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ESTIMATES ON CONJECTURES OF MINKOWSKI AND WOODS II

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Dedicated to Professor R. P. Bambah on his 85th Birthday

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Let $\mathbb{R}^n$ be the n-dimensional Euclidean space. Let $\wedge$ be a lattice of determinant 1 such that there is a sphere $|X| < R$ which contains no point of $\wedge$ other than the origin O and has n linearly independent points of $\wedge$ on its boundary. A well known conjecture in the geometry of numbers asserts that any closed sphere in $\mathbb{R}^n$ of radius $\sqrt{n}/2$ contains a point of $\wedge$. This is known to be true for $n \leq 8$. Recently we gave estimates on a more general conjecture of Woods for $n \geq 9$. This lead to an improvement for $9 \leq n \leq 22$ on estimates of Il'in (1991) to the long standing conjecture of Minkowski on product of n non-homogeneous linear forms. Here we shall refine our method to obtain improved estimates for Woods Conjecture. These give improved estimates of Minkowski's conjecture for $9 \leq n \leq 31$.

Key words : Lattice, Covering, Non-homogeneous, Product of linear forms, Critical determinant, Korkine and Zolotareff reduction, Hermite's constant, centre density.

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1. INTRODUCTION

Let $L_i = a_{i1}x_1 + \cdots + a_{in}x_n$, $1 \leq i \leq n$, be n real linear forms in n variables $x_1, \cdots, x_n$ and determinant $\Delta = \det(a_{ij}) \neq 0$. Let $c_1, \cdots, c_n$ be real numbers. Minkowski is believed to have made the following conjecture

Conjecture I : (Minkowski) There exist integers $u_1, \cdots, u_n$ such that

$$\prod_{i=1}^{n} |(L_i(u_1, \cdots, u_n) + c_i)| \leq \frac{1}{2^n}|\Delta|.$$

This conjecture is known to be true for $n \leq 8$. For a detailed history of Minkowski's conjecture and related results, see Bambah *et al*. [1], Gruber [12], Gruber and Lekkerkerker [13] and Hans-Gill *et al*. [10]. Define $M_n = M_n(|\Delta|)$ by

$$M_n = \mathrm{Sup}_{L_1,\cdots,L_n}\mathrm{Sup}_{(c_1,\cdots,c_n)\in\mathbb{R}^n}\mathrm{Inf}_{(u_1,\cdots,u_n)\in\mathbb{Z}^n}$$
$$\prod_{1\leq i\leq n} |(L_i(u_1, \cdots, u_n) + c_i)|.$$

Minkowski's Conjecture is equivalent to saying that

$$M_n \leq \frac{1}{2^n}|\Delta|. \tag{1}$$

Chebotarev [4] proved the weaker inequality $M_n \leq \frac{1}{2^{n/2}}|\Delta|$. Since then several authors have tried to improve upon this estimate. The bounds have been obtained in the form

$$M_n \leq \frac{1}{\nu_n 2^{n/2}}|\Delta|,$$

where $\nu_n > 1$. Clearly $\nu_n \leq 2^{n/2}$ by considering the linear forms $L_i = x_i$ and $c_i = \frac{1}{2}$ for $1 \leq i \leq n$. For a detailed history see Gruber and Lekkerkerker [13] and Hans-Gill *et al*. [11]. In particular Mordell [18] obtained $\nu_n = 4 - 2(2 - 3\sqrt{2}/4)^n - 2^{-n/2}$ for all n. (Mukhsinov [19] claimed to obtain improvements for small values of n, but it contained mistakes as stated in Mukhsinov [20]). Il'in [14] improved Mordell's estimates for $6 \leq n \leq 17$. In 1991, Il'in [15] obtained further improvements for $6 \leq n \leq 31$. Since Minkowski's Conjecture has been proved for $n \leq 8$, the authors [11] got improvements on results of Il'in for $9 \leq n \leq 22$.

Making more detailed study, we now obtain further improvements for $9 \leq n \leq 31$. For convenience of comparison, we give results by Mordell [18], Il'in [15], authors [11] and our improved results in Table I.

TABLE 1

	Estimates of Mordell	Estimates of Il'in	Our earlier Estimates	Our improved Estimates
n	ν_n	ν_n	ν_n	ν_n
9	2.8170394	3.3151283	19.3967939	19.9177948
10	2.8990614	3.4798928	22.6239298	24.3627506
11	2.9731018	3.5229055	25.3402874	29.2801145
12	3.0405253	3.5502417	27.2130945	32.2801213
13	3.1023558	3.5785628	27.9834142	34.8475153
14	3.1593729	3.6020935	27.5240464	37.8038391
15	3.2121798	3.6111553	25.871948	40.9051980
16	3.2612520	3.6190753	23.2241420	44.3414913
17	3.3069717	3.6392444	19.8972896	47.2339309
18	3.3496524	3.6617581	16.2628000	46.7645724
19	3.3895562	3.6673429	12.6763203	47.2575897
20	3.4269065	3.6723611	9.4205568	46.8640155
21	3.4618973	3.6769169	6.6737319	46.0522028
22	3.4946990	3.684080	4.5063277	43.6612034
23	3.5254641	3.6863331		37.8802374
24	3.5543297	3.6897821		32.5852958
25	3.5814208	3.6929517		27.8149432
26	3.6068520	3.6958893		23.0801951
27	3.6307288	3.7001150		17.3895105
28	3.6531489	3.7026271		12.9938763
29	3.6742031	3.7049722		9.5796191
30	3.6939760	3.7086731		6.7664335
31	3.7125466	3.7255824		4.7459720

We shall follow the Remak-Davenport approach. For the sake of convenience of the reader we describe this approach and give some basic results which are also stated in [11].

If $L_i = a_{i1}x_1 + \cdots + a_{in}x_n$, $1 \leq i \leq n$, are n real linear forms of determinant $\Delta = \det(a_{ij}) \neq 0$, then the associated lattice

$$\wedge = \{(L_1(u_1, \cdots, u_n), \cdots, L_n(u_1, \cdots, u_n)) \quad : \\ (u_1, \cdots, u_n) \in \mathbb{Z}^n\}$$

is of determinant $d(\wedge) = |\Delta|$. Clearly we can state Minkowski's Conjecture in the terminology of lattices as :

Any lattice $\wedge$ of determinant $d(\wedge)$ in $\mathbb{R}^n$ is a covering lattice for the set

$$S : |x_1 x_2 \dots x_n| \leq \frac{d(\wedge)}{2^n}.$$

Define the homogeneous minimum of $\wedge$ as

$$M_H(\wedge) \; = \; \text{Inf}\{|x_1 x_2 \dots x_n| : \\ X = (x_1, x_2, \cdots, x_n) \in \wedge, X \neq O\}.$$

Birch and Swinnerton-Dyer [2] proved.

Proposition 1 — Suppose that Minkowski Conjecture has been proved for dimensions $1, 2, ..., n-1$. Then it holds for all lattices $\wedge$ in $\mathbb{R}^n$ with $M_H(\wedge) = 0$.

McMullen [17] proved.

Proposition 2 — If $\wedge$ is a lattice in $\mathbb{R}^n$ with $M_H(\wedge) \neq 0$ then there exists an ellipsoid having n linearly independent points of $\wedge$ on its boundary and no point of $\wedge$ other than O in its interior.

It is well known that using these results Minkowski's Conjecture would follow from

Conjecture II : If $\wedge$ is a lattice in $\mathbb{R}^n$ of determinant 1 and there is a sphere $|X| \leq R$ which contains no point of $\wedge$ other than O in its interior and has n linearly independent points of $\wedge$ on its boundary then $\wedge$ is a covering lattice for the closed sphere of radius $\sqrt{n}/2$. Equivalently, every closed sphere of radius $\sqrt{n}/2$ lying in $\mathbb{R}^n$ contains a point of $\wedge$.

Woods [22, 23, 24] proved this conjecture for $n = 4, 5$ and 6 using Korkine and Zolotareff reduction. Korkine and Zolotareff [16] proved that a cartesian coordinate system can be chosen in $\mathbb{R}^n$ in such a way that $\wedge$ has a basis of the form

$$(A_1, 0, \cdots, 0), (a_{21}, A_2, 0, \cdots, 0), \cdots, (a_{n-1,1}, \cdots, A_{n-1}, 0), (a_{n1}, \cdots, A_n),$$

where $A_1, A_2, \cdots, A_n$ are all positive. Further for each $i,\ 1 \leq i \leq n$, the lattice $\wedge_i$ in $\mathbb{R}^{n-i+1}$ generated by

$$(A_i, 0, \cdots, 0), (a_{i+1,i}, A_{i+1}, 0, \cdots, 0), \cdots, (a_{ni}, a_{n,i+1}, \cdots, A_n)$$

has minimum A_i i.e.

$$A_i = \mathrm{Inf}\{|P| : \ P \in \wedge_i, \ P \neq O\}.$$

We shall call such a basis a reduced basis of $\wedge$. Woods [23, 24] made the following Conjecture.

Conjecture III (Woods) : If $d(\wedge) = A_1 \ldots A_n = 1$ and $A_i \leq A_1$ for $i = 2, \cdots, n$, then any closed sphere in $\mathbb{R}^n$ of radius $\sqrt{n}/2$ contains a point of $\wedge$.

Woods [24] showed that Conjecture III implies Conjecture II. He [22, 23, 24] proved Conjecture III for $n = 4, 5$ and 6. Hans-Gill *et al*. [8] gave a unified simpler proof for $n \leq 6$. Authors [9, 10] proved it for $n = 7$ and 8. In [11] we obtained estimates ω_n for Woods Conjecture for $n \geq 9$. Here we shall obtain improved estimates on Woods conjecture. These automatically lead to improvements of estimates on Minkowski's conjecture for $9 \leq n \leq 31$.

