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COMPLEX LINES IN COMPLEX HYPERBOLIC SPACE $H^2_{\mathbb{C}}$ [1]

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Let p_1, p_2, p_3, p_4 be four pairwise distinct points in the boundary of complex hyperbolic 2-space $H^2_{\mathbb{C}}$ and any three points do not lie in the same $\mathbf{C}$-circle. We show that we are always able to group the four points into two classes such that each class contains two points, the two complex lines spanned by each class are ultra-parallel or intersect. As an application, we can simplify the discussion in the paper [7], in which Parker and Platis used the global geometry coordinates to describe the Falbel's cross-ratio variety of the four pairwise distinct points on the $\partial H^2_{\mathbb{C}}$.

Key words : Complex hyperbolic space; Hermitian cross-product; $\mathbf{C}$-Circle.

1. INTRODUCTION

An interesting problem in complex hyperbolic geometry is the classification of ordered m-tuples of distinct points in the complex hyperbolic n-space, $H^n_{\mathbb{C}}$, or in

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its boundary, $\partial H^n_{\mathbb{C}}$ up to congruence in the holomorphic isometry group $PU(n,1)$ of $H^n_{\mathbb{C}}$. This problem has recently been the object of series of papers [1, 2]. For $n=2$ and $m=4$, this problem was considered by Falbel [3], see also Falbel-Platis [4], and Parker-Platis [7]. In these papers, they obtained the cross-ratio variety $\mathfrak{X}$ as the configuration space. They proved that the cross-ratio variety $\mathfrak{X}$ can be parameterised by three complex number $\mathbb{X}_1, \mathbb{X}_2, \mathbb{X}_3$ which are not equal to 0 or 1 and satisfy the following identities:

$$\begin{aligned} |\mathbb{X}_3| &= |\mathbb{X}_2|/|\mathbb{X}_1|, \\ 2|\mathbb{X}_1|^2\Re(\mathbb{X}_3) &= |\mathbb{X}_1|^2 + |\mathbb{X}_2|^2 - 2\Re(\mathbb{X}_1 + \mathbb{X}_2) + 1. \end{aligned} \tag{1}$$

In order to make the topology of the cross-ratio variety more transparent, Parker and Platis introduced global geometrical coordinates to give an alternative description of Falbel's cross-ratio variety $\mathfrak{X}$ in [7]. They grouped the four points into two classes, each class contains two points and spans one complex line. So there are two complex lines, denoted by L_1 and L_2. According to the distance between L_1 and L_2, they considered three kinds of global geometric coordinates to describe Falbel's cross-ratio variety $\mathfrak{X}$.

In this paper, we show that if any three points do not lie in the same **C**-circle, then we are always able to regroup the four points into two classes such that the two complex lines spanned by each class are ultra-parallel or intersect. Thus in essence, we only need two kind of global geometry coordinates to describe the Falbel's cross-ratio variety.

Our main result is the following Theorem.

***Theorem* 1** — *Let p_1, p_2, p_3, p_4 be four pairwise distinct points in $\partial H^2_{\mathbb{C}}$ and any three points do not lie in the same* **C**-*circle. Let L_{ij} denote the complex line spanned by p_i and p_j for $i \neq j$. Then there exist two complex lines L_{ij} and L_{uk} that are ultra-parallel (or intersect).*

This paper is organized as following. In section 2 we introduce the basic general definitions and results in complex hyperbolic geometry. The proof of Theorem 1 appears in section 3. In section 4, as an application, we expound how to simplify the discussion in paper [7] using our results.

2. Complex Hyperbolic Geometry

In this section we recall some basic notions in complex hyperbolic geometry. The general references on complex hyperbolic geometry are [5, 6, 8].

