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FINITE GROUPS WITH SOME PRIMARY SUBGROUPS SS-QUASINORMALLY EMBEDDED[1]

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A subgroup of H of a group G is called ss-quasinormally embedded in G if there exists a subgroup T of G such that $G = HT$ and $H \cap T$ is s-quasinormally embedded in G. In this paper, we shall obtain some characterizations about p-nilpotency of G by assuming that some subgroups of prime power order of G are ss-quasinormally embedded in G.

Key words : s-quasinormally embedded; ss-quasinormally embedded; p-nilpotent; p-nilpotent supplement; formation.

1. INTRODUCTION

All groups considered in this paper will be finite. We use conventional notions and notation, as in Huppert [2]. G denotes always a group and $|G|$ stands for the order of G. Let $\mathcal{F}$ be a class of groups. We call $\mathcal{F}$ a formation provided that (i) if $G \in \mathcal{F}$ and $H \trianglelefteq G$, then $G/H \in \mathcal{F}$, and (ii) if G/M and G/N are in $\mathcal{F}$, then $G/(M \cap N)$

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is in $\mathcal{F}$ for any normal subgroups M, N of G. A formation $\mathcal{F}$ is said to be saturated if $G/\Phi(G) \in \mathcal{F}$ implies that $G \in \mathcal{F}$. We say a subgroup H of G has a p-nilpotent supplement T in G if G has a p-nilpotent subgroup T such that $G = HT$.

A subgroup H of a group G is said to be s-quasinormal [12] in G if H permutes with every Sylow subgroup of G. From Ballester-Bolinches and Pedraza-Aguilera [1], we know H is said to be s-quasinormally embedded in G if for each prime p dividing $|H|$, a Sylow p-subgroup of H is also a Sylow p-subgroup of some s-quasinormal subgroup of G. Obviously, s-quasinormally embedded subgroup is a generalization of s-quasinormal subgroup. Recently, by considering some special supplemented subgroups (c-supplemented subgroups), Wang [21] has given some characterization theorems for solvable groups and supersolvable groups. Recall that H is c-supplemented in G if there exists a subgroup K_1 such that $G = HK_1$ and $H \cap K_1 \leq H_G$, where H_G is the maximal normal subgroup of G contained in H. In this case, writing $K = H_G K_1$ we have $G = HK$ and $H \cap K = H_G$; of course, $H \cap K$ is s-quasinormal in G. On the basis of this observation, we now give the following new concept of ss-quasinormally embedded subgroup:

Definition 1.1 — A subgroup H of a group G is called ss-quasinormally embedded in G if there exists a subgroup T of G such that $G = HT$ and $H \cap T$ is s-quasinormally embedded in G.

Example 1.2 : Suppose $G = S_4$, the symmetric group of degree 4. Take $H = <(34)>$. Then H is ss-quasinormally embedded in G, but not s-quasinormally embedded in G.

Example 1.3 : For any simple non-abelian group, there always exists a Sylow subgroup which is ss-quasinormally embedded but not c-supplemented.

In the literature, authors usually put the assumptions on either the minimal subgroups (and cyclic subgroups of order 4 when $p = 2$) or the maximal subgroups of the Sylow subgroups when investigating the structure of a group G (see, for example, [4, 6, 8, 10, 11, 14, 18, 20, 21, 22, 23]). In present paper, we consider not only minimal or maximal subgroups of a Sylow subgroup of a group, but also all possibility order subgroups of a Sylow subgroup. The p-nilpotency of finite groups with ss-quasinormally embedded primary subgroups is investigated. As application, we unify and generalize a series of known results.

2. Preliminaries

Now, we list some basic and known results which will be used below.

Lemma 2.1 — ([1], Lemma 1) Suppose that U is s-quasinormally embedded in a group G and N is a normal subgroup of G. Then

(1) U is s-quasinormally embedded in H whenever $U \leq H \leq G$.

(2) UN is s-quasinormally embedded in G and UN/N is s-quasinormally embedded in G/N.

Lemma 2.2 — Let H be an ss-quasinormally embedded subgroup of a group G.

(1) If $H \leq L \leq G$, then H is ss-quasinormally embedded in L.

(2) If $N \trianglelefteq G$ and $N \leq H \leq G$, then H/N is ss-quasinormally embedded in G/N.

(3) If H is a π-subgroup and N is a normal π'-subgroup of G, then HN/N is ss-quasinormally embedded in G/N.

Proof : By the hypotheses, there is a subgroup K of G such that $G = HK$ and $H \cap K$ is s-quasinormally embedded in G.

(1) $L = L \cap HK = H(L \cap K)$ and $H \cap (L \cap K) = H \cap K$ is s-quasinormally embedded in L by Lemma 2.1. Hence H is ss-quasinormally embedded in L.

(2) $G/N = HK/N = H/N \cdot KN/N$ and $(H/N) \cap (KN/N) = (H \cap KN)/N = (H \cap K)N/N$ is s-quasinormally embedded in G/N by Lemma 2.1. Hence H/N is ss-quasinormally embedded in G/N.

(3) Since $(|G : K|, |N|) = 1$, $N \leq K$. It is easy to see that $G/N = HN/N \cdot KN/N = HN/N \cdot K/N$ and $(HN/N) \cap (K/N) = (HN \cap K)/N = (H \cap K)N/N$ is s-quasinormally embedded in G/N by Lemma 2.1. Hence HN/N is ss-quasinormally embedded in G/N.