To deduce the results on the estimates of Minkowski's conjecture we also need the following generalization of Proposition 1.

Proposition 3 — Suppose that we know

$$M_j \leq \frac{1}{\nu_j 2^{j/2}} \qquad \text{for } \ 1 \leq j \leq n-1.$$

Let $\nu < \min \nu_{k_1}\nu_{k_2} \ldots \nu_{k_s}$, where the minimum is taken over all $(k_1, k_2, \cdots, k_s)$ such that $n = k_1 + k_2 + \cdots + k_s$, k_i positive integers for all i and $s \geq 2$. Then for all lattices $\wedge$ in $\mathbb{R}^n$ of determinant 1 with $M_H(\wedge) = 0$, the estimate ν holds for Minkowski's Conjecture.

To prove this result one can follow the line of proof of Proposition 1, which is Lemma 7 in the paper of Birch and Swinnerton-Dyer [2]. We shall omit the details.

In this paper we shall prove :

***Theorem* 1** — *Let* $9 \leq n \leq 31$. *If* $d(\wedge) = A_1 \ldots A_n = 1$ *and* $A_i \leq A_1$ *for* $i = 2, \cdots, n$, *then any closed sphere in* $\mathbb{R}^n$ *of radius* $\sqrt{\omega_n}/2$ *contains a point of* $\wedge$, *where* ω_n *is as listed in Table 2.*

Remark 1 : As mentioned above $\omega_n = n$ are the best possible estimates for $n \leq 8$. Together with Theorem 1 of [11], this theorem leads to improvement of estimates on Woods Conjecture for all larger n.

Since by arithmetic-geometric inequality the sphere $\left\{ X \in \mathbb{R}^n :| X |\leq \frac{\sqrt{\omega_n}}{2} \right\}$ is a subset of $\left\{ X : |x_1x_2...x_n| \leq \frac{1}{2^{n/2}} (\frac{\omega_n}{2n})^{n/2} \right\}$, Theorem 1 and Proposition 3 immediately imply.

***Theorem* 2** — *The values of ν_n for the estimates of Minkowski's Conjecture can be taken as* $\left(\frac{2n}{\omega_n}\right)^{n/2}$.

For $9 \leq n \leq 31$, these values are listed in Table 1.

In Section 2 we state some preliminary results and in Section 3 we prove Theorem 1.

2. Some Preliminary Results

Let $\Delta(S_n)$ denote the critical determinant of the unit sphere S_n with centre O in $\mathbb{R}^n$ i.e. $\Delta(S_n) = \text{Inf}\{d(\wedge) : \wedge \text{ has no point other than } O \text{ in the interior of } S_n\}$.

Let γ_n be the Hermite's constant i.e. γ_n is the smallest real number such that for any positive definite quadratic form Q in n variables of determinant D, there exist integers $u_1, u_2, \cdots, u_n$ not all zero satisfying

$$Q(u_1, u_2, \cdots, u_n) \leq \gamma_n D^{1/n}.$$

It is well known that $\Delta^2(S_n) = \gamma_n^{-n}$. Let $\mathbb{L}$ be a lattice in $\mathbb{R}^n$ reduced in the sense of Korkine and Zolotareff. Let $A_1, A_2, \cdots, A_n$ be as defined in Section 1. Write $B_i = A_i^2$. We state below some preliminary lemmas. Lemma 1 is due to Korkine and Zolotareff [16], Lemma 2 is due to Pendavingh and Van Zwam [21] and Lemmas 3 and 4 are due to Woods [22]. In Lemma 5, the cases $n = 2$ and 3 are classical results of Lagrange and Gauss; $n = 4$ and 5 are due to Korkine and Zolotareff [16] while $n = 6, 7$ and 8 are due to Blichfeldt [3].

Lemma 1 — For all relevant i, $B_{i+1} \geq \frac{3}{4} B_i$ and $B_{i+2} \geq \frac{2}{3} B_i$.

Lemma 2 — For all relevant i, $B_{i+4} \geq (0.46873) B_i$.

(Notice that what we denote by B_i is denoted by A_i in some papers e.g. [21]).

Lemma 3 — If $2\Delta(S_{n+1})A_1^n \geq d(\mathbb{L})$ then any closed sphere of radius

$$R = A_1\{1 - (A_1^n\Delta(S_{n+1})/d(\mathbb{L}))^2\}^{1/2}$$

in $\mathbb{R}^n$ contains a point of $\mathbb{L}$.

Lemma 4 — For a fixed integer i with $1 \leq i \leq n-1$, denote by $\mathbb{L}_1$ the lattice in $\mathbb{R}^i$ with the reduced basis

$$(A_1, 0, \cdots, 0), (a_{2,1}, A_2, 0, \cdots, 0), \cdots, (a_{i,1}, a_{i,2}, \cdots, a_{i,i-1}, A_i)$$

and denote by $\mathbb{L}_2$ the lattice in $\mathbb{R}^{n-i}$ with the reduced basis

$$(A_{i+1}, 0, \cdots, 0), (a_{i+2,i+1}, A_{i+2}, 0, \cdots, 0), \cdots, (a_{n,i+1}, a_{n,i+2}, \cdots, a_{n,n-1}, A_n).$$

If any closed sphere in $\mathbb{R}^i$ of radius r_1 contains a point of $\mathbb{L}_1$ and if any closed sphere in $\mathbb{R}^{n-i}$ of radius r_2 contains a point of $\mathbb{L}_2$ then any closed sphere in $\mathbb{R}^n$ of radius $\left(r_1^2 + r_2^2\right)^{1/2}$ contains a point of $\mathbb{L}$.

Lemma 5 — $\Delta(S_n) = \sqrt{3}/2,\ 1/\sqrt{2},\ 1/2,\ 1/2\sqrt{2},\ \sqrt{3}/8,\ 1/8$ and $1/16$ for $n =\ 2,\ 3,\ 4,\ 5,\ 6,\ 7$ and 8 respectively.

Lemma 6 —

$$\ell_n = \left\{9^{\frac{1}{5}}\gamma_n^{\frac{1}{n-1}}\gamma_{n-1}^{\frac{1}{n-2}}\cdots\gamma_6^{\frac{1}{5}}\right\}^{-1} \leq B_n \leq m_n, \quad \text{where } m_n = \gamma_{n-1}^{\frac{n-1}{n}}. \qquad (2)$$

This is Theorem 2 of [11]. One may remark here that the lower bound of B_n in Lemma 6 can be improved slightly by making use of Lemma 2 i.e.

$$\left\{(8.5337)^{\frac{1}{5}}\gamma_n^{\frac{1}{n-1}}\gamma_{n-1}^{\frac{1}{n-2}}\cdots\gamma_6^{\frac{1}{5}}\right\}^{-1} \leq B_n \leq \gamma_{n-1}^{\frac{n-1}{n}}. \qquad (3)$$

Lemma 7 — For any integer $s,\ 1 \leq s \leq n-1$

$$B_1B_2\ldots B_s \leq \mu_s^{(n)}, \quad \text{where } \mu_s^{(n)} = \left(\gamma_n^{\frac{1}{n-1}}\gamma_{n-1}^{\frac{1}{n-2}}\cdots\gamma_{n-s+1}^{\frac{1}{n-s}}\right)^{n-s}. \qquad (4)$$

This is Lemma 4 of [11].

3. PROOF OF THEOREM 1

We assume that Theorem 1 is false and derive a contradiction. Let $\mathbb{L}$ be a lattice satisfying the hypothesis of the conjecture. Suppose that there exists a closed sphere of radius $\sqrt{\omega_n}/2$ in $\mathbb{R}^n$ that contains no point of $\mathbb{L}$. Since $B_i = A_i^2$ and $d(\mathbb{L}) = 1$, we have $B_1 B_2 \dots B_n = 1$.

We give some examples of inequalities that arise. Let $\mathbb{L}_1$ be a lattice in $\mathbb{R}^1$ with basis (A_1), $\mathbb{L}_2$ be a lattice in $\mathbb{R}^4$ with basis $(A_2, 0, 0, 0)$, $(a_{3,2}, A_3, 0, 0)$, $(a_{4,2}, a_{4,3}, A_4, 0)$, $(a_{5,2}, a_{5,3}, a_{5,4}, A_5)$, and $\mathbb{L}_i$, for $6 \leq i \leq n$ be lattices in $\mathbb{R}^1$ with basis (A_i). Applying Lemma 4 repeatedly and using Lemma 3, we see that if $2\Delta(S_5)A_2^4 \geq A_2 A_3 A_4 A_5$ then any closed n-sphere of radius

$$\left(\frac{1}{4}A_1^2 + A_2^2 - \frac{A_2^{10}\Delta(S_5)^2}{A_2^2 A_3^2 A_4^2 A_5^2} + \frac{1}{4}A_6^2 + \cdots + \frac{1}{4}A_n^2\right)^{1/2}$$

contains a point of $\mathbb{L}$. By the initial hypothesis this radius exceeds $\sqrt{\omega_n}/2$. Since $\Delta(S_5) = 1/2\sqrt{2}$ and $B_1 B_2 ... B_n = 1$, this results in the conditional inequality

$$\text{if } B_2^4 B_1 B_6 \dots B_n \geq 2 \text{ then } B_1 + 4B_2 - \frac{1}{2}B_2^5 B_1 B_6 \dots B_n + B_6 + B_7 + \cdots + B_n > \omega_n. \tag{5}$$

We call this inequality $(1, 4, 1, \cdots, 1)$, since it corresponds to the ordered partition $(1, 4, 1, \cdots, 1)$ of n for the purpose of applying Lemma 4. Similarly the conditional inequality $(1, \cdots, 1, 2, 1, \cdots, 1)$ corresponding to the ordered partition $(1, \cdots, 1, 2, 1, \cdots, 1)$ is

$$\text{if } 2B_i \geq B_{i+1} \text{ then } \quad B_1 + \cdots + B_{i-1} + 4B_i - \frac{2B_i^2}{B_{i+1}} + B_{i+2} + \cdots + B_n > \omega_n. \tag{6}$$

Since $4B_i - \dfrac{2B_i^2}{B_{i+1}} \leq 2B_{i+1}$, the second inequality in (6) gives

$$B_1 + \cdots + B_{i-1} + 2B_{i+1} + B_{i+2} + \cdots + B_n > \omega_n. \tag{7}$$

One may remark here that the condition $2B_i \geq B_{i+1}$ is necessary only if we want to use inequality (6), but it is not necessary if we want to use the weaker inequality (7). This is so because if $2B_i < B_{i+1}$, using the partition $(1, 1)$ in place of (2) for the relevant part, we get the upper bound $B_i + B_{i+1}$ which is clearly less than $2B_{i+1}$. We shall call inequalities of type (7) as weak inequalities.

