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SOME BOUNDS FOR INTEGRALS WITH REFINEMENTS OF THE GRÜSS INEQUALITY

R. Sharma, R. Bhandari and A. Thakur

Department of Mathematics, Himachal Pradesh University,
Shimla, 171 005 *India*
e-mail: r.hpu.math@gmail.com

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Some integral bounds are obtained which provide refinements and extensions of the Grüss type inequalities. We also get an alternative formulation of the Grüss inequality.

Key words : Grüss inequality, Karamata inequality, Moments and Kurtosis

1. INTRODUCTION

Let $f(x)$ and $g(x)$ be real and integrable functions on $[a, b]$, and

$$D\left(f,g\right)=\frac{1}{b-a}\int\limits_{a}^{b}f\left(x\right)g\left(x\right)dx-\frac{1}{b-a}\int\limits_{a}^{b}f\left(x\right)dx\frac{1}{b-a}\int\limits_{a}^{b}g\left(x\right)dx. \quad (1.1)$$

Let the functions $f(x)$ and $g(x)$ both be either increasing or decreasing. Then Čebyšev's inequality says that [1]

$$D\left(f,g\right)\geq 0. \quad (1.2)$$

If one function is increasing and the other is decreasing, $D(f,g) \leq 0$. Further, if $m_f \leq f(x) \leq M_f$ and $m_g \leq g(x) \leq M_g$, then the complementary Grüss inequality asserts that [2]

$$|D(f,g)| \leq \frac{1}{4}(M_f - m_f)(M_g - m_g). \tag{1.3}$$

For more details, generalizations and further related developments see [3-8]. The inequalities (1.2) and (1.3), their further refinements and extensions can be investigated in more generality, respectively as the special cases of the following inequalities [9]:

$$\mu_{fg} \geq 0 \tag{1.4}$$

and

$$|\mu_{fg}| \leq \frac{1}{4}(M_f - m_f)(M_g - m_g), \tag{1.5}$$

where

$$\mu_{fg} = \mu'_{fg} - \mu'_{f1}\mu'_{1g}, \tag{1.6}$$

$$\mu'_{fg} = \int_a^b f(x)\, g(x)\, \phi(x)\, dx \tag{1.7}$$

and $\phi(x)$ is a probability density function. For the special case when $\phi(x) = \frac{1}{b-a}$, we put $\mu'_{fg} = D'(f,g)$ and in this case $\mu_{fg} = D(f,g)$. We note that $\mu'_{ff} = \mu'_{f^2 1} = \mu'_{1f^2}$ and $\mu'_{f^2 f} = \mu'_{ff^2} = \mu'_{f^3 1} = \mu'_{1f^3}$, but in the general case $\mu_{ff} \neq \mu_{f^2 1}$.

The Grüss inequality provides the bounds for the difference $D(f,g)$. An analogous result for the ratio

$$R(f,g) = \frac{\int_a^b f(x)\, g(x)\, dx}{\int_a^b f(x)\, dx \int_a^b g(x)\, dx}, \tag{1.8}$$

in the special case when $a = 0$ and $b = 1$ was obtained by Karamata [10]. This states that if m_f and m_g are positive then [9]

$$\frac{1}{K^2} \leq R(f,g) \leq K^2 \tag{1.9}$$

where

$$K = \frac{\sqrt{m_f m_g} + \sqrt{M_f M_g}}{\sqrt{m_f M_g} + \sqrt{m_g M_f}}. \tag{1.10}$$

Some other related inequalities

$$\mu'_{f^3 1} \geq \frac{\mu'^2_{f^2 1}}{\mu'_{f1}} \geq \mu'_{f^2 1}\mu'_{f1} \geq \mu'^3_{f1} \quad , \ f > 0, \tag{1.11}$$

follow from the Cauchy inequality,

$$\mu'_{f^2 1}\mu'_{1g^2} \geq \mu'^2_{fg}. \tag{1.12}$$

The complementary inequality says that if m_f and m_g are positive then [11]

$$\mu'_{f^2 1}\mu'_{1g^2} \leq \frac{(m_f m_g + M_f M_g)^2}{4 m_f m_g M_f M_g}\mu'^2_{fg}. \tag{1.13}$$