Lemma 2.3 — Let G be a group and p a prime such that $p^{n+1} \nmid |G|$ for some integer $n \geqslant 1$. If $(|G|, (p-1)(p^2-1) \cdots (p^n-1)) = 1$, then G is p-nilpotent.

Proof : Suppose that the statement is not true and let G be a counterexample of minimal order. Obviously, every subgroup of G satisfies the hypothesis of the theorem. The minimal choice of G implies that G is a minimal non-p-nilpotent group. By [2, III, 5.2 and IV, 5.4], $G = P \rtimes Q$ is a semidirect product of two Sylow subgroups. It is easy to see that every proper quotient group of G satisfies the hypothesis. Thus $\Phi(P) = \Phi(G) = 1$ and so P is an elementary abelian p-group. Since $N_G(P)/C_G(P)$ is isomorphic to a p'-subgroup of Aut(P) and $|$ Aut$(P)|$ divides $p^{n(n-1)/2}(p-1)(p^2-1)\cdots(p^n-1)$ for $|P| \leqslant p^n$, we have $N_G(P)/C_G(P) = 1$. This induces that G is p-nilpotent by [3, Theorem 10.1.8]. The contradiction completes the proof.

Lemma 2.4 — ([8], A, 1.2) Let U, V, and W be subgroups of a group G. Then the following statements are equivalent:

(1) $U \cap VW = (U \cap V)(U \cap W)$.

(2) $UV \cap UW = U(V \cap W)$.

Lemma 2.5 — ([9], Lemma 2.3.) Suppose that H is s-quasinormal in G, P a Sylow p-subgroup of H, where p is a prime. If $H_G = 1$, then P is s-quasinormal in G.

Lemma 2.6 — ([9], Lemma 2.4.) Suppose P is a p-subgroup of G contained in $O_p(G)$. If P is s-quasinormally embedded in G, then P is s-quasinormal in G.

Lemma 2.7 — ([5], Lemma A.) If P is an s-quasinormal p-subgroup of G for some prime p, then $N_G(P) \geq O^p(G)$.

Lemma 2.8 — ([15], Lemma 2.8.) Let M be a maximal subgroup of G and P a normal p-subgroup of G such that $G = PM$, where p is a prime. Then $P \cap M$ is a normal subgroup of G.

Lemma 2.9 — ([1], III, 5.2 and IV, 5.4) Suppose G is a group which is not p-nilpotent but whose proper subgroups are all p-nilpotent. Then

(a) G has a normal Sylow p-subgroup P for some prime p and $G = PQ$, where Q is a non-normal cyclic q-subgroup for some prime $q \neq p$.

(b) $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$.

(c) The exponent of P is p or 4.

Lemma 2.10 — ([4], Lemma 2.3) Let p be a prime dividing the order of a group G with $(|G|, p-1) = 1$. If G has cyclic Sylow p-subgroups, then G is p-nilpotent.

Lemma 2.11 — Suppose a subgroup H of a group G has a p-nilpotent supplement T in G.

(1) If $N \trianglelefteq G$, then HN/N has a p-nilpotent supplement TN/N in G/N;

(2) If $H \leq L \leq G$, then H has a p-nilpotent supplement $T \cap L$ in L.

Lemma 2.12 — Let p be a prime and G a group with $(|G|, p-1) = 1$. Suppose that P is a Sylow p-subgroup of G such that every maximal subgroup of P has a p-nilpotent supplement in G, then G is p-nilpotent.

PROOF : If $p^2 \nmid |G|$, then G is p-nilpotent by Lemma 2.10. Now we assume that $p^2||G|$. Let P_1 be a maximal subgroup of P. By the hypothesis, P_1 has a p-nilpotent supplement K_1 in G. Let $K_{1p'}$ be a normal Hall p'-subgroup of K_1. Then, obviously, $K_{1p'}$ is a Hall p'-subgroup of G. Hence $G = P_1K_1 = P_1N_G(K_{1p'})$. We claim that $K_{1p'}$ is normal in G. Indeed, if $K_{1p'}$ is not normal in G, then $P \cap N_G(K_{1p'}) < P$. It follows that P has a maximal subgroup P_2 such that $P \cap N_G(K_{1p'}) \leq P_2$. It is clear $P_1 \neq P_2$. By the hypothesis, P_2 has also a p-nilpotent supplement K_2 in G. By repeating the above argument, we can find a Hall p'-subgroup $K_{2p'}$ of G such that $G = P_2K_2 = P_2N_G(K_{2p'})$. If $p = 2$, then $K_{1p'}$ and $K_{2p'}$ are conjugate in G by applying a deep result of Gross ([5], Main Theorem). If $p > 2$, then G is a soluble group by Feit-Thompson's Theorem and so $K_{1p'}$ and $K_{2p'}$ are conjugate in G. Since $K_{2p'}$ is normalized by K_2, there exists an element $g \in P_2$ such that $K_{2p'}^g = K_{1p'}$. Then $G = (P_2N_G(K_{2p'}))^g = P_2N_G(K_{1p'})$. This induces that $P = P \cap G = P \cap P_2N_G(K_{1p'}) = P_2(P \cap N_G(K_{1p'})) = P_2$. This contradiction completes the proof.