If $(\lambda_1, \lambda_2, \cdots, \lambda_s)$ is an ordered partition of n, then the conditional inequality arising from it, by using Lemmas 3 and 4, is also denoted by $(\lambda_1, \lambda_2, \cdots, \lambda_s)$. If the conditions in an inequality $(\lambda_1, \lambda_2, \cdots, \lambda_s)$ are satisfied then we say that $(\lambda_1, \lambda_2, \cdots, \lambda_s)$ holds.

Sometimes, instead of Lemma 4, we are able to use induction. The use of this is indicated by putting $(^*)$ on the corresponding part of the partition. For example, if for some s, $1 \leq s \leq n-1$, $B_{s+1} \geq B_{s+j}$ for all $j,\ 2 \leq j \leq n-s$ then the inequality $(s^*, (n-s)^*)$ holds, which gives

$$\phi_{s,n-s}(B_1 B_2 \ldots B_s) = \omega_s (B_1 B_2 \ldots B_s)^{\frac{1}{s}} + \omega_{n-s} \left(\frac{1}{B_1 B_2 \ldots B_s} \right)^{\frac{1}{n-s}} > \omega_n. \tag{8}$$

In particular the inequality $((n-1)^*, 1)$ always holds. This can also be written as

$$f(B_n) = \omega_{n-1} (B_n)^{\frac{-1}{(n-1)}} + B_n > \omega_n. \tag{9}$$

We have $B_1 \geq 1$, because if $B_1 < 1$, we have $B_i \leq B_1 < 1$ for each i contradicting $B_1 B_2 ... B_n = 1$.

Lemma 8 — For any integer $s,\ 1 \leq s \leq n-1$

$$B_1 B_2 \ldots B_s \geq \begin{cases} \dfrac{(0.46873)^{k(2k-2)} B_1^s}{4^k} & \text{if} \quad s = 4k \\[2ex] \dfrac{(0.46873)^{k(2k-1)} B_1^s}{4^k} & \text{if} \quad s = 4k+1 \\[2ex] \dfrac{3(0.46873)^{k(2k)} B_1^s}{4 \times 4^k} & \text{if} \quad s = 4k+2 \\[2ex] \dfrac{(0.46873)^{k(2k+1)} B_1^s}{2 \times 4^k} & \text{if} \quad s = 4k+3. \end{cases}$$

This follows easily by induction on k, using Lemmas 1 and 2.

Lemma 9 — For any integer $s,\ 1 \leq s \leq n-1$

$$B_1 B_2 \dots B_s \geq \begin{cases} \dfrac{(0.46873)^{k(2k-2)}}{4^k B_n^{n-s}} & \text{if} \quad n-s = 4k \\[2ex] \dfrac{(0.46873)^{k(2k-1)}}{4^k B_n^{n-s}} & \text{if} \quad n-s = 4k+1 \\[2ex] \dfrac{3(0.46873)^{k(2k)}}{4 \times 4^k B_n^{n-s}} & \text{if} \quad n-s = 4k+2 \\[2ex] \dfrac{(0.46873)^{k(2k+1)}}{2 \times 4^k B_n^{n-s}} & \text{if} \quad n-s = 4k+3. \end{cases}$$

PROOF : We have $B_1 B_2 \dots B_s = \dfrac{1}{B_{s+1} B_{s+2} \dots B_n}$. Now the proof follows easily by induction on k and using Lemmas 1 and 2.

Remark 2 : Let

δ_n = the best centre density of packings of unit spheres in $\mathbb{R}^n$,

$\delta_n^\star$ = the best centre density of lattice packings of unit spheres in $\mathbb{R}^n$.

Then it is known that (see [7], page 20)

$$\gamma_n = 4(\delta_n^\star)^{\frac{2}{n}} \leq 4(\delta_n)^{\frac{2}{n}}. \tag{10}$$

$\delta_n^\star$ and hence γ_n is known for $n \leq 8$. Also $\gamma_{24} = 4$ has been proved by Cohn and Kumar [6]. For $9 \leq n \leq 31$, $n \neq 24$, using the bounds on δ_n given by Cohn and Elkies [5] and inequality (10), we find bounds on γ_n which we list in Table 2.

Lemma 10 — For each $n,\ 9 \leq n \leq 31$, we have $B_n < \ell'_n$, where ℓ'_n is a suitably chosen real number listed in Table 2.

PROOF : Suppose $B_n \geq \ell'_n$. By Lemma 6 we have $B_n \leq m_n = \gamma_{n-1}^{\frac{n-1}{n}}$. The inequality $((n-1)^*, 1)$ gives $f(B_n) = \omega_{n-1} B_n^{-1/n-1} + B_n > \omega_n$. A simple calculation shows that the function $f(B_n)$ has its maximum value at one of the end points of the interval $[\ell'_n,\ m_n]$. We find that $\max\{f(\ell'_n), f(m_n)\} \leq \omega_n$, giving thereby a contradiction.

Remark 3 : In Table 2 we list the estimates ω_n on Woods Conjecture which we obtain in Section 3. The suitable real numbers ℓ'_n listed in Table 2 satisfy $\ell_n < \ell'_n < m_n$, where lower bound ℓ_n and upper bound m_n of B_n are as given in Lemma 6. In [11] we used only the inequality $((n-1)^*, 1)$ to obtain estimates for $9 \leq n \leq 22$. Having used $((n-1)^*, 1)$ for $\ell'_n \leq B_n \leq m_n$, here we shall use more inequalities for $B_n < \ell'_n$ to get better estimates. For each n the numbers ω_n and ℓ'_n are chosen by careful scrutiny of the inequalities that arise. Inequalities used for each n are listed in Column 4 of Table 2.

Remark 4 : Sometimes we need to maximize functions of several variables. While doing this we shall find it convenient to name the function involved as $g(x), g(x, y)$ etc. to indicate that it is being regarded as function of that variable and other variables are kept fixed. When we say that a given function of several variables in $x, y, \cdots$ is an increasing/decreasing function of $x, y, \cdots$, it means that the concerned property holds when function is considered as a function of one variable at a time, all other variables being fixed.

TABLE 2

n	$\gamma_n \leq$	ℓ'_n	Inequalities	ω_n
9	2.1326324	0.4723	$(8^*, 1), (1, 2, 2, 2, 2), (2, 1, 2, 2, 2),$ $(1, 4, 2, 2)$	9.2587472
10	2.2636302	0.45489	$(9^*, 1), (2, 2, 2, 2, 2), (6^*, 4), (5^*, 5),$ $(1, 9^*), (2, 8^*), (3, 7^*), (1, 4, 5^*)$	10.5605061
11	2.3933470	0.43871	$(10^*, 1), (7^*, 4), (2, \cdots, 2, 1, 2), (6^*, 4, 1)$ $(1, 10^*), (2, 9^*), (1, 2, \cdots, 2), (1, 4, 6^*)$	11.9061976
12	2.5217871	0.349378	$(11^*, 1), (8^*, 2, 2)$	13.4499927
13	2.6492947	0.3503	$(12^*, 1), (1^*, 12^*), (2^*, 11^*),$ $(2, 1, 2, 2, 2, 2, 2)$	15.0562267
14	2.7758041	0.36	$(13^*, 1), (3^*, 11^*), (8^*, 6^*)$ $(2, 2, \cdots, 2)$	16.6646332
15	2.9014777	0.3867	$(14^*, 1), (s^*, (16-s)^*), s = 4, 6, 7, 8$ $(1, 2, 2, \cdots, 2)$	18.2901579
16	3.0263937	0.3843	$(15^*, 1), (s^*, (16-s)^*), s = 5, 6, 7, 8$ $(2, 2, \cdots, 2)$	19.9204292
17	3.1506793	0.3567496	$(16^*, 1), (s^*, (17-s)^*), s = 6, 7, 8, 9^\dagger$ $(1, 2, 2, \cdots, 2)$	21.6026907
18	3.2743307	0.32386	$(17^*, 1), (2, 2, \cdots, 2)$ $(s^*, (18-s)^*), s = 5, 6, 7, 8, 9$	23.4831402
19	3.3974439	0.3259	$(18^*, 1), (1, 2, 2, \cdots, 2)$ $(s^*, (19-s)^*), s = 2, 3, 5, \cdots, 9$	25.3234826