Complex hyperbolic space $H^2_{\mathbb{C}}$: Let $\mathbb{C}^{2,1}$ deonte the vector space $\mathbb{C}^3$ equipped with the Hermitian form

$$\langle z, w\rangle = z_1\overline{w}_3 + z_2\overline{w}_2 + z_3\overline{w}_1.$$

of signature $(2,1)$. It is given by the Hermitian matrix J

$$J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Let $z \in \mathbb{C}^{2,1}$, then we know that $\langle z, z\rangle$ is real. Thus we may define subsets V_-, V_0, V_+ of $\mathbb{C}^{2,1}$ by

$$\begin{aligned} V_- &= \{z \in \mathbb{C}^{2,1} | \langle z, z\rangle < 0\}, \\ V_0 &= \{z \in \mathbb{C}^{2,1} | \langle z, z\rangle = 0\}, \\ V_+ &= \{z \in \mathbb{C}^{2,1} | \langle z, z\rangle > 0\}. \end{aligned}$$

We say that $z \in \mathbb{C}^{2,1}$ is negative, null, or positive if z is in V_-, V_0 or V_+ respectively. Let $P(\mathbb{C}^{2,1})$ denote the projectivisation of $\mathbb{C}^{2,1} - \{0\}$. We denote the image of $z \in \mathbb{C}^{2,1}$ under the projectivisation map by $[z] = [z_1, z_2, z_3]$ for $z = (z_1, z_2, z_3)$. The complex hyperbolic space $H^2_{\mathbb{C}}$ is the projectivisation of the set of negative vectors in $\mathbb{C}^{2,1}$, that is $H^2_{\mathbb{C}} = P(V_-)$. The ideal boundary of $H^2_{\mathbb{C}}$ is defined as the projectivisation of the set of null vectors in $\mathbb{C}^{2,1}$, that is $\partial H^2_{\mathbb{C}} = P(V_0)$. The complex hyperbolic plane $H^2_{\mathbb{C}}$ is a Kahler manifold of constant holomorphic sectional curvature. The holomorphic isometry group of $H^2_{\mathbb{C}}$ is the projectivisation $PU(2,1)$ of the group $SU(2,1)$ of complex linear transformations, which preserve the above Hermitian form.

The Hermitian cross-product $\boxtimes : \mathbb{C}^{2,1} \times \mathbb{C}^{2,1} \to \mathbb{C}^{2,1}$ is defined by

$$z \boxtimes w = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix} (\overline{z} \times \overline{w}) = \begin{pmatrix} \overline{z_2 w_1 - z_1 w_2} \\ \overline{z_1 w_3 - z_3 w_1} \\ \overline{z_3 w_2 - z_2 w_3} \end{pmatrix}.$$

By simple computation, we have

$$\langle z \boxtimes w, z\rangle = \langle z \boxtimes w, w\rangle = 0$$

and

$$\langle a \boxtimes c, b \boxtimes c\rangle = \overline{\langle a, c\rangle\langle c, b\rangle - \langle a, b\rangle\langle c, c\rangle}.$$

In particular

$$\langle a \boxtimes b, a \boxtimes b\rangle = |\langle a, b\rangle|^2 - \langle a, a\rangle\langle b, b\rangle.$$

In order to show that Theorem 1, we need the following Lemma, which can be obtained in Goldman [5].

Lemma 1 — Let a and b be null vectors, then for any $c \in \mathbb{C}^{2,1}$

$$|\langle a \boxtimes b, c\rangle|^2 = |\langle a, b\rangle|^2\langle c, c\rangle - 2\Re\langle a, c\rangle\langle c, b\rangle\langle b, a\rangle.$$

C-*Circle (Chain)* : There are two kinds of totally geodesic submanifolds of dimension 2 in $H^2_{\mathbb{C}}$, complex geodesics(or complex line, represented by $H^1_{\mathbb{C}} \subset H^2_{\mathbb{C}}$) and real slices. Each of these totally geodesic submanifold is a model of the hyperbolic plane. Complex geodesics are obtained by projectivisation of two-dimensional complex subspaces of $\mathbb{C}^{2,1}$. Given any two points in $H^2_{\mathbb{C}}$, there is a unique complex geodesic containing them. Any positive vector $c \in \mathbb{C}^{2,1}$ determines a two-dimensional complex subspace

$$\{z \in \mathbb{C}^{2,1} | \langle c, z\rangle = 0\}$$

and a complex geodesic, which is projectivisation of this subspace. The vector c is called a polar vector of the complex geodesic. A polar vector can be normalised to $\langle c, c\rangle = 1$. Conversely, any complex geodesic is represented by a polar vector.