Our main results give the refinements of the Grüss type inequality (1.5) (Theorem 2.1- 2.2 and Corollary 2.1-2.2, below). The analogous bounds for the ratio are proved (Theorem 2.3 and Corollary 2.3, below) and as a special case we get the Karamata inequality for the general case when x varies over arbitrary positive real interval (Corollary 2.4, below). We also obtain the complementary lower bounds (Corollary 2.5, below). We show that the further generalizations give some more refinements of the Grüss inequality (Theorem 3.1-3.2 and Corollary 3.1-3.5, below). The inequalities (1.11) give lower bounds for $\mu'_{f^3 1}$, we prove a lower bound for $\mu'_{f^4 1}$ (Theorem 3.3, below). We remark that the special cases of these inequalities give inequalities between the moments of probability distributions and also provide refinements of the inequalities (1.11) (See Remarks, below).

2. Main Results

***Theorem* 2.1** — *Let f and g be integrable functions on $[a,b]$, $m_f \leq f(x) \leq M_f$ and $m_g \leq g(x) \leq M_g$. Then*

$$\mu_{fg} \leq \left(M_f - \mu'_{f1}\right)\left(\mu'_{1g} - m_g\right), \tag{2.1}$$

$$\mu_{fg} \leq \left(M_g - \mu'_{1g}\right)\left(\mu'_{f1} - m_f\right), \tag{2.2}$$

$$\mu_{fg} \geq \left(m_f - \mu^{'}_{f1}\right)\left(\mu^{'}_{1g} - m_g\right) \tag{2.3}$$

and

$$\mu_{fg} \geq \left(\mu^{'}_{f1} - M_f\right)\left(M_g - \mu^{'}_{1g}\right). \tag{2.4}$$

PROOF : For $f(x) \leq M_f$ and $g(x) \geq m_g$, we have

$$(M_f - f(x))(g(x) - m_g) \geq 0. \tag{2.5}$$

The inequality (2.5) gives

$$f(x)\, g(x) \leq g(x)\, M_f + f(x)\, m_g - m_g M_f. \tag{2.6}$$

Multiplying both sides of (2.6) by $\phi(x)$ and integrating between the limits $x = a$ and $x = b$ we get, on using the properties of the definite integrals, the following inequality:

$$\mu^{'}_{fg} - \mu^{'}_{f1}\mu^{'}_{1g} \leq M_f\mu^{'}_{1g} + m_g\mu^{'}_{f1} - m_g M_f - \mu^{'}_{f1}\mu^{'}_{1g}. \tag{2.7}$$

Combining (1.6) and (2.7) we immediately get (2.1), on simplification. On using similar arguments, the inequalities (2.2), (2.3) and (2.4) follow respectively from the inequalities:

$$(f(x) - m_f)(M_g - g(x)) \geq 0, \tag{2.8}$$

$$(f(x) - m_f)(g(x) - m_g) \geq 0 \tag{2.9}$$

and

$$(f(x) - M_f)(g(x) - M_g) \geq 0. \tag{2.10}$$

***Theorem* 2.2** — *For $m_f \leq f(x) \leq M_f$ and $m_g \leq g(x) \leq M_g$, we have*

$$|\mu_{fg}| \leq \frac{M_g - m_g}{M_f - m_f}\left(M_f - \mu^{'}_{f1}\right)\left(\mu^{'}_{f1} - m_f\right) \tag{2.11}$$

and

$$|\mu_{fg}| \leq \frac{M_f - m_f}{M_g - m_g}\left(M_g - \mu^{'}_{1g}\right)\left(\mu^{'}_{1g} - m_g\right). \tag{2.12}$$

PROOF : Let

$$\mu_{1g}^{'} \leq \frac{M_g - m_g}{M_f - m_f}\mu_{f1}^{'} + \frac{m_g M_f - m_f M_g}{M_f - m_f}. \tag{2.13}$$

Combining (2.1) and (2.13) we easily get (2.11) for μ_{fg}. If the inequality (2.13) occurs in reverse order then (2.2) implies (2.11). On using similar arguments we find that (2.3) and (2.4) give (2.11) for $(-\mu_{fg})$. The inequality (2.12) can be obtained similarly by interchanging f and g.