TABLE 2 (continued)

n	$\gamma_n \leq$	ℓ'_n	Inequalities	ω_n
20	3.5200620	0.3185	$(19^*, 1), (2, 2, \cdots, 2)$ $(s^*, (20-s)^*), s = 1, 2, \cdots, 9$	27.2255111
21	3.6422432	0.3182	$(20^*, 1), (s^*, (21-s)^*), s = 1, \cdots, 9, 10^\dagger$ $(2, 1, 2, \cdots, 2)$	29.1638254
22	3.7640371	0.2925501	$(21^*, 1), (s^*, (22-s)^*), s = 1, \cdots, 9, 10^\dagger$ $(2, 2, \cdots, 2)$	31.2142617
23	3.8854763	0.2420327	$(22^*, 1), (s^*, (23-s)^*), s = 1, \cdots, 9, 10^\dagger$ $(2, 1, 2, \cdots, 2)$	33.5354821
24	4	0.2431596	$(23^*, 1), (s^*, (24-s)^*), s = 1, \cdots, 10, 11^\dagger$ $(2, 2, \cdots, 2)$	35.9050965
25	4.1274438	0.2444457	$(24^*, 1), (s^*, (25-s)^*), s = 1, \cdots, 11, 12^\dagger$ $(2, 1, 2, \cdots, 2)$	38.3201985
26	4.2480446	0.2342451	$(25^*, 1), (s^*, (26-s)^*), s = 1, \cdots, 11, 12^\dagger$ $(2, 2, \cdots, 2)$	40.8449876
27	4.3684312	0.1932462	$(26^*, 1), (s^*, (27-s)^*), s = 1, \cdots, 11, 12^\dagger$ $(2, 1, 2, \cdots, 2)$	43.7039431
28	4.488631	0.1950064	$(27^*, 1), (s^*, (28-s)^*), s = 1, \cdots, 12, 13^\dagger$ $(2, 2, \cdots, 2)$	46.6267624
29	4.6086676	0.19685	$(28^*, 1), (s^*, (29-s)^*), s = 1, \cdots, 13, 14^\dagger$ $(2, 1, 2, \cdots, 2)$	49.6305176
30	4.7285667	0.1815776	$(29^*, 1), (s^*, (30-s)^*), s = 1, \cdots, 13, 14^\dagger$ $(2, 2, \cdots, 2)$	52.8194566
31	4.8483483	0.1835723	$(30^*, 1), (s^*, (31-s)^*), s = 1, \cdots, 14, 15^\dagger$ $(2, 1, 2, \cdots, 2)$	56.0735184

3.1 *Proof of Theorem 1 for* $9 \leq n \leq 12$

Recall that $\mathbb{L}$ is a lattice for which Theorem 1 is false. We derive a contradiction for each n, $9 \leq n \leq 12$, in the following subsections.

3.1.1 — n = 9

Here we have $\omega_9 = 9.2587472$, $B_1 \leq \gamma_9 \leq 2.1326324$ and $B_9 < 0.4723 = \ell'_9$ (see Table 2 and Lemma 10).

Claim (i) $B_1 > 1.85916$.

The weak inequality $(1, 2, 2, 2, 2)$ gives $B_1 + 2B_3 + 2B_5 + 2B_7 + 2B_9 > 9.2587472$. Using Lemmas 1 and 2 we get $B_1 + 2\left\{\dfrac{1.5}{0.46873} + \dfrac{1}{0.46873} + 1.5 + 1\right\}$

$(0.4723) > 9.2587472$. This gives $B_1 > 1.85916$.

Claim (ii) $B_2 > 1.7008$.

Suppose $B_2 \leq 1.7008$. By Lemma 7, we get $B_8 \geq \left(\gamma_9^{\frac{1}{8}}\gamma_8^{\frac{1}{7}}\ldots\gamma_3^{\frac{1}{2}}\right)^{-2} B_9{}^{-1} \geq (2.0745451)^{-2}(0.4723)^{-1} > 0.49196$. Since $2B_1 > B_2$ and $2B_8 > B_9$, the inequality $(2,1,2,2,2)$ holds which gives $4B_1 - \dfrac{2B_1^2}{B_2} + B_3 + 2B_5 + 2B_7 + 4B_8 - \dfrac{2B_8^2}{B_9} > 9.2587472$. Left side is a decreasing function of B_1, B_8 and an increasing function of B_2, B_9. Since $B_2 \leq 1.7008$, $B_9 < 0.4723$, $B_1 > 1.85916$ and $B_8 > 0.49196$, this gives $4(1.85916) - \dfrac{2(1.85916)^2}{1.7008} + B_3 + 2B_5 + 2B_7 + 4(0.49196) - \dfrac{2(0.49196)^2}{0.4723} > 9.2587472$. This is not true for $B_3 \leq \dfrac{1.5}{0.46873}B_9$, $B_5 \leq \dfrac{1}{0.46873}B_9$, $B_7 \leq 1.5B_9$ and $B_9 < 0.4723$. This gives a contradiction.

Final contradiction

Here $B_2^4B_6B_7B_8B_9B_1 = \dfrac{B_2^3}{B_3B_4B_5} \geq \dfrac{B_2^3}{2B_9^3/(0.46873)^3} > 2$ since $B_2 > 1.7008$ and $B_9 \leq 0.4723$. Also $2B_6 \geq 2 \times 0.46873B_2 > B_7$ as $B_7 \leq 1.5B_9$. Similarly $2B_8 > B_9$, therefore the inequality $(1,4,2,2)$ holds. This gives $B_1 + 4B_2 - \dfrac{1}{2}B_2^5B_6B_7B_8B_9B_1 + 4B_6 - \dfrac{2B_6^2}{B_7} + 4B_8 - \dfrac{2B_8^2}{B_9} > 9.2587472$. Using AM-GM inequality we get $B_1 + 4B_2 + 4B_6 + 4B_8 - 3 \times 2^{\frac{1}{3}}B_1^{\frac{1}{3}}B_2^{\frac{5}{3}}B_6B_8 > 9.2587472$. Left side is a decreasing function of B_6 and B_8, and $B_6 \geq 0.46873B_2$, $B_8 \geq \dfrac{2}{3} \times 0.46873B_2$ so we can replace B_6 by $0.46873B_2$ and B_8 by $\dfrac{2}{3}(0.46873)B_2$ to get $B_1 + 4(1.78121)B_2 - 2^{\frac{4}{3}}B_1^{\frac{1}{3}}B_2^{\frac{11}{3}}(0.46873)^2 > 9.2587472$. Left side of this inequality is a decreasing function of B_2 and an increasing function of B_1. So we can replace B_2 by 1.7008 and B_1 by 2.1326324 to get $2.1326324 + 4(1.78121)(1.7008) - 2^{\frac{4}{3}}(2.1326324)^{\frac{1}{3}}(1.7008)^{\frac{11}{3}}(0.46873)^2 < 9.2587472$, which gives a contradiction. □

3.1.2 — n = 10

Here we have $\omega_{10} = 10.5605061$, $B_1 \leq \gamma_{10} \leq 2.2636302$ and $B_{10} < \ell'_{10} =$

0.45489 (see Table 2 and Lemma 10).

Claim (i) $B_2 > 1.71684$.

The weak inequality $(2,2,2,2,2)$ gives $2B_2 + 2B_4 + 2B_6 + 2B_8 + 2B_{10} > 10.5605061$. Using Lemmas 1 and 2 we get $2B_2 + 2\left\{\frac{1.5}{0.46873} + \frac{1}{0.46873} + 1.5 + 1\right\}(0.45489) > 10.5605061$. This gives $B_2 > 1.71684$.

Claim (ii) $B_7 < 0.7663$

Suppose $B_7 \geq 0.7663$. Then $B_7^4 B_1 B_2 \ldots B_6 = \dfrac{B_7^3}{B_8 B_9 B_{10}} > \dfrac{(0.7663)^3}{2B_{10}^3} > 2$. Therefore the inequality $(6^*, 4)$ holds. This gives $6(B_1 B_2 \ldots B_6)^{1/6} + 4B_7 - \frac{1}{2} B_7^5 B_1 B_2 \ldots B_6 > 10.5605061$. Left side is a decreasing function of B_7, so we can replace B_7 by 0.7663 to get $g(x) = 6x^{1/6} + 4(0.7663) - \frac{1}{2}(0.7663)^5 x > 10.5605061$ where $x = B_1 B_2 \ldots B_6$. The function $g(x)$ has its maximum value at $x = \dfrac{2^{6/5}}{(0.7663)^6}$, where it is less than 10.5605061. This gives a contradiction.

Claim (iii) $B_6 < 0.9366$

Suppose $B_6 \geq 0.9366$. Then $B_6^5 B_1 B_2 \ldots B_5 = \dfrac{B_6^4}{B_7 B_8 B_9 B_{10}} > \dfrac{(0.9366)^4}{2B_{10}^3 (0.7663)} > \dfrac{16}{3}$. Therefore the inequality $(5^*, 5)$ holds. This gives $5(B_1 B_2 \ldots B_5)^{1/5} + 4B_6 - \frac{3}{16} B_6^6 B_1 B_2 \ldots B_5 > 10.5605061$. Left side is a decreasing function of B_6, so we can replace B_6 by 0.9366 to get $g(x) = 5x^{1/5} + 4(0.9366) - \frac{3}{16}(0.9366)^6 x > 10.5605061$ where $x = B_1 B_2 \ldots B_5$. The function $g(x)$ has its maximum value at $x = \left(\dfrac{16}{3(0.9366)^6}\right)^{(5/4)}$, where it is less than 10.5605061. This gives a contradiction.

