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K_0 AND RINGS OVER WHICH THE CLASS OF FINITELY GENERATED STRONGLY GORENSTEIN PROJECTIVE MODULES IS CLOSED UNDER EXTENSIONS

Chaoling Huang

Department of Mathematics, Jiangxi Agricultural University, Nanchang 330045 *P.R. China*
and
Department of Mathematics, Nanjing University, Nanjing, Jiangsu, 210093
P.R. China
e-mail: huangchaoling43@yahoo.com.cn

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In this paper, we consider the rings over which the class of finitely generated strongly Gorenstein projective modules is closed under extensions (called fs-closed rings). We give a characterization about the Grothendieck groups of the category of the finitely generated strongly Gorenstein projective R-modules and the category of the finitely generated R-modules with finite strongly Gorenstein projective dimensions for any left Noetherian fs-closed ring R.

Key words : Strongly Gorenstein projective module, Grothendieck group, extension, pullback, projectively resolving.

1. Introduction

Throughout this note, all rings are associative with a unit $1 \neq 0$ and all modules are unitary.

Recall that an R-module M is said to be Gorenstein projective if there is an exact sequence

$$\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$$

of projective modules with $M = Ker(P^0 \to P^1)$ such that the sequence remains exact when $Hom(-, P)$ is applied to it for any projective R-module P. Such exact sequence is called a complete projective resolution. Gorenstein projective modules were introduced by Enochs and Jenda in [4]. An R-module M is said to be strongly Gorenstein projective if there is a complete projective resolution of the form

$$\cdots \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} \cdots .$$

such that $M = Ker(f)$, see [2, Definition 2.2]. It is clear that any projective module is strongly Gorenstein projective, and any strongly Gorenstein projective module is Gorenstein projective. Conversely, the Gorenstein projective module is not necessarily strongly Gorenstein projective [2, Example 2.13], and the strongly Gorenstein projective module is not necessarily projective [2, Example 2.5]. It was proved in [2, Theorem 2.7] that a module is Gorenstein projective if and only if it is a direct summand of a strongly Gorenstein projective module. In [4], Enochs and Jenda defined a homological dimension, namely the Gorenstein projective dimension, $Gpd_R(-)$, for any R-module. Similarly, we can define the strongly Gorenstein projective dimension, $SGpd_R(-)$, for any R-modules. It is clear that $Gpd_R(M) \leq SGpd_R(M) \leq pd_R(M)$ for any R-module M. In [6] the concept of Gorenstein flat modules was introduced and in [2] so was the concept of strongly Gorenstein flat modules. It is well-known that the above originated from Auslander and Bridger's ideas; see [1], since, reference [1] and [6] are not directly mentioned in the paper.

For any ring R, two finitely generated projective left R-modules A, B are stably isomorphic if $A \oplus nR \cong B \oplus nR$ for some positive integer n. We denote by $[A]$ the stable isomorphism class of A, and by $K_0(R)^+$ the set of all stable isomorphism classes on $\mathcal{P}(R)$, where $\mathcal{P}(R)$ means the category of all finitely generated projective left R-modules. The set $K_0(R)^+$, endowed with the operation $[A]+[B] = [A \oplus B]$, is a monoid with zero element $[0]$. By formally adjoining additive inverses for the elements of $K_0(R)^+$, we embed $K_0(R)^+$ in an abelian group, the Grothendieck group of R, denoted by $K_0(R)$. Every element of $K_0(R)$ has the form $[A] - [B]$ for suitable $A,\ B \in \mathcal{P}(R)$ [7]. There are other definitions of $K_0(R)$, for instance, see [11, Definition 1.1.5, or Definition 3.1.6] and [3, Definiton 3.7.1]. In Section 2, we consider the rings over which the class of finitely generated strongly Gorenstein projective modules is closed under extensions (called fs-closed). In Section 3, we give a characterization about the Grothendieck groups of the category of the finitely generated strongly Gorenstein projective R-modules and category of finitely generated left R-modules with finite strongly Gorenstein projective dimensions for any left Noetherian fs-closed ring R.

In this paper, by $\mathcal{M}(R)$ we denote the category of all finitely generated left R-modules, and by $\mathcal{P}(R)$ and $\mathcal{SGP}(R)$ we denote the category of all finitely generated projective R-modules and finitely generated strongly Gorenstein projective R-modules respectively. We let $\overline{\mathcal{P}}(R)$ and $\overline{\mathcal{SGP}}(R)$ denote the category of the finitely generated left R-modules with finite projective dimensions and with finite strongly Gorenstein projective dimensions respectively.