3. MAIN RESULTS

***Theorem* 3.1** — *Let G be a group and p a prime such that $(|G|, (p-1)(p^2-1) \cdot \cdot \cdot (p^n-1)) = 1$ for some integer $n \geqslant 1$. If there exists a Sylow p-subgroup P of G such that every n-maximal subgroup $($if exists$)$ of P not having a p-nilpotent supplement in G is ss-quasinormally embedded in G, then G is p-nilpotent.*

PROOF : Suppose that the theorem is false and let G be a counterexample of minimal order. We will derive a contradiction in several steps.

(1) G is not a non-abelian simple group.

By Lemma 2.3, $p^{n+1}||P|$ and so there exists a non-identity n-maximal subgroup of P.

By Lemma 2.12, P has a n-maximal subgroup P_n which has no p-nilpotent supplement in G. By the hypothesis of the theorem, there is a non-p-nilpotent subgroup T of G such that $G = P_nT$ and $P_n \cap T$ is s-quasinormally embedded in G. Thus there is an s-quasinormal subgroup K of G such that $P_n \cap T$ is a Sylow p-subgroup of K. Since K is s-quasinormal in G, we have that K is subnormal in G. If G is simple, then $K = 1$ or $K = G$. But the case that $K = G$ is impossible, because $P_n \cap T$ is not a Sylow p-subgroup of G. Thus we have $K = 1$ and so $P_n \cap T = 1$. By Lemma 2.3, T is p-nilpotent, a contradiction.

(2) G has a unique minimal normal subgroup N. Moreover G/N is p-nilpotent, and $\Phi(G) = 1$.

Let N be a minimal normal subgroup of G. We will consider G/N and show G/N satisfies the hypothesis of the theorem. Since P is a Sylow p-subgroup of G, clearly PN/N is a Sylow p-subgroup of G/N. If $|PN/N| \leqslant p^n$, then G/N is p-nilpotent by Lemma 2.3. Thus we may suppose that $|PN/N| \geqslant p^{n+1}$. Let M_n/N be a n-maximal subgroup of PN/N. Then $M_n = N(M_n \cap P)$. Write $P_n = M_n \cap P$. It follows that $P_n \cap N = M_n \cap P \cap N = P \cap N$ is a Sylow p-subgroup of N. Since $p^n = |PN/N : M_n/N| = |PN : (M_n \cap P)N| = |P : M_n \cap P| = |P : P_n|$, P_n is a n-maximal subgroup of P. If P_n has a p-nilpotent supplement in G, then M_n/N has a p-nilpotent supplement in G/N by Lemma 2.11. If P_n is ss-quasinormally embedded in G, then there is a subgroup T of G such that $G = P_nT$ and $P_n \cap T$ is s-quasinormally embedded in G. Therefor $G/N = M_n/N \cdot TN/N = P_nN/N \cdot TN/N$. Since $(|N : P_n \cap N|, |N : T \cap N|) = 1$, $(P_n \cap N)(T \cap N) = N = N \cap G = N \cap (P_nT)$. By Lemma 2.4, $(P_nN) \cap (TN) = (P_n \cap T)N$. It follows that $(P_nN/N) \cap (TN/N) = (P_nN \cap TN)/N = (P_n \cap T)N/N$. Since $(P_n \cap T)N/N$ is s-quasinormally embedded in G/N by Lemma 2.1, we have M_n/N is ss-quasinormally embedded in G. Therefore G/N satisfies the hypothesis of the theorem. The minimal choice of G yields that G/N is p-nilpotent. Since the class of all p-nilpotent groups is a saturated formation, the uniqueness of N and the fact that $\Phi(G) = 1$ are obvious.

(3) $O_{p'}(G) = 1$.

If $O_{p'}(G) \neq 1$, then $N \leq O_{p'}(G)$ by step (2). Since $G/O_{p'}(G) \cong (G/N) /(O_{p'}(G)/N)$ is p-nilpotent, G is p-nilpotent, a contradiction.

(4) $O_p(G) = 1$.

If $O_p(G) \neq 1$, step (3) yields that $N \leq O_p(G)$ and $\Phi(O_p(G)) \leq \Phi(G) = 1$. Therefore, G has a maximal subgroup M such that $G = MN$ and $G/N \cong M$ is