Claim (iv) $B_1 > 1.977081$

Suppose $B_1 \leq 1.977081$. Since $B_5 \leq 1.5B_7 < 1.14945$, $B_4 \leq 1.5B_6 <$

1.4049, $B_3 \leq \dfrac{1}{0.46873}B_7 < 1.63485$ and $B_2 > 1.71684$ we find that B_2 is larger than each of $B_3, B_4, \cdots, B_{10}$. Therefore inequality $(1, 9^*)$ holds. This gives $B_1 + 9.2587472(B_1)^{-1/9} > 10.5605061$ which is not true for $B_1 \leq 1.977081$.

Claim (v) $B_2 > 1.892$

Suppose $B_2 \leq 1.892$.

Case (i) $B_3 \geq B_4$.

As $B_3 \geq \dfrac{2}{3}B_1 > 1.318054$, it is larger than each of $B_4, B_5, \cdots, B_{10}$. Also $2B_1 \geq B_2$, therefore inequality $(2, 8^*)$ holds. This gives $g(B_1, B_2) = 4B_1 - \dfrac{2B_1^2}{B_2} + 8(B_1B_2)^{(-1/8)} > 10.5605061$. $g(B_1, B_2)$ is a decreasing function of B_1 and is an increasing function of B_2, therefore $g(B_1, B_2) \leq g(1.977081, 1.892) < 10.5605061$. This gives a contradiction.

Case (ii) $B_3 < B_4$.

As $B_4 > B_3 \geq \dfrac{2}{3}B_1 > 1.318054$, B_4 is larger than each of $B_5, B_6, \cdots, B_{10}$. Also $B_1^2 \geq B_2B_3$, therefore inequality $(3, 7^*)$ holds. This gives $g(B_1, B_2, B_3) = 4B_1 - \dfrac{B_1^3}{B_2B_3} + 7(B_1B_2B_3)^{(-1/7)} > 10.5605061$. Left side is a decreasing function of B_1 and is an increasing function of B_2 and B_3, therefore $g(B_1, B_2, B_3) \leq g(1.977081, 1.892, 1.4049) < 10.5605061$. This gives a contradiction.

Final contradiction

Here $B_2^4B_6B_7B_8B_9B_{10}B_1 = \dfrac{B_2^3}{B_3B_4B_5} \geq \dfrac{B_2^3}{1.63485 \times 1.4049 \times 1.14945} > 2$ for $B_2 > 1.892$. Also $B_6 \geq 0.46873B_2 > 0.8868$. Therefore B_6 is larger than each of $B_7, \cdots, B_{10}$. Hence inequality $(1, 4, 5^*)$ holds. This gives $B_1 + 4B_2 - \dfrac{B_2^4}{2B_3B_4B_5} + 5(B_1B_2B_3B_4B_5)^{(-1/5)} > 10.5605061$. Left side is an increasing function of $B_3B_4B_5$; a decreasing function of B_2 and an increasing function of B_1. One easily checks that this inequality is not true for $B_1 < 2.2636302$; $B_2 > 1.892$ and $B_3B_4B_5 < 1.63485 \times 1.4049 \times 1.14945$. $\square$

3.1.3 — n = 11

Here we have $\omega_{11} = 11.9061976$, $B_1 \leq \gamma_{11} \leq 2.393347$ and $B_{11} < 0.43871 = \ell'_{11}$ (see Table 2 and Lemma 10).

Claim (i) $B_8 < 0.69641$

Suppose $B_8 \geq 0.69641$. Then $B_8^4 B_1 B_2 \dots B_7 = \dfrac{B_8^3}{B_9 B_{10} B_{11}} > \dfrac{(0.69641)^3}{2B_{11}^3} > 2$. Therefore the inequality $(7^*, 4)$ holds. This gives $7(B_1 B_2 \dots B_7)^{1/7} + 4B_8 - \dfrac{1}{2}B_8^5 B_1 B_2 \dots B_7 > 11.9061976$. Left side is a decreasing function of B_8, so we can replace B_8 by 0.69641 to get $g(x) = 7x^{1/7} + 4(0.69641) - \dfrac{1}{2}(0.69641)^5 x > 11.9061976$, where $x = B_1 B_2 \dots B_7$. The function $g(x)$ has its maximum value at $x = \dfrac{2^{7/6}}{(0.69641)^{35/6}}$, where it is less than 11.9061976. This gives a contradiction.

Claim (ii) $B_2 > 1.95859$.

The weak inequality $(2, 2, 2, 2, 1, 2)$ gives $2B_2 + 2B_4 + 2B_6 + 2B_8 + B_9 + 2B_{11} > 11.9061976$. Using Lemmas 1 and 2 we get $2B_2 + 2\{\dfrac{1}{0.46873} + 1.5 + 1\}(0.69641) + \{1.5 + 2\}(0.43871) > 11.9061976$. This gives $B_2 > 1.95859$.

Claim (iii) $B_7 < 0.8525$

Suppose $B_7 \geq 0.8525$. Then $B_7^4 B_{11} B_1 B_2 \dots B_6 = \dfrac{B_7^3}{B_8 B_9 B_{10}} > \dfrac{(0.8525)^3}{2B_{11}^2(0.69641)} > 2$. Therefore the inequality $(6^*, 4, 1)$ holds. This gives $6(B_1 B_2 \dots B_6)^{1/6} + 4B_7 - \dfrac{1}{2}B_7^5 B_{11} B_1 B_2 \dots B_6 + B_{11} > 11.9061976$. Left side is a decreasing function of B_{11}, so we can replace B_{11} by $0.46873B_7$ to get $g(x) = 6x^{1/6} + 4.46873B_7 - \dfrac{1}{2}B_7^6(0.46873)x > 11.9061976$, where $x = B_1 B_2 \dots B_6$. The function $g(x)$ has its maximum value at $x = (\dfrac{2}{0.46873B_7^6})^{6/5}$. Therefore $g(x) \leq 5(\dfrac{2}{(0.46873)B_7^6})^{1/5} + 4.46873B_7 = \psi(B_7)$, say. $\psi(B_7)$ is a decreasing function of B_7 as $B_7 \leq \dfrac{4}{3}B_8 < 1$, therefore $\psi(B_7) \leq \psi(0.8525)$ which is less than 11.9061976. This gives a contradiction.

Claim (iv) $B_1 > 2.1016$

Suppose $B_1 \leq 2.1016$. Since $B_6 \leq 1.5B_8 < 1.04462$; $B_5 \leq 1.5B_7 < 1.27875$, $B_4 \leq \frac{1}{0.46873}B_8 < 1.486$, $B_3 \leq \frac{1}{0.46873}B_7 < 1.8188$ and $B_2 > 1.9585$ we find that B_2 is larger than each of $B_3, B_4, \cdots, B_{11}$. Therefore the inequality $(1, 10^*)$ holds. This gives $B_1 + 10.5605061(B_1)^{-1/10} > 11.9061976$ which is not true for $B_1 \leq 2.1016$.

Claim (v) $B_2 > 2.013$

Suppose $B_2 \leq 2.013$.

Case (i) $B_3 \geq 1.486$ $(> B_4)$.

As $B_5 \leq \frac{3}{2}B_7 < 1.27875$, B_3 is larger than each of $B_4, B_5, \cdots, B_{11}$. Therefore the inequality $(2, 9^*)$ holds. This gives $g(B_1, B_2) = 4B_1 - \frac{2B_1^2}{B_2} + 9.2587472$ $(B_1B_2)^{(-1/9)} > 11.9061976$. $g(B_1, B_2)$ is a decreasing function of B_1 and is an increasing function of B_2, therefore $g(B_1, B_2) \leq g(2.1016, 2.013) < 11.9061976$. This gives a contradiction.

Case (ii) $B_3 < 1.486$.

The weak inequality $(1, 2, 2, 2, 2, 2)$ gives $B_1 + 2B_3 + 2B_5 + 2B_7 + 2B_9 + 2B_{11} \geq 11.9061976$. Using Lemma 1 and Claim (iii) we see that the left side is $< 2.393347 + 2(1.486) + 2\{1.5 + 1\}B_7 + 2\{1.5 + 1\}B_{11} < 11.9061976$. This gives a contradiction.

Final contradiction

Here $B_2^4B_6 \ldots B_{11}B_1 = \frac{B_2^3}{B_3B_4B_5} \geq \frac{B_2^3}{1.8188 \times 1.486 \times 1.27875} > 2$ for $B_2 > 2.013$. Also $B_6 \geq 0.46873B_2 > 0.94355$. Therefore B_6 is larger than each of $B_7, \cdots, B_{11}$. Hence inequality $(1, 4, 6^*)$ holds. This gives $B_1 + 4B_2 - \frac{B_2^4}{2B_3B_4B_5} + 6(B_1B_2B_3B_4B_5)^{(-1/6)} > 11.9061976$. Left side is an increasing function of $B_3B_4B_5$; a decreasing function of B_2 and an increasing function of B_1. One easily checks that the inequality is not true for $B_1 < 2.393347$; $B_2 > 2.013$ and $B_3B_4B_5 < 1.8188 \times 1.486 \times 1.27875$. □

3.1.4 — n = 12

Here we have $\omega_{12} = 13.4499927$, $B_1 \leq \gamma_{12} \leq 2.5217871$ and $B_{12} < 0.349378 =$

ℓ'_{12} (see Table 2 and Lemma 10). From Lemma 7 we have $B_1B_2\ldots B_8 < \left(\gamma_{12}^{\frac{1}{11}}\gamma_{11}^{\frac{1}{10}}\ldots\gamma_5^{\frac{1}{4}}\right)^4$ which, by Table 2, is less than $(2.14004)^4$. Now $8(B_1B_2\ldots B_8)^{1/8}+2B_{10}+2B_{12} < 13.4499927$ as $B_{10}\le \dfrac{3}{2}B_{12}$. This gives a contradiction to the weak inequality $(8^*,2,2)$. □

3.2 *Proof of Theorem 1 for* $13\le n\le 31$

As mentioned earlier we assume that Theorem 1 is false and derive a contradiction. For fixed n and $s,\ 1\le s\le n-1$, define

$$\phi_{s,n-s}(x)=\omega_s x^{\frac{1}{s}}+\omega_{n-s}(1/x)^{\frac{1}{n-s}}.$$

Let $\lambda_s^{(n)}$ be the larger of the lower bounds of $B_1B_2\ldots B_s$ given in Lemmas 8 and 9. Recall $\mu_s^{(n)}$ is the upper bound of $B_1B_2\ldots B_s$ given in Lemma 7.