2. Rings Over Which the Class of Finitely Generated Strongly Gorenstein Projective Modules is Closed Under Extensions

Recall that a class $\mathcal{X}$ is closed under extensions if for every short exact sequence $0 \to G_1 \to G_2 \to G_3 \to 0$ with $G_1 \in \mathcal{X}$ and $G_3 \in \mathcal{X}$, then $G_2 \in \mathcal{X}$, and $\mathcal{X}$ is projectively resolving if all projective modules are in $\mathcal{X}$ and for every short exact sequence $0 \to G_1 \to G_2 \to G_3 \to 0$ with $G_3 \in \mathcal{X}$ the condition $G_1 \in \mathcal{X}$ and $G_2 \in \mathcal{X}$ are equivalent. It is well-known that the class of all Gorenstein projective R-modules is projectively resolving [8, Theorem 2.5]. But for the class of strongly Gorenstein projective R-modules it is not true, see [13, P2660].

Definition 2.1 — For any ring R, we call it fs-closed, if the class of the finitely generated strongly Gorenstein projective R-modules is closed under extensions.

For any left Noetherian ring R, we have the following characterizations of fs-closed rings:

***Theorem* 2.2** — *The following conditions are equivalent:*

(1) R is fs-closed.

(2) The class of the finitely generated strongly Gorenstein projective R-modules is resolving, i.e., for every short exact sequence $0 \to G_1 \to G_2 \to G_3 \to 0$ *with* $G_3 \in \mathcal{SGP}(R)$ *the condition* $G_1 \in \mathcal{SGP}(R)$ *and* $G_2 \in \mathcal{SGP}(R)$ *are equivalent.*

(3) For any short exact sequence of left R-modules $0 \to G_1 \to G_0 \to M \to 0$*, where* G_1 *and* G_0 *are finitely generated strongly Gorenstein projective, if* $Ext^1_R(M,\ P) = 0$ *for any projective R-module* P*, then* M *is strongly Gorenstein projective.*

PROOF : (1) $\Rightarrow$ (2). Let $0 \to A \to B \to C \to 0$ be an exact sequence of finitely generated left R-modules, where B and C are strongly Gorenstein projective, it is sufficient to prove that A is strongly Gorenstein projective. Since C is strongly Gorenstein projective, by [2, Proposition 2.9] there is an exact sequence of left R-modules $0 \to C \to P \to C \to 0$, where P is projective. Consider the following pullback diagram:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & C & = & C & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & D & \longrightarrow & P & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

Since C and B are strongly Gorenstein projective, by (1), D is strongly Gorenstein projective. By [13, Theorem 2.1], A is strongly Gorenstein projective.

(2) $\Rightarrow$ (1). Clear.

(1) $\Rightarrow$ (3). Since G_1 is strongly Gorenstein projective, there is an exact

sequence of left R-modules $0 \to G_1 \to P \to G_1 \to 0$, where P is projective. Consider the following pushout diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & G_1 & \longrightarrow & G_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & P & \longrightarrow & D & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & G_1 & = & G_1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Since G_1 and G_0 are strongly Gorenstein projective, so is D by (1). By hypothesis, $Ext^1_R(M,\ P) = 0$, thus $0 \to P \to D \to M \to 0$ is split, i.e., $D \cong P \oplus M$, which is strongly Gorenstein projective. By [13, Theorem 2.1], M is strongly Gorenstein projective.

(3) $\Rightarrow$ (1). Let $0 \to A \to B \to C \to 0$ be an exact sequence of finitely generated left R-modules, where A and C are strongly Gorenstein projective, we prove that B is strongly Gorenstein projective. Since C is strongly Gorenstein projective, there is an exact sequence $0 \to C \to P \to C \to 0$, where P is projective. Consider the following pullback diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & C & = & C & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & D & \longrightarrow & P & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Since the sequence $0 \to A \to D \to P \to 0$ is exact, where P is projective and A is strongly Gorenstein projective, by [13, Theorem 2.1], D is strongly Gorenstein projective. On the other hand, for the exact sequence $0 \to A \to B \to C \to 0$, we have the long exact sequence

$$Ext^i_R(C,\ Q) \to Ext^i_R(B,\ Q) \to Ext^i_R(A,\ Q)$$

for any positive integer i and any projective R-module Q. Since A and C are strongly Gorenstein projective, we have that $Ext^i_R(C,\ Q) = Ext^i_R(A,\ Q) = 0$ by [2, Proposition 2.9]. Thus $Ext^i_R(B,\ Q) = 0$. By (3), B is strongly Gorenstein projective. □

Now we consider the change of rings with fs-closed.