p-nilpotent. By Lemma 2.8, $O_p(G) \cap M$ is normal in G. Then the uniqueness of N yields $N = O_p(G)$. Let P_1 be an arbitrary maximal subgroup of P. We will prove that P_1 has a p-nilpotent supplement in G. Take a n-maximal P_n of P such that $P_n \leq P_1$. If P_n has a p-nilpotent supplement in G, obviously P_1 has also a p-nilpotent supplement in G. Thus we may assume that P_n is ss-quasinormally embedded in G. Then there is a subgroup T of G such that $G = P_nT$ and $P_n \cap T$ is s-quasinormally embedded in G. Hence there is an s-quasinormal subgroup K of G such that $P_n \cap T$ is a Sylow p-subgroup of K. If $K_G \neq 1$, then $N \leq K_G \leq K$. It follows that $N \leq P_n \cap T \leq P_1$ and so $G = NM = P_1M$, i.e., P_1 has the p-nilpotent supplement M. Now we suppose $K_G = 1$. By Lemma 2.5, $P_n \cap T$ is s-quasinormal in G. From Lemma 2.7 we have $O^p(G) \leq N_G(P_n \cap T)$. Since $P_n \cap T$ is subnormal in G, $P_n \cap T \leq O_p(G) = N$ by [15, Corollary 1.10.17]. Thus, $P_n \cap T \leq P_1 \cap N$ and $P_n \cap T \leq (P_n \cap T)^G = (P_n \cap T)^{O^p(G)P} = (P_n \cap T)^P \leq (P_1 \cap N)^P = P_1 \cap N \leq N$. It follows that $(P_n \cap T)^G = 1$ or $(P_n \cap T)^G = P_1 \cap N = N$. If $(P_n \cap T)^G = 1$, then $P_n \cap T = 1$ and so $|T|_p = p^n$. Hence T is p-nilpotent by Lemma 2.3. It follows that P_1 has the p-nilpotent supplement T. If $(P_n \cap T)^G = P_1 \cap N = N$, then $N \leq P_1$ and so P_1 has the p-nilpotent supplement M. The arbitrary choice of P_1 implies that every maximal subgroup of P has a p-nilpotent supplement in G, therefore G is p-nilpotent by Lemma 2.12, a contradiction.

(5) N is not p-nilpotent.

Assume N is p-nilpotent and let $N_{p'}$ be the normal p-complement of N. Since $N_{p'}\ Char\ N \trianglelefteq G$, we have $N_{p'} \trianglelefteq G$ and so $N_{p'} \leq O_{p'}(G) = 1$ by step (3). It follows that N is a p-group. Then $N \leq O_p(G) = 1$ by step (4), a contradiction.

(6) $G = NP$.

If $NP < G$, then NP satisfies the hypothesis of the theorem. The minimal choice of G yields that N is p-nilpotent, which contradicts step (5).

(7) G has a Hall p'-subgroup and any two Hall p'-subgroups of G are conjugate in G.

If $N \cap P \leq \Phi(P)$, then N is p-nilpotent by J. Tate's Theorem ([2], IV, 4.7), a contradiction. Therefore, there is a maximal subgroup P_1 of P such that $P = (N \cap P)P_1$. We take a n-maximal subgroup P_n of P such that $P_n \leq P_1$. First we show G has a Hall p'-subgroup. If P_n has a p-nilpotent supplement in G, obviously G has a Hall p'-subgroup. Thus we may assume P_n is ss-quasinormally embedded in G. Then there is a subgroup T of G such that $G = P_nT$ and $P_n \cap T$ is s-quasinormally embedded in G. Thus there is an s-quasinormal subgroup K of G such that $P_n \cap T$ is a Sylow p-subgroup of K. If $K_G \neq 1$, then $N \leq K_G \leq K$ and so $P_n \cap T \cap N = P_n \cap N$ is a Sylow p-subgroup of N. We know $P_n \cap N \leq P \cap N$ and $P \cap N$ is a Sylow p-subgroup of N, so $P_n \cap N = P \cap N$. Consequently, $P = (N \cap P)P_1 = (P_n \cap N)P_1 = P_1$, a contradiction. Therefore we may assume that $K_G = 1$. By Lemma 2.5, $P_n \cap T$ is s-quasinormal in G and so $P_n \cap T \lhd \lhd G$. Hence $P_n \cap T \leq O_p(G) = 1$. Since $|T|_p = p^n$, T is p-nilpotent by Lemma 2.3. Let $T_{p'}$ be the normal p-complement of T. Then $T_{p'}$ is a Hall p'-subgroups of G. If p is odd, then G is solvable by Feit-Thompson's Theorem, contrary to steps (3) and (4). Thus $p = 2$. By applying a deep result of Gross ([5], main Theorem), any two Hall p'-subgroups of G are conjugate in G.

(8) The final contradiction.

By step (7), G has a Hall p'-subgroup. By step (6), we may suppose that N has a Hall p'-subgroup $N_{p'}$. By Frattini's argument, $G = NN_G(N_{p'}) = (P \cap N)N_{p'}N_G(N_{p'}) = (P \cap N)N_G(N_{p'})$ and so $P = P \cap G = P \cap (P \cap N)N_G(N_{p'}) = (P \cap N)(P \cap N_G(N_{p'}))$. Since $N_G(N_{p'}) < G$, $P \cap N_G(N_{p'}) < P$. We take a maximal subgroup P_1 of P such that $P \cap N_G(N_{p'}) \leq P_1$. Then $P = (P \cap N)P_1$. Pick a n-maximal subgroup P_n of P contained in P_1. We will show P_n has a p-nilpotent supplement in G. We may assume that P_n is ss-quasinormally embedded in G. Then there is a subgroup T of G such that $G = P_nT$ and $P_n \cap T$ is s-quasinormally embedded in G. Therefore there is an s-quasinormal subgroup K of G such that $P_n \cap T$ is a Sylow p-subgroup of K. If $K_G \neq 1$, then $N \leq K_G \leq K$ and so $P_n \cap T \cap N = P_n \cap N$ is a Sylow p-subgroup of N. We know $P_n \cap N \leq P \cap N$ and $P \cap N$ is a Sylow p-subgroup of N, so $P_n \cap N = P \cap N$. Consequently, $P = (N \cap P)P_1 = (P_n \cap N)P_1 = P_1$, a contradiction. Therefore $K_G = 1$. By Lemma 2.5, $P_n \cap T$ is s-quasinormal in G and so $P_n \cap T \lhd \lhd G$. Hence $P_n \cap T \leq O_p(G) = 1$. It follows that $|T|_p = p^n$. By Lemma 2.3, T is p-nilpotent. Let $T_{p'}$ be the normal p-complement of T, then $T_{p'}$ is a Hall p'-subgroup of G. By step (7), $T_{p'}$ and $N_{p'}$ are conjugate in G. Since $T_{p'}$ is normalized by T, there exists $g \in P_n$ such that $T_{p'}^g = N_{p'}$. Hence $G = (P_nT)^g = P_nT^g = P_nN_G(T_{p'}^g) = P_nN_G(N_{p'})$ and $P = P \cap G = P \cap P_nN_G(N_{p'}) = P_n(P \cap N_G(N_{p'})) \leq P_1$, a contradiction.