Consider the complex hyperbolic space $H^2_{\mathbb{C}}$ and its boundary $\partial H^2_{\mathbb{C}} = S^3$. We call **C**-circle(or **chain**) the intersections of S^3 with boundaries of totally geodesic complex submanifold $H^1_{\mathbb{C}}$ in $H^2_{\mathbb{C}}$.

Cartan's angular invariant : Let (p_1, p_2, p_3) be an ordered triple of distinct points on the boundary $\partial H^2_{\mathbb{C}}$. Then Cartan's angular invariant $A(p_1, p_2, p_3)$ is defined to be

$$A(p_1, p_2, p_3) = \arg(-\langle P_1, P_2\rangle\langle P_2, P_3\rangle\langle P_3, P_1\rangle).$$

where P_i are corresponding lifts of p_i. It is verified that $A(p_1, p_2, p_3)$ is independent of the chosen lifts and satisfies

$$-\frac{\pi}{2} \leq A(p_1, p_2, p_3) \leq \frac{\pi}{2}.$$

The following result can be found in Goldman [5].

Proposition 1 — Suppose (p_1, p_2, p_3) be an ordered triple of distinct points on the boundary $\partial H^2_{\mathbb{C}}$. Then

$$(p_1, p_2, p_3) \in \textbf{chain} \Leftrightarrow \mathrm{A}(\mathrm{p}_1, \mathrm{p}_2, \mathrm{p}_3) = \pm\frac{\pi}{2}.$$

Distance to complex lines : Suppose L_1 and L_2 be two complex lines of $H^2_{\mathbb{C}}$, with polar vectors $\mathbf{n}_1$ and $\mathbf{n}_2$. Let

$$N(L_1, L_2) = \frac{|\langle \mathbf{n}_1, \mathbf{n}_2 \rangle|^2}{\langle \mathbf{n}_1, \mathbf{n}_1 \rangle \langle \mathbf{n}_2, \mathbf{n}_2 \rangle}.$$

Obviously, $N(L_1, L_2)$ does not depend the choice of the lift in the pair of polar vectors, and is $PU(2, 1)$-invariant. Following proposition gives the geometric interpretation of $N(L_1, L_2)$, and we refer to [8] for details.

Proposition 2 — Let L_1 and L_2 be two complex lines of $H^2_{\mathbb{C}}$, with polar vectors $\mathbf{n}_1$ and $\mathbf{n}_2$. Let

$$N(L_1, L_2) = \frac{|\langle \mathbf{n}_1, \mathbf{n}_2 \rangle|^2}{\langle \mathbf{n}_1, \mathbf{n}_1 \rangle \langle \mathbf{n}_2, \mathbf{n}_2 \rangle}.$$

(1) If $N(L_1, L_2) > 1$, then L_1 and L_2 are ultra-parallel and

$$\cosh^2(\frac{\rho(L_1, L_2)}{2}) = N(L_1, L_2).$$

(2) If $N(L_1, L_2) = 1$, then L_1 and L_2 are asymptotic or coincide.

(3) If $N(L_1, L_2) < 1$, then L_1 and L_2 intersect.

3. THE PROOF OF THEOREM 1

In this section, we give the proof of our result. In order to obtain our result, we need the following straightforward lemma.

Lemma 2 — Let p_1, p_2, p_3, p_4 be four points in the boundary $\partial H^2_{\mathbb{C}}$, then on can apply an element of $PU(2,1)$ taking p_1 to $[0,0,1]$ and p_2 to $[1,0,0]$. Then denote the remaining points to $p_3 = [a_1, a_2, a_3], p_4 = [b_1, b_2, b_3]$, where a_i, b_i are complex numbers.

Proof of Theorem 1 : Let p_1, p_2, p_3, p_4 be four points in the boundary $\partial H^2_{\mathbb{C}}$ and $\mathbf{n}_{ij} = p_i \boxtimes p_j$ denote the polar vector of the complex line L_{ij}.

