text{rest commute} \rangle$, $p \geq n \geq 3$. Then $G/G' \simeq C_p \times C_p$, $\gamma_i(G) = \langle x_i, \ldots, x_n \rangle$, $2 \leq i \leq n$, $Z(G) = \langle x_n \rangle \simeq C_p$ and $Z_2(G) = \langle x_{n-1}, x_n \rangle \simeq C_p \times C_p$. Thus G is of order p^{n+1} and maximal class.

2. It is clear that if G is a non-abelian group of order p^4 such that $|Z_2(G)| = p^2$, then G is of maximal class. Also if G is a non-abelian group of order p^5 such that $|Z_2(G)| = p^2$, then G is of maximal class. Assume that this is false. Hence $\mathrm{cl}(G) = 3$. Thus $G/Z(G)$ is a non-abelian group of order p^4 with nilpotency class 2. Hence $|Z(G/Z(G))| = |Z_2(G)/Z(G)| = p^2$, whence $|Z_2(G)| = p^3$ which is a contradiction. Therefore the p-group G of the smallest order such that $|Z_2(G)| = p^2$ and G is not of maximal class has order p^6. For example let $G = \langle x_1, x_2, \ldots, x_6 \mid x_2^p = x_3^p = \cdots = x_6^p = 1, x_1^p = x_5, [x_2, x_1] = x_3, [x_3, x_1] = [x_2, x_6] = x_4, [x_4, x_1] = [x_3, x_6] = x_5, \text{ rest commute} \rangle$, $p \geq 5$. Then $G/G' \simeq C_p \times C_p \times C_p$, $G' = \gamma_2(G) = \langle x_3, x_4, x_5 \rangle \simeq C_p \times C_p \times C_p$, $\gamma_3(G) = \langle x_4, x_5 \rangle \simeq C_p \times C_p$ and $\gamma_4(G) = \langle x_5 \rangle \simeq C_p$. Hence G is a group of order p^6 and nilpotency class 4. In particular, $Z(G) = \langle x_5 \rangle \simeq C_p$ and $Z_2(G) = \langle x_4, x_5 \rangle \simeq C_p \times C_p$.

3. Let $G = \langle x_1, x_2, x_3 \mid x_1^{2^n} = x_2^{2^n} = x_3^4 = 1, [x_1, x_3] = x_2, [x_2, x_3] = x_2^{-2}x_1^{-2}, [x_1, x_2] = 1\rangle$, $n \geq 2$. Then $G/G' \simeq C_2 \times C_4$, $\gamma_{2i}(G) = \langle x_1^{2^i}, x_2^{2^{i-1}}\rangle$ and $\gamma_{2i+1}(G) = \langle x_1^{2^i}, x_2^{2^i}\rangle$, $i \geq 1$. Thus G is of order 2^{2n+2} and nilpotency class $2n$. In particular, $Z(G) = \langle x_2^{2^{n-1}}\rangle \simeq C_2$ and $Z_2(G) = \langle x_1^{2^{n-1}}, x_2^{2^{n-1}}\rangle \simeq C_2 \times C_2$.

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