Corollary 2.1 — The inequality (2.11) provides a refinement of the Grüss type inequality (1.5), that is

$$|\mu_{fg}| \leq \frac{M_g - m_g}{M_f - m_f}\left(M_f - \mu_{f1}^{'}\right)\left(\mu_{f1}^{'} - m_f\right) \leq \frac{1}{4}\left(M_f - m_f\right)\left(M_g - m_g\right). \tag{2.14}$$

The inequality (2.12) also provides the refinement of the Grüss type inequality (1.5).

PROOF : On using Arithmetic-Geometric mean inequality, we find that

$$\left(M_f - \mu_{f1}^{'}\right)\left(\mu_{f1}^{'} - m_f\right) \leq \frac{1}{4}\left(M_f - m_f\right)^2. \tag{2.15}$$

Combining (2.11) and (2.15) we get (2.14).

Corollary 2.2 — A refinement of the Grüss inequality (1.3) is

$$|D\left(f,g\right)| \leq \frac{M_g - m_g}{M_f - m_f}\left(M_f - D^{'}\left(f,1\right)\right)\left(D^{'}\left(f,1\right) - m_f\right). \tag{2.16}$$

PROOF : The inequality (2.16) is a special case of the inequality (2.14), $\phi\left(x\right) = \frac{1}{b-a}$.

***Theorem* 2.3** — *For* $0 < m_f \leq f\left(x\right) \leq M_f$ *and* $0 < m_g \leq g\left(x\right) \leq M_g$, *we have*

$$\frac{\mu_{fg}}{\mu_{f1}^{'}\mu_{1g}^{'}} \leq \frac{\left(M_f - m_f\right)\left(M_g - m_g\right)}{\left(\sqrt{m_f M_g} + \sqrt{m_g M_f}\right)^2}. \tag{2.17}$$

PROOF : We note that the inequality (2.11) is valid when

$$\mu_{1g}^{'} \geq \frac{M_g - m_g}{M_f - m_f}\mu_{f1}^{'} + \frac{m_g M_f - m_f M_g}{M_f - m_f}. \tag{2.18}$$

Therefore, from (2.11) and (2.18), we get that

$$\frac{\mu_{fg}}{\mu'_{f1}\mu'_{1g}} \leq \frac{(M_g - m_g)\left(M_f - \mu'_{f1}\right)\left(\mu'_{f1} - m_f\right)}{\mu'_{f1}\left((M_g - m_g)\,\mu'_{f1} + m_g M_f - m_f M_g\right)}. \tag{2.19}$$

Let

$$h\left(\mu'_{f1}\right) = \frac{(M_g - m_g)\left(M_f - \mu'_{f1}\right)\left(\mu'_{f1} - m_f\right)}{\mu'_{f1}\left((M_g - m_g)\,\mu'_{f1} + m_g M_f - m_f M_g\right)}. \tag{2.20}$$

The derivative

$$\frac{\partial h}{\partial \mu'_{f1}} = (m_g - M_g)$$

$$\frac{(M_f M_g - m_f m_g)\,\mu'^2_{f1} - 2m_f M_f\,(M_g - m_g)\,\mu'_{f1} + m_f M_f\,(m_f M_g - m_g M_f)}{\mu'^2_{f1}\left(\mu'_{f1} M_g - \mu'_{f1} m_g + m_g M_f - m_f M_g\right)^2} \tag{2.21}$$

vanishes at

$$\mu'_{f1} = \frac{m_f M_f\,(M_g - m_g) + (M_f - m_f)\sqrt{m_f m_g M_f M_g}}{M_f M_g - m_f m_g} = \alpha_1\,(\text{say})\,. \tag{2.22}$$

and

$$\mu'_{f1} = \frac{m_f M_f\,(M_g - m_g) - (M_f - m_f)\sqrt{m_f m_g M_f M_g}}{M_f M_g - m_f} = \alpha_2\,(\text{say})\,. \tag{2.23}$$

Now $\alpha_1 \geq \alpha_2$ if and only if

$$(M_f - m_f)\sqrt{m_f m_g M_f M_g} \geq 0. \tag{2.24}$$

This is true. Therefore, $\alpha_1 \geq \alpha_2$ and we find that the function $h\left(\mu'_{f1}\right)$ has maximum at α_1. On substituting the value of μ'_{f1} from (2.22) in (2.19), we get the inequality (2.17).