Thus

$$N(L_{ij}, L_{uk}) = \frac{|\langle \mathbf{n}_{ij}, \mathbf{n}_{uk}\rangle|^2}{\langle \mathbf{n}_{ij}, \mathbf{n}_{ij}\rangle\langle \mathbf{n}_{uk}, \mathbf{n}_{uk}\rangle}$$

where

$$\begin{aligned}
&|\langle \mathbf{n}_{ij}, \mathbf{n}_{uk}\rangle|^2 \\
&= |\langle p_i \boxtimes p_j, p_u \boxtimes p_k\rangle|^2 \\
&= |\langle p_i, p_j\rangle|^2 \langle p_u \boxtimes p_k, p_u \boxtimes p_k\rangle - 2\Re\langle p_i, p_u \boxtimes p_k\rangle\langle p_u \boxtimes p_k, p_j\rangle\langle p_j, p_i\rangle \\
&= |\langle p_i, p_j\rangle|^2 (|\langle p_u, p_k\rangle|^2 - \langle p_u, p_u\rangle\langle p_k, p_k\rangle) - 2\Re\langle p_i, p_u \boxtimes p_k\rangle\langle p_u \boxtimes p_k, p_j\rangle\langle p_j, p_i\rangle \\
&= |\langle p_i, p_j\rangle|^2 |\langle p_u, p_k\rangle|^2 - 2\Re\langle p_i, p_u \boxtimes p_k\rangle\langle p_u \boxtimes p_k, p_j\rangle\langle p_j, p_i\rangle
\end{aligned} \tag{2}$$

and

$$\langle \mathbf{n}_{ij}, \mathbf{n}_{ij}\rangle = \langle p_i \boxtimes p_j, p_i \boxtimes p_j\rangle = |\langle p_i, p_j\rangle|^2 - \langle p_i, p_i\rangle\langle p_j, p_j\rangle.$$

Since $\langle p_i, p_i\rangle = 0$ for all $i = 1, 2, 3, 4$, we have the following equation:

$$\langle \mathbf{n}_{ij}, \mathbf{n}_{ij}\rangle = |\langle p_i, p_j\rangle|^2.$$

Therefore

$$\begin{aligned}
&N(L_{ij}, L_{uk}) \\
&= \frac{|\langle \mathbf{n}_{ij}, \mathbf{n}_{uk}\rangle|^2}{\langle \mathbf{n}_{ij}, \mathbf{n}_{ij}\rangle\langle \mathbf{n}_{uk}, \mathbf{n}_{uk}\rangle} \\
&= \frac{|\langle p_i, p_j\rangle|^2 |\langle p_u, p_k\rangle|^2 - 2\Re\langle p_i, p_u \boxtimes p_k\rangle\langle p_u \boxtimes p_k, p_j\rangle\langle p_j, p_i\rangle}{|\langle p_i, p_j\rangle|^2 |\langle p_u, p_k\rangle|^2} \\
&= 1 - \frac{2\Re\langle p_i, p_u \boxtimes p_k\rangle\langle p_u \boxtimes p_k, p_j\rangle\langle p_j, p_i\rangle}{|\langle p_i, p_j\rangle|^2 |\langle p_u, p_k\rangle|^2}
\end{aligned} \tag{3}$$

Let $P_{iukj} = \Re\langle p_i, p_u \boxtimes p_k\rangle\langle p_u \boxtimes p_k, p_j\rangle\langle p_j, p_i\rangle$. Thus

$$N(L_{ij}, L_{uk}) = 1 - \frac{2P_{iukj}}{|\langle p_i, p_j\rangle|^2 |\langle p_u, p_k\rangle|^2}.$$

Since p_1, p_2, p_3, p_4 are null vector. From Lemma 1, we obtain

$$\begin{aligned} P_{iukj} &= \Re\langle p_i, p_u \boxtimes p_k\rangle\langle p_u \boxtimes p_k, p_j\rangle\langle p_j, p_i\rangle \\ &= \Re\langle p_u, p_i\rangle\langle p_i, p_k\rangle\langle p_k, p_u\rangle\Re\langle p_u, p_j\rangle\langle p_j, p_k\rangle\langle p_k, p_u\rangle\Re\langle p_j, p_i\rangle. \end{aligned} \tag{4}$$

From Lemma 2, we can assume that

$$p_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad p_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad p_3 = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \quad p_4 = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix},$$

where a_i, b_i are complex numbers.