Theorem 2.3 — *Let R be a commutative ring, and let A be a Noetherian subring of R such that $R \cong A \oplus E$ as A-module for some projective A-module E, If R is fs-closed and the class of the finitely generated strongly Gorenstein projective A-modules is closed under the direct summands, then A is fs-closed.*

PROOF : By the proof of [9, Theorem 2.4], for any finitely generated strongly Gorenstein projective A-module M, $M \otimes_A R$ is a strongly Gorenstein projective R-module. Let

$$0 \to M_1 \to M \to M_2 \to 0$$

be an exact sequence of finitely generated left A-modules, where M_1 and M_2 are strongly Gorenstein projective. We prove that M is a strongly Gorenstein projective A-module. We first prove that for any finitely generated strongly Gorenstein projective R-module M, it is also a strongly Gorenstein projective A-module. Since M is a strongly Gorenstein projective R-module, by [2, Proposition 2.9], there is an exact sequence

$$0 \to M \to P \to M \to 0$$

where P is projective R-module and $Ext_R(M, Q) = 0$ for any projective R-module Q. By [10, Lemma 7.2.2],

$$0 \to M \to P \to M \to 0$$

is an exact sequence of A-modules, where P is projective A-module. By [5, Theorem 3.2.5] and [12, Ex 9.20, P 258],

$$Ext_A(M,R)\otimes_A R = Ext_R(M\otimes_A R, R\otimes_A R) = Hom_A(R, Ext_R(M, R\otimes_A R)) = 0,$$

since $R\otimes_A R$ is a projective R-module. By [5, Ex 2.1.13, P 43], R is faithfuly flat A module since E is projectice A-module. Thus $Ext_A(M,R) = 0$. Since $Ext_A(M,A)$ is the direct summand of $Ext_A(M,R)$, $Ext_A(M,A) = 0$. From [2, Proposition 2.12], M is a strongly Gorenstein projective A-module. For R is a projective A-module, we have that

$$0 \to M_1 \otimes_A R \to M \otimes_A R \to M_2 \otimes_A R \to 0$$

is an exact sequence of finitely generated left R-modules, where $M_1 \otimes_A R$ and $M_2 \otimes_A R$ are strongly Gorenstein projective R-modules. By hypothesis, $M \otimes_A R$ is a strongly Gorenstein projective R-module. By the above discussion, $M \otimes_A R$ is also strongly Gorenstein projective A-module. Since M is the direct summand of $M \otimes_A R, M$ is strongly Gorenstein projective A-module. That is A is fs-closed. □

Corollary 2.4 — Let A be a commutative Noetherian ring and X an indeterminate over A. If $A[X]$ is fs-closed and the class of the finitely generated strongly Gorenstein projective A-modules is closed under the direct summands, then A is fs-closed.

Corollary 2.5 — Let A be a commutative Noetherian ring and E a projective A-module. If the trivial ring extension $R = A \propto E$ of A by E is fs-closed and the class of the finitely generated strongly Gorenstein projective A-modules is closed under direct summands, then A is fs-closed.

Proposition 2.6 — Let R be a commutative ring, and S be a multiplicative set. If $S^{-1}R$ is a finitely generated free R-module, R is fs-closed if and only if $S^{-1}R$ is fs-closed.

Proof : $\Rightarrow$. It is clear from [13, Proposition 3.17 (2)].