Corollary 2.3 — For $0 < m_f \leq f(x) \leq M_f$ and $0 < m_g \leq g(x) \leq M_g$, we have

$$\frac{\mu'_{fg}}{\mu'_{f1}\mu'_{1g}} \leq \left(\frac{\sqrt{m_f m_g} + \sqrt{M_f M_g}}{\sqrt{m_f M_g} + \sqrt{m_g M_f}}\right)^2. \tag{2.25}$$

PROOF : This is immediate, adding one to both sides of (2.17) and simplifying the resulting expressions, we get the inequality (2.25).

Corollary 2.4 — For $0 < m_f \leq f(x) \leq M_f$ and $0 < m_g \leq g(x) \leq M_g$, we have

$$R(f,g) \leq \frac{1}{b-a}\left(\frac{\sqrt{m_f m_g}+\sqrt{M_f M_g}}{\sqrt{m_f M_g}+\sqrt{m_g M_f}}\right)^2, \tag{2.26}$$

where $R(f,g)$ is given by (1.8).

PROOF : The inequality (2.26) is a special case of the inequality (2.25), $\phi(x) = \frac{1}{b-a}$.

Corollary 2.5 — For $0 < m_f \leq f(x) \leq M_f$ and $0 < m_g \leq g(x) \leq M_g$, we have

$$\mu_{fg}^{'2} \geq \mu_{f^2 1}^{'} \mu_{1g^2}^{'} - \frac{1}{4}\left(\frac{M_f}{m_g} - \frac{m_f}{M_g}\right)^2 \mu_{1g^2}^{'2}. \tag{2.27}$$

PROOF : For $0 < m_f \leq f(x) \leq M_f$ and $0 < m_g \leq g(x) \leq M_g$, we have

$$\left(f(x) - \frac{m_f}{M_g} g(x)\right)\left(f(x) - \frac{M_f}{m_g} g(x)\right) \leq 0. \tag{2.28}$$

As in proof of Theorem 2.1, inequality (2.28) gives

$$\mu_{f^2 1}^{'} \mu_{1g^2}^{'} - \mu_{fg}^{'2} \leq \left(\mu_{fg}^{'} - \frac{m_f}{M_g}\mu_{1g^2}^{'}\right)\left(\frac{M_f}{m_g}\mu_{1g^2}^{'} - \mu_{fg}^{'}\right). \tag{2.29}$$

From (2.29) we get, on using Arithmetic-Geometric mean inequality, inequality (2.27).

3. SOME FURTHER GENERALIZATIONS

We obtain generalizations of the Grüss type inequality (1.5) involving higher powers of f and g. This also provides some further refinements of the inequalities (1.3) and (1.5).

***Theorem* 3.1** — *Let* $f(x)$ *and* $g(x)$ *be real and integrable functions on* $[a, b]$ *such that* $m_f \leq f(x) \leq M_f$. *Then*

$$\mu'_{fg^2} \leq M_f \mu'_{1g^2} - \frac{\left[M_f \mu'_{1g} - \mu'_{fg}\right]^2}{M_f - \mu'_{f1}} \tag{3.1}$$

and

$$\mu'_{fg^2} \geq m_f \mu'_{1g^2} + \frac{\left[m_f \mu'_{1g} - \mu'_{fg}\right]^2}{\mu'_{f1} - m_f}. \tag{3.2}$$

PROOF : For $f(x) \leq M_f$, the inequality

$$(g(x) - \alpha)^2 (f(x) - M_f) \leq 0 \tag{3.3}$$

holds good for every real number α, therefore

$$f(x) g^2(x) - M_f g^2(x) + \alpha^2 f(x) - \alpha^2 M_f - 2\alpha f(x) g(x) + 2\alpha M_f g(x) \leq 0. \tag{3.4}$$

On multiplying both sides of (3.4) with $\phi(x)$ and integrating over the corresponding limits, we get on simplification

$$\left(M_f - \mu'_{f1}\right) \alpha^2 - 2\left(M_f \mu'_{1g} - \mu'_{fg}\right) \alpha + \left(M_f \mu'_{1g^2} - \mu'_{fg^2}\right) \geq 0. \tag{3.5}$$