Lemma 11 — If for some $s,\ 1\le s\le n-1,\ \phi_{s,n-s}(\lambda_s^{(n)})\le\omega_n$ and $\phi_{s,n-s}(\mu_s^{(n)})\le\omega_n$, then we must have $B_{s+1}<max\{B_{s+2},\cdots,B_n\}$.

PROOF : Suppose $B_{s+1}\ge\max\{B_{s+2},\cdots,B_n\}$, then the inequality $(s^*,(n-s)^*)$ holds which gives inequality (8) i.e.

$$\phi_{s,n-s}(B_1B_2\ldots B_s)=\omega_s(B_1B_2\ldots B_s)^{\frac{1}{s}}+\omega_{n-s}\left(\frac{1}{B_1B_2\ldots B_s}\right)^{\frac{1}{n-s}}>\omega_n.$$

It is easy to see that the function $\phi_{s,n-s}(x)$ has maximum at one of the end points of the interval in which x lies. For $x=B_1B_2\ldots B_s$ and $\lambda_s^{(n)}\le B_1B_2\ldots B_s\le\mu_s^{(n)}$, the above inequality contradicts the hypothesis.

Remark 5 : We find that $\max\left\{\phi_{s,n-s}(\lambda_s^{(n)}),\ \phi_{s,n-s}(\mu_s^{(n)})\right\}$ is $\phi_{s,n-s}(\mu_s^{(n)})$ in all the cases. So for simplicity of verification of $\phi_{s,n-s}(\lambda_s^{(n)})\le\omega_n$, we sometimes choose a convenient number $a,\ 0<a\le\lambda_s^{(n)}$ for which $\phi_{s,n-s}(a)\le\omega_n$. In particular $\phi_{s,n-s}(1)=\omega_s+\omega_{n-s}<\omega_n$ always.

Lemma 12 — Suppose that for some $s,\ 1\le s\le n-1,\ \phi_{s,n-s}(\lambda_s^{(n)})\le\omega_n$ but $\phi_{s,n-s}(\mu_s^{(n)})>\omega_n$. Let a real number $\sigma_s^{(n)}$ be such that $\lambda_s^{(n)}<\sigma_s^{(n)}<\mu_s^{(n)}$ and $\phi_{s,n-s}(\sigma_s^{(n)})\le\omega_n$.

(i) If $B_1B_2\dots B_s < \sigma_s^{(n)}$, then $B_{s+1} < max\{B_{s+2},\cdots,B_n\}$,

(ii) If $B_1B_2\dots B_s \geq \sigma_s^{(n)}$, then $B_{s+1} \leq \dfrac{\mu_{s+1}^{(n)}}{\sigma_s^{(n)}}$.

PROOF : In Case (i), Lemma 11 gives the result. In Case (ii), since $B_{s+1} = \dfrac{B_1\dots B_{s+1}}{B_1\dots B_s}$ we use Lemma 7 to get the desired result.

Remark 6 : We will use Lemma 12 for $n = 17, 21, 22, \cdots, 31$. Those s for which we apply Lemma 12 are indicated by putting a † on s in Table 2.

Remark 7 : For the proof of Theorem 1 in the cases $13 \leq n \leq 31$, we apply Lemmas 10, 11 and 12 successively (Lemma 12 is used in cases mentioned in Remark 6) and get a final contradiction by applying weak inequality $(2, 2, \cdots, 2)$ for even n and $(1, 2, 2, \cdots, 2)$ or $(2, 1, 2, \cdots, 2)$ for odd n. Using upper bounds of B_s for different s obtained in Lemmas 10, 11, 12 and using Lemmas 1 and 2, we find that $2B_2 + 2B_4 + \cdots + 2B_n$ or $B_1 + 2B_3 + \cdots + 2B_n$ or $2B_2 + B_3 + 2B_5 + \cdots + 2B_n$ is less than ω_n, giving thereby a contradiction.

We illustrate the method for $n = 13, 14, 17, 21, 31$. Others are similar and can be verified with the help of the inequalities listed in Table 2.

3.2.1 — n = 13

Here $\omega_{13} = 15.0562267$ and $m_{13} = 2.3485931$. By Lemma 10 we can take $B_{13} < 0.3503 = \ell'_{13}$. By Lemmas 7 and 8 we have

$$1 \leq \lambda_1^{(13)} \leq B_1 \leq \mu_1^{(13)} = 2.6492947,$$

$$0.75 \leq \lambda_2^{(13)} \leq B_1B_2 \leq \mu_2^{(13)} = (1.1797224)^{11}$$

.

Further $\max\left\{\phi_{1,12}(\lambda_1^{(13)}),\ \phi_{1,12}(\mu_1^{(13)}),\ \phi_{2,11}(\lambda_2^{(13)}),\ \phi_{2,11}(\mu_2^{(13)})\right\} = \phi_{2,11}(\mu_2^{(13)})$ which is $\leq 15.0562267 = \omega_{13}$. Therefore by Lemma 11 we can take $B_3 < \max\{B_4, B_5, \cdots, B_{13}\}$ which is $\leq \dfrac{4/3}{(0.46873)^2}B_{13}$ (by Lemmas 1 and 2) and $B_2 < \max\{B_3, B_4, \cdots, B_{13}\}$ which in turn is also $\leq \dfrac{4/3}{(0.46873)^2}B_{13}$. Now

$$2B_2 + B_3 + 2B_5 + 2B_7 + 2B_9 + 2B_{11} + 2B_{13}$$
$$\leq 2\left\{\frac{4/3}{(0.46873)^2} + \frac{2/3}{(0.46873)^2} + \frac{1}{(0.46873)^2} + \frac{1.5}{0.46873} + \frac{1}{0.46873} + \frac{3}{2} + 1\right\} B_{13}$$
$$= 2\{\frac{3}{(0.46873)^2} + \frac{2.5}{0.46873} + 2.5\}B_{13} < 15.0562267 \text{ for } B_{13} < 0.3503.$$

This is a contradiction to the weak inequality $(2, 1, 2, 2, 2, 2, 2)$. □

3.2.2 — n = 14

Here $\omega_{14} = 16.6646332$ and $m_{14} = 2.4711931$. By Lemma 10 we can take $B_{14} < \ell'_{14} = 0.36$. By Lemmas 7, 8, and 9 we have

$$0.5 \leq \lambda_3^{(14)} \leq B_1 B_2 B_3 \leq \mu_3^{(14)} = (1.2761057)^{11},$$

$$1 < \frac{3(0.46873)^2}{16B_{14}^6} \leq \lambda_8^{(14)} \leq B_1 B_2 \ldots B_8 \leq \mu_8^{(14)} = (2.0433047)^6.$$

Further $\max\left\{\phi_{3,11}(\lambda_3^{(14)}),\ \phi_{3,11}(\mu_3^{(14)}),\ \phi_{8,6}(\lambda_8^{(14)}),\ \phi_{8,6}(\mu_8^{(14)})\right\} = \phi_{3,11}(\mu_3^{(14)})$ which is $\leq 16.6646332 = \omega_{14}$. Therefore by Lemma 11 we can take $B_9 < \max\{B_{10}, B_{11}, \cdots, B_{14}\}$ which is $\leq \frac{1}{0.46873}B_{14}$ (by Lemmas 1 and 2) and $B_4 < \max\{B_5, B_6, \cdots, B_{14}\}$ which is $\leq \frac{1}{(0.46873)^2}B_{14}$, as $B_5 \leq \frac{1}{0.46873}B_9,\ B_6 \leq 2B_9$, $B_7 \leq \frac{3}{2}B_9$ and $B_8 \leq \frac{4}{3}B_9$. Now

$$2B_2 + 2B_4 + 2B_6 + 2B_8 + 2B_{10} + 2B_{12} + 2B_{14}$$
$$\leq 2\left\{\frac{1.5}{(0.46873)^2} + \frac{1}{(0.46873)^2} + \frac{2}{0.46873} + \frac{4/3}{0.46873} + \frac{1}{0.46873} + \frac{3}{2} + 1\right\} B_{14}$$
$$= 2\{\frac{2.5}{(0.46873)^2} + \frac{13/3}{0.46873} + 2.5\}B_{14} < 16.6646332 \text{ for } B_{14} < 0.36.$$

This is a contradiction to the weak inequality $(2, 2, 2, 2, 2, 2, 2)$. □

3.2.3 — n = 17

Here $\omega_{17} = 21.6026907$ and $m_{17} = 2.8355395$. By Lemma 10, we can take $B_{17} < \ell'_{17} = 0.3567496$. By Lemmas 7, 8 and 9 we have

$$0.04 \le \lambda_6^{(17)} \le B_1B_2\ldots B_6 \le \mu_6^{(17)} = (1.5927325)^{11},$$

$$1 < \frac{3(0.46873)^8}{64B_{17}^{10}} \le \lambda_7^{(17)} \le B_1B_2\ldots B_7 \le \mu_7^{(17)} = (1.7379745)^{10},$$

$$1 < \frac{(0.46873)^6}{16B_{17}^{9}} \le \lambda_8^{(17)} \le B_1B_2\ldots B_8 \le \mu_8^{(17)} = (1.9031203)^{9},$$

$$1 < \frac{(0.46873)^4}{16B_{17}^{8}} \le \lambda_9^{(17)} \le B_1B_2\ldots B_9 \le \mu_9^{(17)} = (2.0920918)^{8}.$$