Then we have

$$\begin{aligned} N(L_{12}, L_{34}) &= 1 - \frac{2\Re\langle p_1, p_3 \boxtimes p_4\rangle\langle p_3 \boxtimes p_4, p_2\rangle\langle p_2, p_1\rangle}{|\langle p_1, p_2\rangle|^2 |\langle p_3, p_4\rangle|^2}, \\ N(L_{13}, L_{24}) &= 1 - \frac{2\Re\langle p_1, p_2 \boxtimes p_4\rangle\langle p_2 \boxtimes p_4, p_3\rangle\langle p_3, p_1\rangle}{|\langle p_1, p_3\rangle|^2 |\langle p_2, p_4\rangle|^2}, \\ N(L_{14}, L_{23}) &= 1 - \frac{2\Re\langle p_1, p_2 \boxtimes p_3\rangle\langle p_2 \boxtimes p_3, p_4\rangle\langle p_4, p_1\rangle}{|\langle p_1, p_4\rangle|^2 |\langle p_2, p_3\rangle|^2}. \end{aligned} \tag{5}$$

By computing, we obtain

$$p_3 \boxtimes p_4 = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \boxtimes \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} \overline{a_2 b_1 - a_1 b_2} \\ \overline{a_1 b_3 - a_3 b_1} \\ \overline{a_3 b_2 - a_2 b_3} \end{bmatrix},$$

$$p_2 \boxtimes p_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \boxtimes \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} -\bar{b}_2 \\ \bar{b}_3 \\ 0 \end{bmatrix}$$

and

$$p_2 \boxtimes p_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \boxtimes \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} -\bar{a}_2 \\ \bar{a}_3 \\ 0 \end{bmatrix}.$$

Thus

$$P_{1342} = \Re\langle p_1, p_3 \boxtimes p_4\rangle\langle p_3 \boxtimes p_4, p_2\rangle\langle p_2, p_1\rangle = \Re(a_2b_1 - a_1b_2)(\overline{b_2a_3 - a_2b_3}),$$

$$P_{1243} = \Re\langle p_1, p_2 \boxtimes p_4\rangle\langle p_2 \boxtimes p_4, p_3\rangle\langle p_3, p_1\rangle = -\Re b_2a_1(\overline{a_2b_3 - b_2a_3}),$$

$$P_{1234} = \Re\langle p_1, p_2 \boxtimes p_3\rangle\langle p_2 \boxtimes p_3, p_4\rangle\langle p_4, p_1\rangle = -\Re a_2b_1(\overline{b_2a_3 - a_2b_3}).$$

Thus, we have

$$\begin{aligned} &\Re\langle p_1, p_3 \boxtimes p_4\rangle\langle p_3 \boxtimes p_4, p_2\rangle\langle p_2, p_1\rangle + \\ &\Re\langle p_1, p_2 \boxtimes p_4\rangle\langle p_2 \boxtimes p_4, p_3\rangle\langle p_3, p_1\rangle + \\ &\Re\langle p_1, p_2 \boxtimes p_3\rangle\langle p_2 \boxtimes p_3, p_4\rangle\langle p_4, p_1\rangle = 0, \end{aligned} \tag{6}$$

that is

$$P_{1342} + P_{1243} + P_{1234} = 0. \tag{7}$$

If every term P_{iujk} in the left sides of the equations (7) is zero, for example if $P_{1342} = 0$, that is

$$P_{1342} = \Re\langle p_3, p_1\rangle\langle p_1, p_4\rangle\langle p_4, p_3\rangle\Re\langle p_3, p_2\rangle\langle p_2, p_4\rangle\langle p_4, p_3\rangle\Re\langle p_2, p_1\rangle = 0.$$

Since $\langle p_2, p_1\rangle = 1$, we obtain

$$\Re\langle p_3, p_1\rangle\langle p_1, p_4\rangle\langle p_4, p_3\rangle = 0$$

or

$$\Re\langle p_3, p_2\rangle\langle p_2, p_4\rangle\langle p_4, p_3\rangle = 0,$$

which imply either the Cartan angular invariant of (p_3, p_1, p_4) or (p_3, p_2, p_4) is $\pm\frac{\pi}{2}$. Thus p_3, p_1, p_4 or p_3, p_2, p_4 lies in the same **C**-circle by Proposition 1, which is a contradiction. Thus there at least exists one term P_{iujk} that is negative (or positive) in the left sides of the equations (7). So $N(L_{ij}, L_{uk}) > 1$ (or $N(L_{ij}, L_{uk}) < 1$), this show that the two complex lines L_{ij}, L_{uk} are ultra-parallel (or intersect) by Proposition 2. □

Obviously, in this case the four points p_1, p_2, p_3, p_4 can span three pairs of complex lines. From the proof of Theorem 1, we know if there exists a pair of complex lines which are asymptotic, then the other two pairs of complex lines are not asymptotic.