***Theorem* 3.2** — *Let p be a prime and $\mathcal{F}$ a saturated formation containing all p-nilpotent groups. Suppose that G is a group with $(|G|, (p-1)(p^2-1)\cdots(p^n-1)) = 1$ for some integer $n \geqslant 1$. Then $G \in \mathcal{F}$ if and only if G has a normal subgroup E such that $G/E \in \mathcal{F}$ and E has a Sylow p-subgroup P such that every n-maximal subgroup (if exists) of P not having a p-nilpotent supplement in G is*

ss-quasinormally embedded in G.

PROOF : The necessity is obvious. We need only to prove the sufficiency. Suppose that the assertion is not true and let G be a counterexample of minimal order. By Lemma 2.2, every n-maximal subgroup of P not having a p-nilpotent supplement in G is ss-quasinormally embedded in E. Hence by Theorem 3.1, E is p-nilpotent. Obviously $E \neq G$. Let T be a normal Hall p'-subgroup of E. Now we divide the proof into the following steps:

(1) $T = 1$, and so $P = E \trianglelefteq G$.

Assume that $T \neq 1$. Since T is a normal Hall p'-subgroup of E and $E \trianglelefteq G$, we have $T \trianglelefteq G$. We claim that G/T (with respect to E/T) satisfies the hypothesis of the theorem. In fact, $(G/T)/(E/T) \cong G/E \in \mathcal{F}$ and E/T is a p-group. Suppose that M_n/T is a n-maximal subgroup of PT/T which has no p-nilpotent supplement in G/T and $P_n = M_n \cap P$. Then P_n is a n-maximal subgroup of P and $M_n = P_nT$. By the hypothesis, P_n is ss-quasinormally embedded in G. By Lemma 2.2, $M_n/T = P_nT/T$ is ss-quasinormally embedded in G/T. The minimal choice of G implies that $G/T \in \mathcal{F}$. It is easy to see that $G \in \mathcal{F}$ from [7, Proposition IV. 3.11], a contradiction. Hence $T = 1$ and so $P = E \trianglelefteq G$.

(2) Suppose that Q is a Sylow q-subgroup of G, where q is a prime divisor of $|G|$ and $q \neq p$, then $PQ = P \times Q$.

By (1), $P = E \trianglelefteq G$. Thus PQ is a subgroup of G. Obviously by Lemma 2.2, every n- maximal subgroup of P not having a p-nilpotent supplement in PQ is ss-quasinormally embedded in PQ. Hence by Theorem 3.1, we have that PQ is p-nilpotent. It follows that $Q \trianglelefteq PQ$ and so $PQ = P \times Q$.

(3) Final contradiction.

Let H be an arbitrary non-identity normal subgroup of G contained in P and G_p a Sylow p-subgroup of G. By (2), we have $HQ = H \times Q$ for any Sylow q-subgroup of G with $q \neq p$. This induces that $O^p(G) \leq C_G(H)$ and $[H, G] = [H, G_pO^p(G)] = [H, G_p] \trianglelefteq G$. We claim that $[H, G_p] < H$. Indeed, if $[H, G_p] = H$, then for any non-negative integer t, $H = [H, G_p, \cdots, G_p] \leq G_p^{t+1}$, where the number of G_p in $[H, G_p, \cdots, G_p]$ is t, which contradicts [7, Theorem A.10.3]. Thus $[H, G_p] < H$ and consequently there exists a normal subgroup K of G such that H/K is a chief factor of G and $[H, G_p] \leq K$. This implies that $H/K \leq Z(G/K)$. Then since $G/P \in \mathcal{F}$, we obtain that $G \in \mathcal{F}$. The final contradiction completes the proof.

***Theorem* 3.3** — *Let G be a group and p a prime such that $(|G|,(p-1)(p^2-1)\cdots(p^n-1))=1$ for some integer $n\geqslant 1$. Assume that one of the following conditions is satisfied:*

(a) either p is odd or $n\geqslant 2$, and for every subgroup H of G with $|H|=p^n$, either H has a p-nilpotent supplement in G, or H is ss-quasinormally embedded in G;

(b) $p=2$ and $n=1$, and for every subgroup H of G with $|H|=2$ or 4, either H has a p-nilpotent supplement in G, or H is ss-quasinormally embedded in G. Then G is p-nilpotent.