The quadratic inequality (3.5) holds for all real values of α therefore its discriminant must be non-positive,

$$\left(M_f \mu'_{1g} - \mu'_{fg}\right)^2 - \left(M_f - \mu'_{f1}\right)\left(M_f \mu'_{1g^2} - \mu'_{fg^2}\right) \leq 0. \tag{3.6}$$

This immediately gives the inequality (3.1). On using the similar arguments we find that the inequality

$$(g(x) - \beta)^2 (f(x) - m_f) \geq 0 \tag{3.7}$$

holds for all real values of β and

$$\left(m_f \mu'_{1g} - \mu'_{fg}\right)^2 - \left(\mu'_{f1} - m_f\right)\left(\mu'_{fg^2} - m_f \mu'_{1g^2}\right) \leq 0. \tag{3.8}$$

From (3.8) we easily get (3.2).

Corollary 3.1 — For $m_g \leq g(x) \leq M_g$, we have

$$\mu'_{f^2g} \leq M_g\mu'_{f^21} - \frac{\left[M_g\mu'_{f1} - \mu'_{fg}\right]^2}{M_g - \mu'_{1g}} \tag{3.9}$$

and

$$\mu'_{f^2g} \geq m_g\mu'_{f^21} + \frac{\left[m_g\mu'_{f1} - \mu'_{fg}\right]^2}{\mu'_{1g} - m_g}. \tag{3.10}$$

PROOF : On interchanging $f(x)$ and $g(x)$ and using the analysis similar to that in the proof of Theorem 3.1, we easily get the inequalities (3.9) and (3.10).

***Theorem* 3.2** — *Let $f(x)$ and $g(x)$ be real and integrable functions on $[a,b]$* such that $m_f \leq f(x) \leq M_f$, then

$$M_f \geq \frac{\mu_{fg^2} + \mu'_{f1}\mu_{1g^2} + \sqrt{\left(\mu_{fg^2} - \mu'_{f1}\mu_{1g^2}\right)^2 + 4\mu_{fg}^2\mu_{1g^2}}}{2\mu_{1g^2}} \tag{3.11}$$

and

$$m_f \leq \frac{\mu_{fg^2} + \mu'_{f1}\mu_{1g^2} - \sqrt{\left(\mu_{fg^2} - \mu'_{f1}\mu_{1g^2}\right)^2 + 4\mu_{fg}^2\mu_{1g^2}}}{2\mu_{1g^2}}. \tag{3.12}$$

where μ_{fg} is given by (1.6) and

$$\mu_{fg^2} = \mu'_{fg^2} - 2\mu'_{1g}\mu'_{fg} + \mu'^2_{1g}\mu'_{f1}. \tag{3.13}$$

PROOF : From (3.1), we have

$$\left(\mu'_{1g^2} - \mu'^2_{1g}\right) M_f^2 - \left(\mu'_{fg^2} - 2\mu'_{1g}\mu'_{fg} + \mu'_{f1}\mu'_{1g^2}\right) M_f - \left(\mu'^2_{fg} - \mu'_{f1}\mu'_{fg^2}\right) \geq 0. \tag{3.14}$$

From (3.14) we find that either $M_f \geq \beta_1$ or $M_f \leq \beta_2$, where

$$\beta_1 = \frac{\mu_{fg^2} + \mu'_{f1}\mu_{1g^2} + \sqrt{\left(\mu_{fg^2} - \mu'_{f1}\mu_{1g^2}\right)^2 + 4\mu_{fg}^2\mu_{1g^2}}}{2\mu_{1g^2}} \tag{3.15}$$

and

$$\beta_2 = \frac{\mu_{fg^2} + \mu'_{f1}\mu_{1g^2} - \sqrt{\left(\mu_{fg^2} - \mu'_{f1}\mu_{1g^2}\right)^2 + 4\mu^2_{fg}\mu_{1g^2}}}{2\mu_{1g^2}}. \tag{3.16}$$

But, if $M_f \leq \beta_2$

$$M_f - \mu'_{f1} \leq \frac{\left(\mu_{fg^2} - \mu'_{f1}\mu_{1g^2}\right) - \sqrt{\left(\mu_{fg^2} - \mu'_{f1}\mu_{1g^2}\right)^2 + 4\mu^2_{fg}\mu_{1g^2}}}{2\mu_{1g^2}}. \tag{3.17}$$