So we obtain the following result.

Corollary 1 — Suppose that p_1, p_2, p_3 and p_4 are four distinct points in $\partial H^2_{\mathbb{C}}$ and any three ones of them are not in the same **C**-circle. Let L_{ij} denote the six possible complex lines spanned by p_i and p_j for $i \neq j$. If there exists a pair of complex lines are asymptotic, then the other two pairs are not asymptotic. Furthermore, if one of the two pairs of asymptotic lines is ultra-parallel, the other one is intersected.

Remark 1 : When there exists three points are in the same **C**-circle. These four points only are able to span four complex lines. For example if p_1, p_2, p_3 are in the same **C**-circle, we have $L_{12} = L_{23} = L_{13}$, so there only have four complex lines $L_{12}, L_{14}, L_{24}, L_{34}$, obviously, every two complex lines are asymptotic at one of points p_i for some $i \in \{1, 2, 3, 4\}$.

4. APPLICATION

In this section, as an application of our results, we expound how to simplify the discussion in paper [7]. Firstly, we state the results of Parker-Platis's paper (details can be found in [7]). Let p_1, q_1, p_2, q_2 to denote the four pairwise distinct points of $\partial H^2_{\mathbb{C}}$. Let L_i denote the complex line spanned by p_i and q_i for $i = 1, 2$.

Writing $\mathbf{p}_i$ and $\mathbf{q}_i$ for lifts of p_i and q_i to $\mathbb{C}^3 - \{0\}$, we can define the three cross-ratios of our points p_i and q_i as follows

$$
\begin{aligned}
\mathbb{X}_1 &= [p_2, p_1, q_1, q_2] = \frac{\langle \mathbf{q}_1, \mathbf{p}_2 \rangle \langle \mathbf{q}_2, \mathbf{p}_1 \rangle}{\langle \mathbf{q}_2, \mathbf{p}_2 \rangle \langle \mathbf{q}_1, \mathbf{p}_1 \rangle}, \\
\mathbb{X}_2 &= [p_2, q_1, p_1, q_2] = \frac{\langle \mathbf{p}_1, \mathbf{p}_2 \rangle \langle \mathbf{q}_2, \mathbf{q}_1 \rangle}{\langle \mathbf{q}_2, \mathbf{p}_2 \rangle \langle \mathbf{p}_1, \mathbf{q}_1 \rangle}, \qquad (8) \\
\mathbb{X}_3 &= [p_1, q_1, p_2, q_2] = \frac{\langle \mathbf{p}_2, \mathbf{p}_1 \rangle \langle \mathbf{q}_2, \mathbf{q}_1 \rangle}{\langle \mathbf{q}_2, \mathbf{p}_1 \rangle \langle \mathbf{p}_2, \mathbf{q}_1 \rangle}.
\end{aligned}
$$

Suppose that L_1 and L_2 are not asymptotic. Parker and Platis introduced the first kind of global geometrical coordinates $r, \theta_1, \theta_2, \psi$ to express the above three cross-ratios $\mathbb{X}_1, \mathbb{X}_2, \mathbb{X}_3$ and obtained the following theorem.