$\Leftarrow$. Let $0 \to A \to B \to C \to 0$ be an exact sequence of finitely generated R-modules, where A and C are strongly Gorenstein projective, we prove that B is a strongly Gorenstein projective R-module. At first, since every strongly Gorenstein projective module is Gorenstein projective, by [8, Proposition 2.5], B is Gorenstein projective. Thus $Ext(B,\ Q) = 0$ for any projective R-module Q by [8, Proposition 2.3]. Since $S^{-1}R$ is a finitely generated projective R-module,

$$0 \to A \otimes_R S^{-1}R \to B \otimes_R S^{-1}R \to C \otimes_R S^{-1}R \to 0$$

is an exact sequence of $S^{-1}R$-modules. By [13, Proposition 3.17 (1)], $A \otimes_R S^{-1}R$ and $C \otimes_R S^{-1}R$ are strongly Gorenstein projective $S^{-1}R$-modules. By hypothesis, $B \otimes_R S^{-1}R$ is strongly Gorenstein projective $S^{-1}R$-module. Thus there is an exact sequence of the form:

$$0 \to B \otimes_R S^{-1}R \to P \to B \otimes_R S^{-1}R \to 0,$$

where P is a projective $S^{-1}R$-module. By [12, Lemma 3.75], $P \cong S^{-1}P \cong P \otimes_R S^{-1}R$. Since $S^{-1}R$ is faithfully flat by [5, Lemma 2.1.13], $0 \to B \to P \to B \to 0$ is exact as R-modules, where P is a projective R-module. By [2, Proposition 2.9], B is a strongly Gorenstein projective R-module. □

Proposition 2.7 — Let R be a commutative local Noetherian ring, and $I \subseteq J(R)$ be an ideal. If I-adic completion $\hat{R}$ is fs-closed, then R is fs-closed.

PROOF : Let $0 \to A \to B \to C \to 0$ be an exact sequence of finitely generated R-modules, where A and C are strongly Gorenstein projective. By [5, Theorem 2.5.11], $0 \to \hat{A} \to \hat{B} \to \hat{C} \to 0$ is an exact sequence of $\hat{R}$-modules. By [13, Proposition 3.1 (1)], $\hat{A}$ and $\hat{C}$ are strongly Gorenstein projective $\hat{R}$-modules. By hypothesis, $\hat{B}$ is a strongly Gorenstein projective $\hat{R}$-module. From [13, Proposition 3.1 (1)], B is a strongly Gorenstein projective R-module.

Example 2.8 : (1) Since a strongly Gorenstein projective R-module is Gorenstein projective, by [8, Theorem 2.27], a strongly Gorenstein projective R-module is projective if and only if it has finite projective dimension. Thus, over any ring R with $lD(R) < \infty$, where $lD(R)$ means the left global dimension of R, the class of the strongly Gorenstein projective R-modules is closed under extensions.

(2) Recall that a ring R is called an $(n,\ d)$-ring if every R-module having a finite n-presentation has projective dimension at most d. If R is an $(n,\ d)$-ring, then the class of the finitely generated strongly Gorenstein projective R-modules is closed under extensions.

(3) For any principal ideal domain A, and every nonzero prime ideal $\mathfrak{M}$ of A, set $R = A/\mathfrak{M}^2$. The class of the finitely generated strongly Gorenstein projective R-modules is closed under extensions and direct summands, since every R-module is the strongly Gorenstein projective. In particular, $\mathbb{Z}/4\mathbb{Z}$ is the case, moreover, the global dimension of it is infinite.

3. K_0 Groups

Definition 3.1 — Let $\mathcal{P}$ be a subcategory of $\mathcal{M}(R)$, which is closed under extensions, i.e., if $0 \to P_1 \to P \to P_2 \to 0$ is an exact sequence in $\mathcal{M}(R)$ and $P_1, P_2 \in \mathcal{P}$, then $P \in \mathcal{P}$, and let $\mathcal{P}_0 = \{< M > \mid < M >$ is the isomorphism class of $M \in \mathcal{P}\}$. We define $K_0(\mathcal{P})$ to the free abelian group on $\mathcal{P}_0$ modulo the following relations:

(1) $[P] = [P']$ if $< P >=< P' >$ in $\mathcal{P}_0$;

(2) $[P] = [P_1] + [P_2]$ if $0 \to P_1 \to P \to P_2 \to 0$ in $\mathcal{P}$.

Lemma 3.2 — Let R be a left Noetherian fs-closed ring. Then

(1) if $0 \to M_1 \to M_2 \to Q \to 0$ is an exact sequence in $\mathcal{M}(R)$ with $Q \in \mathcal{SGP}(R)$, then $M_1 \in \overline{\mathcal{SGP}}(R)$ if and only if $M_2 \in \overline{\mathcal{SGP}}(R)$;

(2) if $0 \to M_1 \to Q \to M_2 \to 0$ is an exact sequence in $\mathcal{M}(R)$ with $Q \in \mathcal{SGP}(R)$, then $M_1 \in \overline{\mathcal{SGP}}(R)$ if and only if $M_2 \in \overline{\mathcal{SGP}}(R)$;

(3) if $0 \to Q \to M_1 \to M_2 \to 0$ is an exact sequence in $\mathcal{M}(R)$ with $Q \in \mathcal{SGP}(R)$, then $M_1 \in \overline{\mathcal{SGP}}(R)$ if and only if $M_2 \in \overline{\mathcal{SGP}}(R)$.