This is not possible as right hand side expression in (3.17) is negative while left hand side expression is positive. Hence $M_f \geq \beta_1$ and we conclude that (3.11) must hold good. Similarly from (3.2), we have

$$\left(\mu'_{1g^2} - \mu'^2_{1g}\right) m_f^2 - \left(\mu'_{fg^2} - 2\mu'_{1g}\mu'_{fg} + \mu'_{f1}\mu'_{1g^2}\right) m_f - \left(\mu'^2_{fg} - \mu'_{f1}\mu'_{fg^2}\right) \geq 0. \tag{3.18}$$

The inequality (3.18) implies that either $m_f \geq \gamma_1$ or $m_f \leq \gamma_2$, where

$$\gamma_1 = \frac{\mu_{fg^2} + \mu'_{f1}\mu_{1g^2} + \sqrt{\left(\mu_{fg^2} - \mu'_{f1}\mu_{1g^2}\right)^2 + 4\mu^2_{fg}\mu_{1g^2}}}{2\mu_{1g^2}} \tag{3.19}$$

and

$$\gamma_2 = \frac{\mu_{fg^2} + \mu'_{f1}\mu_{1g^2} - \sqrt{\left(\mu_{fg^2} - \mu'_{f1}\mu_{1g^2}\right)^2 + 4\mu^2_{fg}\mu_{1g^2}}}{2\mu_{1g^2}}. \tag{3.20}$$

But, if $m_f \geq \gamma_1$ then

$$m_f - \mu'_{f1} \geq \frac{\left(\mu_{fg^2} - \mu'_{f1}\mu_{1g^2}\right) + \sqrt{\left(\mu_{fg^2} - \mu'_{f1}\mu_{1g^2}\right)^2 + 4\mu^2_{fg}\mu_{1g^2}}}{2\mu_{1g^2}}. \tag{3.21}$$

This is not possible as right hand side expression in (3.21) is positive while left hand side expression is negative. Hence $m_f \leq \gamma_2$ and the inequality (3.12) follows immediately.

Corollary 3.2 — Let $f(x)$ and $g(x)$ be real and integrable functions on $[a,b]$ such that $m_g \leq g(x) \leq M_g$. Then

$$M_g \geq \frac{\mu_{f^2g} + \mu'_{1g}\mu_{f^21} + \sqrt{\left(\mu_{f^2g} - \mu'_{1g}\mu_{f^21}\right)^2 + 4\mu^2_{fg}\mu_{f^21}}}{2\mu_{f^21}} \tag{3.22}$$

and

$$m_g \leq \frac{\mu_{f^2g} + \mu'_{1g}\mu_{f^21} - \sqrt{\left(\mu_{f^2g} - \mu'_{1g}\mu_{f^21}\right)^2 + 4\mu^2_{fg}\mu_{f^21}}}{2\mu_{f^21}}. \tag{3.23}$$

where

$$\mu_{f^2g} = \mu'_{f^2g} - 2\mu'_{f1}\mu'_{fg} + \mu'^2_{f1}\mu'_{1g}, \tag{3.24}$$

and μ_{fg} is given by (1.6).

PROOF : The inequalities (3.22) and (3.23) follow easily on applying the similar arguments as in the proof of Theorem 3.2.

Corollary 3.3 — For $m_f \leq f(x) \leq M_f$ and $m_g \leq g(x) \leq M_g$, we have

$$\frac{\mu^2_{fg}}{\mu_{1g^2}} + \frac{1}{4}\left(\frac{\mu_{fg^2}}{\mu_{1g^2}} - \mu'_{f1}\right)^2 \leq \frac{(M_f - m_f)^2}{4} \tag{3.25}$$

and

$$\frac{\mu^2_{fg}}{\mu_{f^21}} + \frac{1}{4}\left(\frac{\mu_{f^2g}}{\mu_{f^21}} - \mu'_{1g}\right)^2 \leq \frac{(M_g - m_g)^2}{4}. \tag{3.26}$$

PROOF : On combining the inequalities (3.11) and (3.12), we get the inequality (3.25). Similarly, on combining the inequalities (3.22) and (3.23), we get the inequality (3.26).