We find that

$$\max\left\{\phi_{s,17-s}(\lambda_s^{(17)}),\ \phi_{s,17-s}(\mu_s^{(17)}),\ \text{for } s = 6,7,8\right\} = \phi_{6,11}(\mu_6^{(17)}) < \omega_{17}.$$

Therefore by Lemma 11 we can take $B_9 < \max\{B_{10},\cdots,B_{17}\}$, $B_8 < \max\{B_9,\cdots,B_{17}\}$ and $B_7 < \max\{B_8,\cdots,B_{17}\} = \max\{B_{10},\cdots,B_{17}\}$. Since by Lemmas 1 and 2, $B_{11} \le \dfrac{1.5}{0.46873}B_{17} \le \dfrac{1.5}{0.46873}\ell'_{17} = u_{17}\ell'_{17}$ (say) and $\max\{B_{11},\cdots,B_{17}\} \le \dfrac{1.5}{0.46873}\ell'_{17}$ therefore each of B_7 and B_9 is $\le \max\{B_{10}, u_{17}\ell'_{17}\}$.

We notice that $\phi_{9,8}(\lambda_9^{(17)}) < \omega_{17}$ but $\phi_{9,8}(\mu_9^{(17)}) > \omega_{17}$. So we use Lemma 12 with $\sigma_9^{(17)} = (2.08)^8$. Here $\phi_{9,8}(\sigma_9^{(17)}) < \omega_{17}$.

In Case (i) i.e. for $B_1B_2\ldots B_9 \le (2.08)^8$ we have $B_{10} < \max\{B_{11},\cdots,B_{17}\}$ which in turn is $\le u_{17}\ell'_{17}$.

In Case (ii) i.e. for $B_1B_2\ldots B_9 > 2.08^8$ we have $B_{10} < \dfrac{\mu_{10}^{(17)}}{\sigma_9^{(17)}} = \dfrac{(2.3098565)^7}{(2.08)^8}$ < 1.001356 which is easily seen to be $< u_{17}\ell'_{17}$.

Now in both the cases

$$B_1 + 2B_3 + 2B_5 + 2B_7 + 2B_9 + 2B_{11} + 2B_{13} + 2B_{15} + 2B_{17}$$
$$\leq 3.1506793 + 2\left\{\frac{1}{0.46873}u_{17} + \frac{3}{2}u_{17} + u_{17} + u_{17} + u_{17}\right\}\ell'_{17}$$
$$+ 2\left\{\frac{1}{0.46873} + \frac{3}{2} + 1\right\}B_{17}$$
$$\leq 3.1506793 + 2\left\{\frac{1.5}{(0.46873)^2} + \frac{31/4}{0.46873} + \frac{3}{2} + 1\right\}\ell'_{17} < 21.6026907 = \omega_{17}$$

as $\ell'_{17} = 0.3567496$ and $B_1 \leq \gamma_{17} \leq 3.1506793$ (see Table 2). This is a contradiction to the weak inequality $(1, 2, 2, \cdots, 2)$. □

3.2.4 — n = 21

Here $\omega_{21} = 29.16383254$ and $m_{21} = 3.3153098$. By Lemma 10 we can take $B_{21} < \ell'_{21} = 0.3182$. By Lemmas 7, 8 and 9 we have

$$1 \leq \lambda_1^{(21)} \leq B_1 \leq \gamma_{21} \leq 3.6422432 = \mu_1^{(21)},$$

$$0.75 \leq \lambda_2^{(21)} \leq B_1B_2 \leq \mu_2^{(21)} = (1.1398147)^{19},$$

$$0.5 \leq \lambda_3^{(21)} \leq B_1B_2B_3 \leq \mu_3^{(21)} = (1.219952)^{18},$$

$$0.25 \leq \lambda_4^{(21)} \leq B_1B_2 \ldots B_4 \leq \mu_4^{(21)} = (1.3081094)^{17},$$

$$\frac{(0.46873)}{4} \leq \lambda_5^{(21)} \leq B_1B_2 \ldots B_5 \leq \mu_5^{(21)} = (1.4053818)^{16},$$

$$\frac{3(0.46873)^2}{16} \leq \lambda_6^{(21)} \leq B_1B_2 \ldots B_6 \leq \mu_6^{(21)} = (1.5130595)^{15},$$

$$0.1281 < \frac{3(0.46873)^{18}}{256B_{21}^{14}} \leq \lambda_7^{(21)} \leq B_1B_2 \ldots B_7 \leq \mu_7^{(21)} = (1.6326769)^{14},$$

$$0.5279 < \frac{(0.46873)^{15}}{64B_{21}^{13}} \leq \lambda_8^{(21)} \leq B_1B_2 \ldots B_8 \leq \mu_8^{(21)} = (1.7660666)^{13},$$

$$1 < \frac{(0.46873)^{12}}{64B_{21}^{12}} \leq \lambda_9^{(21)} \leq B_1B_2 \ldots B_9 \leq \mu_9^{(21)} = (1.9154372)^{12}.$$

$$1 < \frac{(0.46873)^{10}}{32B_{21}^{11}} \leq \lambda_{10}^{(21)} \leq B_1B_2 \ldots B_{10} \leq \mu_{10}^{(21)} = (2.0834682)^{11}.$$

It is easy to see that

$$\max\left\{\phi_{s,21-s}(\lambda_s^{(21)}),\ \phi_{s,21-s}(\mu_s^{(21)}),\ \text{for } 1 \le s \le 9\right\} = \phi_{1,20}(\mu_s^{(21)}) < \omega_{21}.$$

Therefore by Lemma 11 we can take $B_i < \max\{B_{i+1}, \cdots, B_{21}\}$ for $i = 2, 3, \cdots, 10$. This implies that each of $B_2, \cdots, B_{10}$ is less than $\max\{B_{11}, \cdots, B_{21}\}$. Again by Lemmas 1 and 2 we get $\max\{B_{12}, \cdots, B_{21}\} \le \frac{4/3}{(0.46873)^2} B_{21} < \frac{4/3}{(0.46873)^2} \ell'_{21} = u_{21}\ell'_{21}$ (say); so each of $B_2, B_3, \cdots, B_{10}$ is $\le \max\{B_{11}, u_{21}\ell'_{21}\}$. We notice that $\phi_{10,11}(\lambda_{10}^{(21)}) < \omega_{21}$ but $\phi_{10,11}(\mu_{10}^{(21)}) > \omega_{21}$. So we apply Lemma 12 with $\sigma_{10}^{(21)} = (2.058)^{11}$. Here $\phi_{10,11}(\sigma_{10}^{(21)}) < \omega_{21}$.

In Case (i) i.e. for $B_1 B_2 \ldots B_{10} \le (2.058)^{11}$ we have $B_{11} < \max\{B_{12}, \cdots, B_{21}\}$ which in turn is $\le u_{21}\ell'_{21}$.

In Case (ii) i.e. for $B_1 B_2 \ldots B_{10} > (2.058)^{11}$ we have $B_{11} < \frac{\mu_{11}^{(21)}}{\sigma_{10}^{(21)}} = \frac{(2.2734606)^{10}}{(2.058)^{11}} < 1.31515$ which is easily seen to be $< u_{21}\ell'_{21}$.

Now in both the cases

$$\begin{aligned}
&2B_2 + B_3 + 2B_5 + 2B_7 + 2B_9 + 2B_{11} + 2B_{13} + 2B_{15} + 2B_{17} + 2B_{19} + 2B_{21} \\
&\le 2\left\{u_{21} + \frac{1}{2}u_{21} + u_{21} + u_{21} + u_{21} + u_{21}\right\}\ell'_{21} \\
&\qquad + 2\left\{\frac{1}{(0.46873)^2} + \frac{1.5}{0.46873} + \frac{1}{0.46873} + \frac{3}{2} + 1\right\} B_{21} \\
&< 2\left\{\frac{25/3}{(0.46873)^2} + \frac{2.5}{0.46873} + 2.5\right\}\ell'_{21} < 29.1638254 = \omega_{21} \quad \text{as} \quad \ell'_{21} = 0.3182.
\end{aligned}$$