Proof : (1) $\Leftarrow$. Suppose $M_2 \in \overline{\mathcal{SGP}}(R)$. Then there is an exact sequence in $\mathcal{M}(R)$

$$0 \to Q_m \to \cdots \to Q_1 \to Q_0 \to M_2 \to 0$$

with each $Q_i \in \mathcal{SGP}(R)$ $(0 \leq i \leq m)$. Let $K = Ker(Q_0 \to M_2)$. Then $K \in \overline{\mathcal{SGP}}(R)$. Consider the pullback of $M_1 \to M_2$ and $Q_0 \to M_2$:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & K & = & K & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & T_0 & \to & Q_0 & \to & Q & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & M_1 & \to & M_2 & \to & Q & \to & 0. \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Since R is left Noetherian fs-closed, $T_0 \in SGP(R)$. So $0 \to Q_m \to \cdots \to Q_1 \to T_0 \to M_1 \to 0$ is an exact sequence in $\mathcal{M}(R)$ with each $Q_i \in \mathcal{SGP}(R)$ $(0 \leq i \leq m)$. Thus $M_1 \in \overline{\mathcal{SGP}}(R)$.

$\Rightarrow$. Suppose $M_1 \in \overline{\mathcal{SGP}}(R)$. Then there are exact sequences in $\mathcal{M}(R)$ $0 \to Q_n \to \cdots \to Q_1 \to Q_0 \to M_1 \to 0$ and $0 \to Q \to P \to Q \to 0$ with each $Q_i \in \mathcal{SGP}(R)$ $(0 \leq i \leq m)$ and P being finitely generated projective. Consider the following commutative diagram with all rows and all columns being exact

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Q_n & \longrightarrow & K_n & \longrightarrow & Q & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Q_{n-1} & \longrightarrow & Q_{n-1} \oplus P & \longrightarrow & P & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Q_1 & \longrightarrow & Q_1 \oplus P & \longrightarrow & P & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Q_0 & \longrightarrow & Q_0 \oplus P & \longrightarrow & P & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_1 & \longrightarrow & M_2 & \longrightarrow & Q & \longrightarrow & 0. \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

From the top row, $K_n \in \mathcal{SGP}(R)$. Then the middle column gives that $M_2 \in \overline{\mathcal{SGP}}(R)$.

(2) $\Leftarrow$. Suppose $M_2 \in \overline{\mathcal{SGP}}(R)$. Then there is an exact sequence in $\mathcal{M}(R)$: $0 \to K \to Q_0 \to M_2 \to 0$ with $Q_0 \in \mathcal{SGP}(R)$ and $K \in \overline{\mathcal{SGP}}(R)$.

Consider the pullback of $Q \to M_2$ and $Q_0 \to M_2$:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & K & = & K & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & M_1 & \to & T & \to & Q_0 & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & M_1 & \to & Q & \to & M_2 & \to & 0. \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

From the middle column, by (1), $T \in \overline{\mathcal{SGP}}(R)$. The middle row gives that $M_1 \in \overline{\mathcal{SGP}}(R)$ by (1) again.

$\Rightarrow$. Clear.

(3) $\Leftarrow$. Suppose $M_2 \in \overline{\mathcal{SGP}}(R)$. Then there is an exact sequence in $\mathcal{M}(R)$: $0 \to M_2' \to Q_2 \to M_2 \to 0$ with $Q_2 \in \mathcal{SGP}(R)$ and $M_2' \in \overline{\mathcal{SGP}}(R)$. Consider the pullback of $M_1 \to M_2$ and $Q_2 \to M_2$:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & M_2' & = & M_2' & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & Q & \to & T & \to & Q_2 & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & Q & \to & M_1 & \to & M_2 & \to & 0. \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

Since $Q,\ Q_2 \in \mathcal{SGP}(R)$, $T \in \mathcal{SGP}(R)$ by the middle row. Then the middle column gives that $M_1 \in \overline{\mathcal{SGP}}(R)$.