Corollary 3.4 — Under the above conditions:

$$D^2(f,g) + \frac{1}{4}\left(\frac{D\left(f,g^2\right)}{D\left(1,g^2\right)} - D'(f,1)\right)^2 D\left(1,g^2\right) \leq \frac{(M_f - m_f)^2}{4} D\left(1,g^2\right) \tag{3.27}$$

and

$$D^2(f,g) + \frac{1}{4}\left(\frac{D\left(f^2,g\right)}{D\left(f^2,1\right)} - D'(1,g)\right)^2 D\left(f^2,1\right) \leq \frac{(M_g - m_g)^2}{4} D\left(f^2,1\right) \tag{3.28}$$

where

$$D\left(f,g^2\right) = D'\left(f,g^2\right) + D'^2\left(1,g\right)D'\left(f,1\right) - 2D'\left(1,g\right)D'\left(f,g\right) \tag{3.29}$$

and

$$D\left(f^2,g\right) = D'\left(f^2,g\right) + D'^2\left(f,1\right)D'\left(1,g\right) - 2D'\left(f,1\right)D'\left(f,g\right) \tag{3.30}$$

The inequalities (3.27) and (3.28) provide the refinements of the Grüss inequality (1.3).

PROOF : On substituting $\phi\left(x\right) = \frac{1}{b-a}$ in (3.25) and (3.26), we easily get the inequalities (3.27) and (3.28), respectively.

Corollary 3.5 — Let $f\left(x\right)$ be real and integrable function on $[a,b]$ and $m_f \leq f\left(x\right) \leq M_f$. Then

$$\mu_{ff} + \left(\frac{M_3\left(f\right)}{2\mu_{ff}}\right)^2 \leq \frac{\left(M_f - m_f\right)^2}{4} \tag{3.31}$$

where

$$M_3\left(f\right) = \mu'_{f^2f} - 3\mu'_{f1}\mu'_{ff} + 2\mu'^3_{f1}. \tag{3.32}$$

The inequality (3.31) provides refinement of the following Grüss type inequality:

$$\mu_{ff} \leq \frac{\left(M_f - m_f\right)^2}{4}.$$

PROOF : The inequality (3.31) follows easily from (3.25), on substituting $g\left(x\right) = f\left(x\right).$

***Theorem* 3.3** — *Let $f\left(x\right)$ and $g\left(x\right)$ be real and integrable functions on $[a,b]$. Then,*

$$\mu'_{f^41} \geq \frac{\left(\mu'_{f^2g} - \mu'_{1g}\mu'_{f^21}\right)^2}{\mu'_{1g^2} - \mu'^2_{1g}} + \mu'^2_{f^21}, \qquad \mu'_{1g^2} \neq \mu'^2_{1g} \tag{3.33}$$

and

$$\mu'_{1g^4} \geq \frac{\left(\mu'_{fg^2} - \mu'_{f1}\mu'_{1g^2}\right)^2}{\mu'_{f^21} - \mu'^2_{f1}} + \mu'^2_{1g^2}, \qquad \mu'_{f^21} \neq \mu'^2_{f1}. \tag{3.34}$$

PROOF : The inequality,

$$\left(f^2(x) - \alpha g(x) + \beta\right)^2 \geq 0, \tag{3.35}$$

holds good for all real numbers α and β. Therefore,

$$f^4(x) + \alpha^2 g^2(x) + 2\beta f^2(x) - 2\alpha f^2(x) g(x) - 2\alpha\beta g(x) + \beta^2 \geq 0. \tag{3.36}$$

On multiplying both sides of (3.36) by $\phi(x)$ and integrating, we get for $a \leq x \leq b$, the following inequality:

$$\mu'_{f^4 1} \geq 2\alpha\mu'_{f^2 g} - 2\beta\mu'_{f^2 1} - \alpha^2\mu'_{1g^2} + 2\alpha\beta\mu'_{1g} - \beta^2. \tag{3.37}$$

Let

$$h(\alpha, \beta) = 2\alpha\mu'_{f^2 g} - 2\beta\mu'_{f^2 1} - \alpha^2\mu'_{1g^2} + 2\alpha\beta\mu'_{1g} - \beta^2. \tag{3.38}$$