PROOF : Suppose that the statement is false and let G be a counterexample of minimal order. Then $p^{n+1}||G|$ by Lemma 2.3. We proceed the proof by the following steps.

(1) Every proper subgroup of G is p-nilpotent.

Let L be a proper subgroup of G. Then $(|L|,(p-1)(p^2-1)\cdots(p^n-1))=1$. If $p^{n+1}\nmid|L|$, then by Lemma 2.3, L is p-nilpotent. Now assume that $p^{n+1}||L|$. Let H be a subgroup of L with order p^n (when p is odd or $n\geq 2$), or with order 2 or 4 (when $p=2$ and $n=1$). Then by the hypothesis, H either has a p-nilpotent supplement in G or H is ss-quasinormally embedded in G. By Lemmas 2.11 and 2.2, H has a p-nilpotent supplement in L or ss-quasinormally embedded in L. This shows that L satisfies our hypothesis. The minimal choice of G implies that L is p-nilpotent.

(2) From Lemma 2.9, G has the following properties:

(i) $G=PQ$, where P is a normal Sylow p-subgroup of G and Q is a non-normal cyclic Sylow q-subgroup of G; (ii) $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$; (iii) If $p>2$, then the exponent of P is p; if $p=2$, then the exponent of P is 2 or 4.

(3) P is not cyclic.

If P is cyclic, then G is p-nilpotent by Lemma 2.10, a contradiction.

(4) Suppose $H\leq P$ with $|H|=p^n$ (or $|H|=2$ or 4 when $p=2$ and $n=1$), then H is s-quasinormal in G.

Let T be any supplement of H in G. Then $G=HT$ and $P=P\cap G=$

$P \cap HT = H(P \cap T)$. Since $P/\Phi(P)$ is abelian, we have $(P \cap T)\Phi(P)/\Phi(P) \trianglelefteq G/\Phi(P)$. Since $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$, $P \cap T \leq \Phi(P)$ or $P = (P \cap T)\Phi(P) = P \cap T$ for some supplement T. If $P \cap T \leq \Phi(P)$, then $H = P \trianglelefteq G$. In this case, H is s-quasinormal in G. Now assume that $P = P \cap T$ for any supplement T. Then $T = G$ is the unique supplement of H in G. By the hypothesis, H is ss-quasinormally embedded in G. Since $T = G$ is not p-nilpotent by the minimal choice of G, we have $H = H \cap T$ is s-quasinormally embedded in G. Note that $H \leq P \leq O_p(G)$, it follows that H is s-quasinormal in G by Lemma 2.6.

(5) Final contradiction.

Step (2)(iii) implies that P is generated by elements x of order p or 4, and it suffices to show that each such element x normalizes Q. Note that $|P| \geqslant p^{n+1}$ by Lemma 2.3, and suppose first that either p is odd or $n \geqslant 2$. Then each of the generators x is contained in a subgroup H of order p^n, and $H < P$, so H normalizes Q. It follows that Q is normal in G, a contradiction. We may now suppose that $p = 2$ and $n = 1$, and $|P| \geqslant 2^2$. If $|P| > 4$, then each generator x is contained in a subgroup H of order 4, and again $H < P$, so H normalizes Q. If $|P| = 4$, then step (3) implies that P is generated by subgroups H of order 2, and H normalizes Q as before.

Corollary 3.4 — Let G be a group and p a prime such that $(|G|, (p-1)(p^2-1)\cdots(p^n-1)) = 1$ for some integer $n \geqslant 1$. Assume that one of the following conditions is satisfied:

(a) either p is odd or $n \geqslant 2$, and for every subgroup H of G with $1 < |H| < p^{n+1}$, either H has a p-nilpotent supplement in G, or H is ss-quasinormally embedded in G;

(b) $p = 2$ and $n = 1$, and for every subgroup H of G with $|H| = 2$ or 4, either H has a p-nilpotent supplement in G, or H is ss-quasinormally embedded in G.

Then G is p-nilpotent.

***Theorem* 3.5** — *Let p be a prime and $\mathcal{F}$ a saturated formation containing all p-nilpotent groups. Suppose that G is a group with $(|G|, (p-1)(p^2-1)\cdots(p^n-1)) = 1$ for some integer $n \geqslant 1$ and G has a normal subgroup E such that $G/E \in \mathcal{F}$. Then $G \in \mathcal{F}$ if and only if one of the following conditions is satisfied:*

(a) either p is odd or $n \geqslant 2$, and for every subgroup H of E with $1 < |H| <$

p^{n+1}, either H has a p-nilpotent supplement in G, or H is ss-quasinormally embedded in G;

(b) $p = 2$ and $n = 1$, and for every subgroup H of E with $|H| = 2$ or 4, either H has a p-nilpotent supplement in G, or H is ss-quasinormally embedded in G.