$\Rightarrow$. Suppose $M_1 \in \overline{\mathcal{SGP}}(R)$. Then there is an exact sequence in $\mathcal{M}(R)$: $0 \to M_1' \to Q_1 \to M_1 \to 0$ with $Q_1 \in \mathcal{SGP}(R)$ and $M_1' \in \overline{\mathcal{SGP}}(R)$. Consider the pullback of $Q \to M_1$ and $Q_1 \to M_1$:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & M_1' & = & M_1' & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & T & \longrightarrow & Q_1 & \longrightarrow & M_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & Q & \longrightarrow & M_1 & \longrightarrow & M_2 & \longrightarrow & 0. \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

From the first column, by (1), $T \in \overline{\mathcal{SGP}}(R)$. Using the result of (2) to the middle row, we have $M_2 \in \overline{\mathcal{SGP}}(R)$. □

***Theorem* 3.3** — *Let R be a left Noetherian fs-closed ring, and let $0 \to M_1 \to M_2 \to M_3 \to 0$ be an exact sequence with $M_3 \in \overline{\mathcal{SGP}}(R)$. Then $M_1 \in \overline{\mathcal{SGP}}(R)$ if and only if $M_2 \in \overline{\mathcal{SGP}}(R)$.*

Proof : $\Rightarrow$. Suppose $M_1 \in \overline{\mathcal{SGP}}(R)$. Then there is an exact sequence in $\mathcal{M}$:

$$0 \to Q_m \to \cdots \to Q_1 \to Q_0 \to M_1 \to 0$$

with each $Q_i \in \mathcal{SGP}(R)$ $(0 \leq i \leq m)$. Pick an exact sequence

$$0 \to K_m \to P_{m-1} \to \cdots \to P_1 \to P_0 \to M_3 \to 0$$

where $P_0, \ldots, P_{m-1}$ are finitely generated projective. By Lemma 3.2 (2), we know that $K_m \in \overline{\mathcal{SGP}}(R)$ as $M_3 \in \overline{\mathcal{SGP}}(R)$. Therefore, we have the following commutative diagram with all rows and all columns be exact:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Q_m & \longrightarrow & T_m & \longrightarrow & K_m & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Q_{m-1} & \longrightarrow & Q_{m-1} \oplus P_{m-1} & \longrightarrow & P_{m-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Q_1 & \longrightarrow & Q_1 \oplus P_1 & \longrightarrow & P_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Q_0 & \longrightarrow & Q_0 \oplus P_0 & \longrightarrow & P_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & 0. \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

From the top row, by Lemma 3.2 (3), $T_m \in \overline{\mathcal{SGP}}(R)$. So the middle column gives that $M_2 \in \overline{\mathcal{SGP}}(R)$.

$\Leftarrow$. Suppose $M_2 \in \overline{\mathcal{SGP}}(R)$. Since $M_3 \in \overline{\mathcal{SGP}}(R)$, there is an exact sequence in $\mathcal{M}(R)$: $0 \to M_3' \to Q \to M_3 \to 0$ with $Q \in \mathcal{SGP}(R)$ and $M_3' \in \overline{\mathcal{SGP}}(R)$. Consider the pullback of $M_2 \to M_3$ and $Q \to M_3$:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & M_3' & = & M_3' & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_1 & \longrightarrow & T & \longrightarrow & Q & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & 0. \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

By the middle column and the direction $\Rightarrow$, we have $T \in \overline{\mathcal{SGP}}(R)$. Using Lemma 3.2 (1) to the middle row, we obtain that $M_1 \in \overline{\mathcal{SGP}}(R)$. $\square$

Lemma 3.4 — Let R be a left Noetherian fs-closed ring. Let $M' \to M$ be a morphism in $\overline{\mathcal{SGP}}(R)$ and

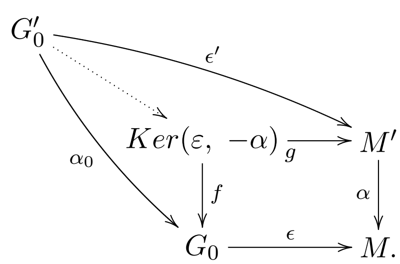
$$0 \to G_n \to G_{n-1} \to \cdots \to G_0 \to M \to 0$$

be a strongly Gorenstein projective resolution of M. Then one can complete these to a commutative diagram:

$$\begin{array}{ccccccccccccccc} 0 & \longrightarrow & \cdots & \longrightarrow & G'_{n+1} & \longrightarrow & G'_n & \longrightarrow & \cdots & \longrightarrow & G'_0 & \xrightarrow{\varepsilon'} & M' & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow{\scriptstyle \alpha_n} & & & & \downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle \alpha} & & \\ 0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & G_n & \longrightarrow & \cdots & \longrightarrow & G_0 & \xrightarrow{\varepsilon} & M & \longrightarrow & 0 \end{array}$$

in $\overline{\mathcal{SGP}}(R)$.