The derivatives are,

$$\frac{dh}{d\alpha} = 2\mu'_{f^2 g} - 2\alpha\mu'_{1g^2} + 2\beta\mu'_{1g}, \tag{3.39}$$

$$\frac{dh}{d\beta} = -2\mu'_{f^2 1} + 2\alpha\mu'_{1g} - 2\beta, \tag{3.40}$$

$$s = \frac{d^2 h}{d\beta d\alpha} = 2\mu'_{1g}, \tag{3.41}$$

$$r = \frac{d^2 h}{d\alpha^2} = -2\mu'_{1g^2} \tag{3.42}$$

and

$$t = \frac{d^2 h}{d\beta^2} = -2. \tag{3.43}$$

We find that $rt - s^2 > 0$ and the function $h(\alpha, \beta)$ achieves its maximum at

$$\alpha = \frac{\mu'_{f^2 g} - \mu'_{1g}\mu'_{f^2 1}}{\mu'_{1g^2} - \mu'^2_{1g}} \tag{3.44}$$

and

$$\beta = \frac{\mu'_{1g}\mu'_{f^2 g} - \mu'_{f^2 1}\mu'_{1g^2}}{\mu'_{1g^2} - \mu'^2_{1g}}, \tag{3.45}$$

where the derivatives $\frac{dh}{d\alpha}$ and $\frac{dh}{d\beta}$ vanish simultaneously. The inequality (3.37) therefore gives the greatest lower bound for $\mu'_{f^4 1}$ when α and β are respectively given by (3.44) and (3.45), and on substituting these values of α and β in (3.37); the inequality (3.33) follows immediately. On interchanging f and g and proceeding as above, we easily get the inequality (3.34).

4. Remarks

The special cases of the above results also provide the inequalities for the moments of a random variable whose probability density function is defined over a finite real interval $[a, b]$. We mention a few examples:

(1) For $f = g = x$, the inequality (3.33) gives

$$\mu'_4 \geq \frac{\left(\mu'_3 - \mu'_1 \mu'_2\right)^2}{\mu'_2 - \mu'^2_1} + \mu'^2_2, \tag{4.1}$$

where $\mu'_r = \int\limits_a^b x^r \phi(x)\, dx, \ \ r = 1, 2, 3, 4$. The bounds for fourth central moment and kurtosis can be deduced from the inequality (4.1), also see [12].

(2) For $f = g$ the inequality (3.2) can be written in the following equivalent forms:

$$\mu'_{f^2 f} \geq m_f^3 + \frac{\left(\mu'_{ff} - m_f \mu'_{f1}\right)\left(\mu'_{ff} - m_f^2\right) + m_f^2 \left(\mu'_{f1} - m_f\right)^2}{\mu'_{f1} - m_f}, \tag{4.2}$$

$$\mu'_{f^2 f} \geq \frac{\mu'^2_{ff}}{\mu'_{f1}} + \frac{m_f \left(\mu'_{ff} - \mu'^2_{f1}\right)\left(\mu'_{ff} - m_f \mu'_{f1}\right)}{\mu'_{f1}\left(\mu'_{f1} - m_f\right)}, \tag{4.3}$$

$$\mu'_{f^2 f} \geq \mu'_{ff} \mu'_{f1} + \frac{\left(\mu'_{ff} - m_f^2\right)\left(\mu'_{ff} - \mu'^2_{f1}\right)}{\mu'_{f1} - m_f} \tag{4.4}$$

and

$$\mu'_{f^2 f} \geq \mu'^3_{f1} + \frac{\left(\mu'_{ff} - \mu'^2_{f1}\right)\left(\mu'_{ff} - m_f \mu'_{f1} + \mu'^2_{f1} - m_f^2\right)}{\mu'_{f1} - m_f}. \tag{4.5}$$

The inequalities (4.3), (4.4) and (4.5) provide refinements of the corresponding inequalities in (1.11), $\mu'_{f^2 f} = \mu'_{f^3 1}$. The bounds for the third order moments in terms of the first and second order moments follow from (3.1) and (3.2) when $f = g = x$.

(3) An upper bound for the variance of a random variable $m \leq x \leq M$,

$$S^2 + \left(\frac{M_3}{2S^2}\right)^2 \leq \frac{(M-m)^2}{4} \tag{4.6}$$

provides the refinement of the Popoviciu inequality and follows as a special case of the inequality (3.31) for $f(x) = x$, see [13, 14].

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