PROOF : The necessity part is obvious. We need only to prove the sufficiency part. Suppose that the statement is false and let G be a counter example of minimal order. Obviously, $(|E|, (p-1)(p^2-1)\cdots(p^n-1)) = 1$ and either H has a p-nilpotent supplement in E by Lemma 2.11 or H is ss-quasinormally embedded in E by Lemma 2.2. Now, Corollary 3.4 implies that E is p-nilpotent. Let P be a Sylow p-subgroup of E and T a normal Hall p'-subgroup of E. Then T is normal in G. We now proceed to prove the theorem via the following steps.

(1) $T = 1$.

If $T \neq 1$, then we first claim that G/T (with respect to E/T) satisfies the hypothesis of the theorem. In fact, $(G/T)/(E/T) \cong G/E \in \mathcal{F}$. Let N/T be an arbitrary subgroup of E/T with $1 < |N/T| < p^{n+1}$ or $|N/T| = 2$ or 4 (when $p = 2$ and $n = 1$). Then $N = [T]L$, where L is a Sylow p-subgroup of N. Thus, $1 < |L| < p^{n+1}$ or $|L| = 2$ or 4 (when $p = 2$ and $n = 1$). By the hypothesis, either L has a p-nilpotent supplement in G or L is ss-quasinormally embedded in G. This means that either $N/T = TL/T$ has a p-nilpotent supplement in G/T by Lemma 2.11 or N/T is ss-quasinormally embedded in G/T by Lemma 2.2. Hence, G/T satisfies the hypothesis. The minimal choice of G implies that $G/T \in \mathcal{F}$. It is easy to see that $G \in \mathcal{F}$ from [7, Proposition IV. 3.11], a contradiction.

(2) Suppose that Q is a Sylow q-subgroup of G, where $q \neq p$ is a prime divisor of $|G|$. Then $PQ = P \times Q$.

By step (1), $P = E \trianglelefteq G$. Hence, PQ is a subgroup of G. Obviously, for every subgroup H of PQ with $1 < |H| < p^{n+1}$ or $|H| = 2$ or 4 (when $p = 2$ and $n = 1$), either H has a p-nilpotent supplement in PQ by Lemma 2.11 or H is ss-quasinormally embedded in PQ by Lemma 2.2. Hence by Corollary 3.4, PQ is p-nilpotent. It follows that $Q \trianglelefteq PQ$ and so $PQ = P \times Q$.

(3) Final contradiction.

Let M be an arbitrary non-identity normal subgroup of G contained in P and G_p a Sylow p-subgroup of G. By step (2), we have $MQ = M \times Q$ for any Sylow q-subgroup of G. This induces that $O^p(G) \leq C_G(M)$ and $[M, G] =$

$[M, G_pO^p(G)] = [M, G_p] \trianglelefteq G$. Since $[M, G_p] < M$, there exists a normal subgroup N of G such that M/N is a chief factor of G and $[M, G] \leq N$. This implies that $M/N \leq Z(G/N)$. Let f be the full and integrated formation function such that $\mathcal{F} = LF(f)$. Then $G/C_G(M/N) = 1 \in f(p)$. The arbitrary choice of M implies that there exists a normal chain of G contained in P such that every chief factor M/N is f-central. It follows that $G \in \mathcal{F}$, a contradiction.

4. Some Applications

Corollary 4.1 — ([9], Theorem 3.1) Let p be a prime dividing the order of a group G with $(|G|, p-1) = 1$. If there exists a Sylow p-subgroup P of G such that every maximal subgroup of P is s-quasinormally embedded in G, then G is p-nilpotent.

Corollary 4.2 — ([10], Theorem 3.1) Let p be the smallest prime dividing the order of a group G. If there exists a Sylow p-subgroup P of G such that every maximal subgroup of P is s-quasinormally embedded in G, then G is p-nilpotent.

Corollary 4.3 — ([13], Theorem 3.4) Let p be the smallest prime dividing the order of a group G. If there exists a Sylow p-subgroup P of G such that every maximal subgroup of P is c-normal in G, then G is p-nilpotent.

Corollary 4.4 — ([4], Theorem 3.1) Let P be a Sylow p-subgroup of a group G, where p is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. If every maximal subgroup of P is either c-normal or s-quasinormally embedded in G, then G is p-nilpotent.

Corollary 4.5 — ([14], Theorem 3.1) Let P be a Sylow p-subgroup of a group G, where p is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. If every maximal subgroup of P is c-supplemented in G, then G is p-nilpotent.

Corollary 4.6 — ([17], Theorem 3.2) Let G be a group and P a Sylow p-subgroup of G, where p is the smallest prime dividing $|G|$. If all maximal subgroups of P are c-supplemented in G, then G is p-nilpotent.

Corollary 4.7 — ([3], Theorem 3.1) Let p be a prime dividing the order of a group G with $(|G|, p-1) = 1$. Suppose that every maximal subgroup of P is c-supplemented in G and $G \in C_{p'}$, then $G/O_p(G)$ is p-nilpotent and $G \in D_{p'}$.

Corollary 4.8 — ([16], Theorem 3.1) Let p be a prime dividing the order of a group G with $(|G|, p-1) = 1$. Assume that H is a normal subgroup of G such that G/H is p-nilpotent. If there exists a Sylow p-subgroup P of H such that every maximal subgroup of P is c^*-normal in G, then G is p-nilpotent.

Corollary 4.9 — ([19], Theorem 2.3) Let p be the smallest prime dividing the order of a group G and N a normal subgroup of G such that G/N is p-nilpotent. If every cyclic subgroup of prime order or order 4 of N is c-supplemented in G, then G is p-nilpotent.