Proof : Since $G_0 \to M$ is epic, $(\varepsilon,\ -\alpha) :\ G_0 \oplus M' \to M$ is an epimorphism, and $0 \to Ker(\varepsilon,\ -\alpha) \to G_0 \oplus M' \to M \to 0$ is exact. Since R is fs-closed, $G_0 \oplus M' \in \overline{\mathcal{SGP}}(R)$. $Ker(\varepsilon,\ -\alpha) \in \overline{\mathcal{SGP}}(R)$, by Theorem 3.3, since R is a left Noetherian fs-closed ring. Since $M' \in \overline{\mathcal{SGP}}(R)$, there is an R-module $G'_0 \in \mathcal{SGP}(R)$ and an epimorphism $\epsilon' : G'_0 \to M'$. It is easy to see that $Ker(\varepsilon,\ -\alpha)$ is the pullback of ε and α. Thus there is an epimorphism $G'_0 \to Ker(\varepsilon,\ -\alpha)$. We get a commutative diagram:



Suppose $j \geq 1$ and α_k has been constructed for $0 \leq k < j$.

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & Kerd'_{j-1} & \longrightarrow & G'_{j-1} & \xrightarrow{d'_{j-1}} & \cdots & \longrightarrow & G'_0 & \xrightarrow{\varepsilon'} & M' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha_{j-1}|_{Kerd'_{j-1}}} & & \downarrow{\scriptstyle \alpha_{j-1}} & & & & \downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle \alpha} & & \\ 0 & \longrightarrow & Kerd_{j-1} & \longrightarrow & G_{j-1} & \xrightarrow{d_{j-1}} & \cdots & \longrightarrow & G_0 & \xrightarrow{\varepsilon} & M & \longrightarrow & 0. \end{array}$$

It is easy to check that $Kerd_{j-1} \in \overline{\mathcal{SGP}}(R)$ and $Kerd'_{j-1} \in \overline{\mathcal{SGP}}(R)$ by the induction and Theorem 3.3. Now we repeat the above construction to fill in the commutative diagram:

$$\begin{array}{ccc} G'_j & \longrightarrow & Kerd'_{j-1} \\ \downarrow{\scriptstyle \alpha_j} & & \downarrow{\scriptstyle \alpha_{j-1}} \\ G_j & \longrightarrow & Kerd_{j-1}. \end{array}$$

We can complete the diagram since $Kerd'_n \in \overline{\mathcal{SGP}}(R)$. □

Lemma 3.5 — Let R be a left Noetherian fs-closed ring. If

$$0 \to M_n \to M_{n-1} \to \cdots \to M_1 \to M_0 \to 0$$

is exact, where $M_i \in \mathcal{SGP}(R)(\overline{\mathcal{SGP}}(R))$ for all i, then $\sum_{j=0}^{n}(-1)^j[M_j] = 0$ in $K_0(\mathcal{SGP}(R))$ $(K_0(\overline{\mathcal{SGP}}(R)))$.

Proof : If $n = 1$, $0 \to M_1 \to M_0 \to 0$ is exact, then $M_1 \cong M_0$. So $[M_1] = [M_0]$. If $n = 2$, $0 \to M_2 \to M_1 \to M_0 \to 0$ is exact. By Definition 3.1, $[M_1] = [M_2]+[M_0]$. So $[M_2]-[M_1]+[M_0] = 0$. Now assume that $n > 3$, and by induction on n the lemma is true for exact sequence of shorter length. The kernel K of $M_1 \to M_0$ lies in $\mathcal{SGP}(R)$ ($\overline{\mathcal{SGP}}(R)$ by Theorem 3.3). So we can split the given exact sequence into two shorter exact sequences $0 \to K \to M_1 \to M_0 \to 0$ and

$$0 \to M_n \to \cdots \to M_2 \to K \to 0$$

By the above discussion $[K]+[M_0] = [M_1]$ and $[K]+\sum_{j=2}^{n}(-1)^{j-1}[M_j] = 0$. Thus $\sum_{j=0}^{n}(-1)^j[M_j] = 0$. □

Proposition 3.6 — Let R be a left Noetherian fs-closed ring. If $M \in \overline{\mathcal{SGP}}(R)$ and $G \to M$ and $G' \to M$ are two different finite strongly Gorenstein projective resolution, then $\sum_j(-1)^j[G_j] = \sum_j(-1)^j[G'_j]$ in $K_0(\mathcal{SGP}(R))$.