***Theorem* 2** — *[Parker and Platis] Let p_1, q_1, p_2 and q_2 be four pairwise distinct points of $\partial H^2_{\mathbb{C}}$. Let L_i be the complex line spanned by p_i and q_i. Suppose that L_1*

and L_2 are not asymptotic. then

$$\begin{aligned}\mathbb{X}_1 &= \frac{r^2e^{i\theta_1+i\theta_2}-2rcos\psi+e^{-i\theta_1-i\theta_2}}{-4sin\theta_1 sin\theta_2},\\ \mathbb{X}_2 &= \frac{r^2e^{-i\theta_1+i\theta_2}-2rcos\psi+e^{i\theta_1-i\theta_2}}{4sin\theta_1 sin\theta_2},\\ \mathbb{X}_3 &= \frac{r^2e^{i\psi}-2rcos(\theta_1-\theta_2)+e^{-i\psi}}{r^2e^{i\psi}-2rcos(\theta_1+\theta_2)+e^{-i\psi}}.\end{aligned} \tag{9}$$

moreover, these expressions satisfy the cross-ratio identities: (1) and when $r \neq 1$, the quantities $re^{i\psi}, \theta_1$ and θ_2 may be written uniquely in terms of $\mathbb{X}_1, \mathbb{X}_2$ and $\mathbb{X}_3$.

Suppose that L_1 and L_2 are asymptotic at a point that is distinct from p_i and q_i. Parker and Platis introduced the second kind of global geometrical coordinates $r^{'}, \theta_1^{'}, \theta_2^{'}, \psi^{'}$ to express the above three cross-ratios $\mathbb{X}_1, \mathbb{X}_2, \mathbb{X}_3$ and obtained the following theorem.

***Theorem* 3** — *[Parker and Platis] Let p_1, q_1, p_2 and q_2 be four pairwise distinct points of $\partial H^2_{\mathbb{C}}$. Let L_i be the complex line spanned by p_i and q_i. Suppose that L_1 and L_2 are asymptotic at a point that distinct from p_i and q_i. Let $(r^{'}, \theta_1^{'}, \theta_2^{'}, \psi^{'})$ be a tangent vector to the corresponding point of the space of coordinates defined above with $\theta_i^{'}$. Then $r^{'} \neq 0$ and*

$$\begin{aligned}\mathbb{X}_1 &= \frac{r^{'2}+2ir^{'}(\theta_1^{'}+\theta_2^{'})-(\theta_1^{'}+\theta_2^{'})^2+\psi^{'2}}{-4\theta_1^{'}\theta_2^{'}},\\ \mathbb{X}_2 &= \frac{r^{'2}-2ir^{'}(\theta_1^{'}-\theta_2^{'})-(\theta_1^{'}-\theta_2^{'})^2+\psi^{'2}}{4\theta_1^{'}\theta_2^{'}},\\ \mathbb{X}_3 &= \frac{r^{'2}-2ir^{'}\psi^{'}-\psi^{'2}+(\theta_1^{'}-\theta_2^{'})^2}{r^{'2}-2ir^{'}\psi^{'}-\psi^{'2}+(\theta_1^{'}+\theta_2^{'})^2}.\end{aligned} \tag{10}$$

Moreover, these expressions satisfy the cross-ratio identities: (1) and $\theta_1^{'}/r^{'}, \theta_2^{'}/r^{'}$ and $\psi^{'}/r^{'}$ may be expressed uniquely in terms of $\mathbb{X}_1, \mathbb{X}_2$ and $\mathbb{X}_3$.

Let $L_1^{'}$ denote the complex line spanned by p_1, p_2, and $L_2^{'}$ denote the complex line spanned by q_1, q_2, while let $L_1^{''}$ denote the complex line spanned by p_1, q_2, and $L_2^{''}$ denote the complex line spanned by p_2, q_1. From Corollary 1, under the assumptions of Theorem 3, we know that if the complex line L_1 and L_2 are asymptotic, then the complex line $L_1^{'}$ and $L_2^{'}$ are not asymptotic, and the complex line $L_1^{''}$

and L_2'' are not asymptotic as well. So we can use the complex lines L_1' and L_2' to replace the complex lines L_1 and L_2, swap p_2 and q_1, introduce the following three cross-ratios

$$\begin{aligned} \mathbb{X}_1' &= [q_1, p_1, p_2, q_2], \\ \mathbb{X}_2' &= [q_1, p_2, p_1, q_2], \\ \mathbb{X}_3' &= [p_1, p_2, q_1, q_2]. \end{aligned} \tag{11}$$

Since L_1' and L_2' are not asymptotic, we can choose the first kind of global geometrical coordinates to express $\mathbb{X}_1', \mathbb{X}_2', \mathbb{X}_3'$ just as in Theorem 2, so we simplify the discussion in the paper [7].

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