Corollary 4.10 — ([20], Lemma 3.8) Let p be the smallest prime dividing the order of a group G. If every cyclic subgroup of prime order or order 4 of G is c-normal in G, then G is p-nilpotent.

Corollary 4.11 — ([9], Theorem 4.1) Let p be a prime dividing the order of a group G with $(|G|, p-1) = 1$ and N a normal subgroup of G such that G/N is p-nilpotent. If every cyclic subgroup of prime order or order 4 of N is s-quasinormally embedded in G, then G is p-nilpotent.

Corollary 4.12 — ([21], Theorem 3.1) Let G be a group and p a prime of $|G|$ such that $(|G|, p^2-1) = 1$. Then G is p-nilpotent if and only if there exists a normal subgroup E of G such that G/E is p-nilpotent and each subgroup of E of order p^2 has a p-nilpotent quotient-supplement in G.

Corollary 4.13 — ([21], Theorem 3.3) Let G be a group and $(|G|, 21) = 1$. Then G is 2-nilpotent if and only if each subgroup of G of order 8 has a 2-nilpotent quotient-supplement in G.

Corollary 4.14 — ([9], Theorem 3.5) Let G be a group and $(|G|, 21) = 1$. If each subgroup of G of order 8 is complemented in G, then G is 2-nilpotent.

Corollary 4.15 — ([22], Theorem 3.1) Let G be a group and p a prime of $|G|$ such that $(|G|, p^2-1) = 1$. If there exists a normal subgroup E of G such that G/E is p-nilpotent and each subgroup of E of order p^2 is complemented in G, then G is 2-nilpotent.

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REFERENCES

1. A. Ballester-Bolinches and M. C. Pedraza-Aguilera, Sufficient conditions for supersolvability of finite groups, *J. Pure Appl. Algebra*, **127** (1998), 113-118.
2. B. Huppert, *Endiche Gruppen I*, Springer-Verlag, Berlin, 1968.
3. D. J. S. Robinson, *A course in the theory of groups*, Spinger-Verlag, New York, 1982.
4. D. Li and X. Guo, The influence of c-normality of subgroups on the structure of finite groups, *J. Pure Appl. Algebra*, **150** (2000), 53-60.
5. F. Gross, Conjugacy of odd order Hall subgroups, *Bull. London Math. Soc.*, **19** (1987), 311-319.
6. H. Wei and Y. Wang, On c^*-normality and its properties, *J. Group Theory*, **10** (2007), 211-223.
7. K. Doerk and T. Hawkes, *Finite Soluble Groups*, Walter de Gruyter, Berlin-New York, 1992.
8. L. Miao, W. Guo and K. P. Shum, New criteria for p-nilpotency of finite groups, *Comm. Algebra*, **35** (2007), 965-974.
9. L. Miao, On complemented subgroups of finite groups, *Czechoslovak Math. J.*, **56** (2006), 1019-1028.
10. M. Asaad and A. A. Heliel, On s-quasinormally embedded subgroups of finite groups, *J. Pure Appl. Algebra*, **165** (2001), 129-135.
11. M. Ramadan, M. E. Mohamed and A. A. Heliel, On c-normality of certain subgroups of prime power order of finite groups, *Arch. Math.*, **85** (2005), 203-210.
12. O. H. Kegel, Sylow-Gruppen and subnormalteiler endlicher Gruppen, *Math. Z.*, **78** (1962), 205-221.
13. P. Schmidt, Subgroups permutable with all Sylow subgroups, *J. Algebra*, **207** (1998), 285-293.
14. S. Li and Y. Li, On s-quasionormal and c-normal subgroups of a finie group, *Czechoslovak Math. J.*, **58** (2008), 1083-1095.
15. W. Guo, *The theory of classes of groups*, Science Press-Kluwer Academic Publishers, Beijing-Boston, 2000.
16. X. Zhong and S. Li, On c-supplemented minimal subgroups of finite groups, *Southeast Asian Bull. Math.*, **28** (2004), 1141-1148.

17. X. Guo and K. P. Shum, Finite p-nilpotent groups with some subgroups c-supplemented, *J. Aust. Math. Soc*., **78** (2005), 429-439.

18. X. Guo and K. P. Shum, On c-normal maximal and minimal subgroups of Sylow p-subgroups of finite groups, *Arch. Math*., **80** (2003), 561-569.

19. X. Guo and K. P. Shum, On p-nilpotency of finite group with some subgroup c-supplemented, *Algebra Colloq*., **10** (2003), 259-266.

20. X. Li and Z. Tao, Weakly normal subgroups of finite groups, *Indian J. Pure Appl. Math*., **41** (2010), 745-753.

21. Y. Wang, Finite groups with some subgroups of Sylow subgroups c-supplemented, *J. Algebra*, **224** (2000), 467-478.

22. Y. Li, Y. Wang and H. Wei, On p-nilpotency of finite groups with some subgroups π-quasinormally embedded, *Acta. Math. Hungar*, **108** (2005), 283-298.

23. Y. Wang, H. Wei and Y. Li, A generalization of Kramer's theorem and its application, *Bull. Aust. Math. Soc*., **65** (2002), 467-475.