This is a contradiction to the weak inequality $(2, 1, 2, \cdots, 2)$. □

3.2.5 — n = 31

Here $\omega_{31} = 56.0735184$ and $m_{31} = 4.4974263$. By Lemma 10 we can take $B_{31} < \ell'_{31} = 0.1835723$. By Lemmas 7, 8 and 9 we have

$$1 \le \lambda_1^{(31)} \le B_1 \le \gamma_{31} \le 4.8483483 = \mu_1^{(31)},$$

$$0.75 \le \lambda_2^{(31)} \le B_1 B_2 \le \mu_2^{(31)} = (1.1120381)^{29},$$

$$0.5 \le \lambda_3^{(31)} \le B_1B_2B_3 \le \mu_3^{(31)} = (1.1744075)^{28},$$
$$0.25 \le \lambda_4^{(31)} \le B_1B_2\dots B_4 \le \mu_4^{(31)} = (1.2415699)^{27},$$
$$\frac{(0.46873)}{4} \le \lambda_5^{(31)} \le B_1B_2\dots B_5 \le \mu_5^{(31)} = (1.3140112)^{26},$$
$$\frac{3(0.46873)^2}{16} \le \lambda_6^{(31)} \le B_1B_2\dots B_6 \le \mu_6^{(31)} = (1.3922802)^{25},$$
$$\frac{(0.46873)^3}{8} \le \lambda_7^{(31)} \le B_1B_2\dots B_7 \le \mu_7^{(31)} = (1.4769984)^{24},$$
$$\frac{(0.46873)^4}{16} \le \lambda_8^{(31)} \le B_1B_2\dots B_8 \le \mu_8^{(31)} = (1.5687601)^{23},$$
$$\frac{(0.46873)^6}{16} \le \lambda_9^{(31)} \le B_1B_2\dots B_9 \le \mu_9^{(31)} = (1.6685893)^{22},$$
$$0.0043 < \frac{(0.46873)^{45}}{1024B_{31}^{21}} \le \lambda_{10}^{(31)} \le B_1B_2\dots B_{10} \le \mu_{10}^{(31)} = (1.7773033)^{21},$$
$$0.0355 < \frac{(0.46873)^{40}}{1024B_{31}^{20}} \le \lambda_{11}^{(31)} \le B_1B_2\dots B_{11} \le \mu_{11}^{(31)} = (1.8959636)^{20},$$
$$0.2701 < \frac{(0.46873)^{36}}{512B_{31}^{19}} \le \lambda_{12}^{(31)} \le B_1B_2\dots B_{12} \le \mu_{12}^{(31)} = (2.0257964)^{19},$$
$$1 < \frac{3(0.46873)^{32}}{1024B_{31}^{18}} \le \lambda_{13}^{(31)} \le B_1B_2\dots B_{13} \le \mu_{13}^{(31)} = (2.1682245)^{18},$$
$$1 < \frac{(0.46873)^{28}}{256B_{31}^{17}} \le \lambda_{14}^{(31)} \le B_1B_2\dots B_{14} \le \mu_{14}^{(31)} = (2.324907)^{17},$$
$$1 < \frac{(0.46873)^{24}}{256B_{31}^{16}} \le \lambda_{15}^{(31)} \le B_1B_2\dots B_{15} \le \mu_{15}^{(31)} = (2.4977895)^{16}.$$

It is easy to see that

$$\max\left\{\phi_{s,31-s}(\lambda_s^{(31)}),\ \phi_{s,31-s}(\mu_s^{(31)}),\ \text{for } 1 \le s \le 14\right\} = \phi_{14,17}(\mu_{14}^{(31)}) < \omega_{31}.$$

Therefore by Lemma 11 we can take $B_i < \max\{B_{i+1}, \cdots, B_{31}\}$ for $i = 2, 3, \cdots, 15$. This implies that each of $B_2, \cdots, B_{15}$ is less than $\max\{B_{16}, \cdots, B_{31}\}$. Again by Lemmas 1 and 2 we have $B_{17} \le \frac{1.5}{(0.46873)^3}B_{31} < \frac{1.5}{(0.46873)^3}\ell'_{31} = u_{31}\ell'_{31}$ (say) and $\max\{B_{17}, \cdots, B_{31}\} \le u_{31}\ell'_{31}$; so each of $B_2, B_3, \cdots, B_{15}$ is $\le \max\{B_{16}, u_{31}\ell'_{31}\}$.

We notice that $\phi_{15,16}(\lambda_{15}^{(31)}) < \omega_{31}$ but $\phi_{15,16}(\mu_{15}^{(31)}) > \omega_{31}$, so we apply Lemma 12 with $\sigma_{15}^{(31)} = (2.47)^{16}$. Here $\phi_{15,16}(\sigma_{15}^{(31)}) < \omega_{31}$.

In Case (i) i.e. for $B_1B_2\ldots B_{15} \leq (2.47)^{16}$ we have $B_{16} < \max\{B_{17}, \cdots, B_{31}\}$ which in turn is $\leq u_{31}\ell'_{31}$.

In Case (ii) i.e. for $B_1B_2\ldots B_{15} > (2.47)^{16}$ we have $B_{16} < \dfrac{\mu_{16}^{(31)}}{\sigma_{15}^{(31)}} < \dfrac{(2.6891655)^{15}}{(2.47)^{16}} < 1.4491446$ which is easily seen to be $< u_{31}\ell'_{31}$.

Now in both the cases

$$
\begin{aligned}
&2B_2 + B_3 + 2B_5 + \cdots + 2B_{15} + 2B_{17} + 2B_{19} + \cdots + 2B_{31} \\
&\leq 2\{u_{31} + \frac{1}{2}u_{31} + u_{31} + u_{31} + u_{31} + u_{31} + u_{31} + u_{31} + u_{31}\}\ell'_{31} \\
&+2\left\{\frac{1}{(0.46873)^3} + \frac{1.5}{(0.46873)^2} + \frac{1}{(0.46873)^2} + \frac{1.5}{0.46873} + \frac{1}{0.46873} + \frac{3}{2} + 1\right\} B_{31} \\
&< 2\left\{\frac{13.75}{(0.46873)^3} + \frac{2.5}{(0.46873)^2} + \frac{2.5}{0.46873} + 2.5\right\}\ell'_{31} < 56.0735184 = \omega_{31}
\end{aligned}
$$

as $\ell'_{31} = 0.1835723$.

This is a contradiction to the weak inequality $(2, 1, 2, \cdots, 2)$. □

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REFERENCES

1. R. P. Bambah, V. C. Dumir and R. J. Hans-Gill, Non-homogeneous problems: Conjectures of Minkowski and Watson, *Number Theory, Trends in Mathematics*, Birkhauser Verlag, Basel (2000), 15-41.

2. B. J. Birch and H. P. F. Swinnerton Dyer, On the inhomogeneous minimum of the product of n linear forms, *Mathematika*, **3** (1956), 25-39.

3. H. F. Blichfeldt, The minimum values of positive quadratic forms in six, seven and eight variables, *Math. Z.*, **39** (1934), 1-15.

4. N. Čebotarev, Beweis des Minkowski'schen Satzes über lineare inhomogene Formen, Vierteljschr. *Naturforsch. Ges. Zurich*, **85** Beiblatt, (1940), 27-30.

5. H. Cohn and N. Elkies, New upper bounds on sphere packings. *I. Ann. of Math.*, **157**(2) (2003), 689-714.

6. H. Cohn and A. Kumar, The densest lattice in twenty-four dimensions. *Electron. Res. Announc. Amer. Math. Soc.*, **10** (2004), 58-67.

7. J. H. Conway and N. J. A. Sloane, *Sphere packings, Lattices and groups*, Springer-verlag, 2nd edition, New York, (1993).

8. R. J. Hans-Gill, Madhu Raka , Ranjeet Sehmi and Sucheta, A unified simple proof of Woods' conjecture for $n \leq 6$, *J. Number Theory*, **129** (2009), 1000-1010.

9. R. J. Hans-Gill, Madhu Raka and Ranjeet Sehmi, On Conjectures of Minkowski and Woods for $n = 7$, *J. Number Theory*, **129** (2009), 1011-1033.

10. R. J. Hans-Gill, Madhu Raka and Ranjeet Sehmi, On Conjectures of Minkowski and Woods for $n = 8$, *Acta Arithmetica*, **147**(4) (2011), 337-385.

11. R. J. Hans-Gill, Madhu Raka and Ranjeet Sehmi, Estimates On Conjectures of Minkowski and Woods, *Indian J. Pure Appl. Math.*, **41**(4) (2010), 595-606.

12. P. Gruber, *Convex and discrete geometry*, Springer Grundlehren Series, Vol. **336** (2007).

13. P. Gruber and C. G. Lekkerkerker, *Geometry of Numbers*, Second Edition, North Holland (1987).

14. I. V. Il'in, A remark on an estimate in the inhomogeneous Minkowski conjecture for small dimensions. (Russian), *Petrozavodsk. Gos. Univ., Petrozavodsk*, **90** (1986), 24-30.

15. I. V. Il'in, Chebotarev estimates in the inhomogeneous Minkowski conjecture for small dimensions. (Russian) Algebraic systems (Russian), *Ivanov. Gos. Univ., Ivanovo*, (1991) 115-125.

16. A. Korkine and G. Zolotareff, Sur les formes quadratiques, *Math. Ann.*, **6** (1873), 366-389 and Sur les formes quadratiques positives, *ibid.*, **11** (1877), 242-292.

17. C. T. McMullen, Minkowski's Conjecture, Well rounded lattices and topological dimension, *J. Amer. Math. Soc.*, **18** (2005), 711-734.

18. L. J. Mordell, Tschebotareff's Theorem on the product of Non-homogeneous Linear Forms (II) *J. London Math. Soc.*, **35** (1960), 91-97.

19. K. Mukhsinov, On the inhomogeneous Minkowski conjecture for small dimensions, *Zap. Naucn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, **106** (1981), 104-133.

20. K. Mukhsinov, Estimates in the inhomogeneous Minkowski conjecture, *Zap. Naucn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, **121** (1983), 195-196.

21. R. A. Pendavingh and S. H. M. Van Zwam, New Korkine-Zolotarev inequalities, *SIAM J. Optim*., **18**(1) (2007), 364-378.

22. A. C. Woods, The densest double lattice packing of four spheres, *Mathematika*, **12** (1965), 138-142.

23. A. C. Woods, Lattice coverings of five space by spheres, *Mathematika* **12** (1965), 143-150.

24. A. C. Woods, Covering six space with spheres, *J. Number Theory*, **4** (1972), 157-180.