Proof : Apply Lemma 3.4 to complete the diagram:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & G''_n & \longrightarrow & \cdots & \longrightarrow & G''_0 & \xrightarrow{\varepsilon''} & M & \longrightarrow & 0\,, \\ & & \downarrow{\scriptstyle (\alpha_n,\ \alpha'_n)} & & & & \downarrow{\scriptstyle (\alpha_0,\ \alpha'_0)} & & \downarrow{\scriptstyle \Delta} & & \\ 0 & \longrightarrow & G_n \oplus G'_n & \longrightarrow & \cdots & \longrightarrow & G_0 \oplus G'_0 & \longrightarrow & M \oplus M & \longrightarrow & 0 \end{array}$$

where G'' is only a chain complex that is acyclic except at G''_n. By the long exact sequence we have that the mapping cones of α and α' are exact. Thus by Lemma 3.5, $\sum_j (-1)^j [G_j] - \sum_j (-1)^j [G''_j] = 0$. By the same discussion $\sum_j (-1)^j [G'_j] - \sum_j (-1)^j [G''_j] = 0$. Therefore, $\sum_j (-1)^j [G_j] = \sum_j (-1)^j [G'_j]$ in $K_0(\mathcal{SGP}(R))$. □

Theorem 3.7 — *Let R be a left Noetherian fs-closed ring. Then*

$$K_0(\mathcal{SGP}(R)) \cong K_0(\overline{\mathcal{SGP}}(R)).$$

Proof : Define $\varphi: K_0(\mathcal{SGP}(R)) \to K_0(\overline{\mathcal{SGP}}(R))$ by $[M]_{\mathcal{SGP}}(R) \mapsto [M]_{\overline{\mathcal{SGP}}(R)}$, and $\psi: K_0(\overline{\mathcal{SGP}}(R)) \to K_0(\mathcal{SGP}(R))$ by $[M]_{\overline{\mathcal{SGP}}(R)} \mapsto \sum (-1)^j [G_j]_{\mathcal{SGP}}(R)$. It is easy to check that φ and ψ are group-isomorphisms and that $\varphi\psi = 1_{K_0(\overline{\mathcal{SGP}}(R))}$ and $\psi\varphi = 1_{K_0(\mathcal{SGP}(R))}$ by the above results. □

Remark 3.8 : The class of the finitely generated Gorenstein projective modules is denoted $\mathcal{GP}(R)$ and the class of finite generated R-modules with finite Gorenstein projective resolutions is denoted $\overline{\mathcal{GP}}(R)$. Let R be a Noetherian ring. Since the class of all Gorenstein projective R-modules is projectively resolving [8, Theorem 2.5], using the analogous method, we can prove that $K_0(\mathcal{GP}(R)) \cong K_0(\overline{\mathcal{GP}}(R))$. It is clear that we have the following inclusion relations:

$$\mathcal{P}(R) \subseteq \mathcal{SGP}(R) \subseteq \mathcal{GP}(R) \subseteq \mathcal{M}(R), \overline{\mathcal{P}}(R) \subseteq \overline{\mathcal{SGP}}(R) \subseteq \overline{\mathcal{GP}}(R) \subseteq \mathcal{M}(R).$$

If R is regular (left Noetherian and $lpd_R M < \infty$ for any finite generated R-module M), we have that

$$\overline{\mathcal{P}}(R) = \overline{\mathcal{SGP}}(R) = \overline{\mathcal{GP}}(R).$$

Therefore, $K_0(\mathcal{P}(R)) \cong K_0(\overline{\mathcal{P}}(R)) = K_0(\mathcal{SGP}(R)) \cong K_0(\overline{\mathcal{SGP}}(R)) = K_0(\mathcal{GP}(R)) \cong K_0(\overline{\mathcal{GP}}(R))$.

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