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A SIMULATION-BASED APPROACH TO THE STUDY OF COEFFICIENT OF VARIATION OF GOMPERTZ DISTRIBUTION UNDER PROGRESSIVE FIRST-FAILURE CENSORING

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In applied statistics, the coefficient of variation is widely used. However, inference concerning the coefficient of variation of non-normal distributions are rarely reported. In this article, a simulation-based Bayesian approach is adopted to estimate the coefficient of variation (CV) under progressive first-failure censored data from Gompertz distribution. The sampling schemes such as, first-failure censoring, progressive type II censoring, type II censoring and complete sample can be obtained as special cases of the progressive first-failure censored scheme. The simulation-based approach will give us a point estimate as well as the empirical sampling distribution of CV. The joint prior density as a product of conditional gamma density and inverted gamma density for the unknown Gompertz parameters are considered. In addition, the results of maximum likelihood and parametric bootstrap techniques are also proposed. An analysis of a real life data set is presented for illustrative purposes. Results from simulation studies assessing the performance of our proposed method are included.

Key words : Coefficient of variation; Gompertz distribution; Progressive first-failure censored scheme; Bayesian and non-Bayesian approaches ; Gibbs and Metropolis sampler; Bootstrap method.

1. INTRODUCTION

The coefficient of variation (CV) of a population is defined as the ratio of the population standard deviation to the population mean. It is regarded as a measure of stability or uncertainty, and can indicate the relative dispersion of data in the population to the population mean. The CV measures the variability of a series of numbers independently of the unit of measurement used for these numbers. In order to do so, the CV eliminates the unit of measurement of the standard deviation of a series of numbers by dividing it by the mean of these numbers. The CV can be used to compare distributions obtained with different units, such as, for example, the variability of the weights of newborns (measured in grams) with the size of adults (measured in centimeters). This approach has been used by several authors to obtain the CV estimates (for details, see Pang *et al*. [24] and Pang *et al*. [25]).

The CV has long been widely used as a descriptive and inferential quantity in several fields such as chemistry, engineering, finance, medical sciences, physics, and telecommunications. In chemical experiments, it is often used as a yardstick of precision of measurements; two measurement methods may be compared on the basis of their respective CV. In finance, the CV can be used as a measure of relative risks (Miller and Karson [23]). In clinical and diagnostic areas, the CV is also often used as a yardstick of the precision of measurements (Reh and Scheffler [27]). In physiological science, the CV can be applied to assess the homogeneity of bone samples (Hamer *et al*. [16]). It has been used as a tool in uncertainty analysis of fault trees (Ahn [1]) and in assessing the strength of ceramics (Gong and Li [15]). Many statistical procedures concerning CV are based on the normal distribution. However, several phenomena do not agree with the normality assumption due to asymmetry or to the presence of heavy-andlight tails in the distribution of the data. Thus, the statistical inference under normal populations cannot be adequate in the mentioned cases.

The Gompertz distribution occupies an important position in modelling human mortality and fitting actuarial tables. Historically, the Gompertz distribution was introduced by Gompertz [14] many authors have contributed to the statistical methodology and characterization of this distribution, for example Read [26] and Frances [8]. Garg *et al*. [10] studied the properties of the Gompertz distribution and obtained the maximum likelihood estimates for the parameters. Chen [4] developed an exact confidence interval and exact joint confidence region for the parameters of Gompertz distribution under type II censored. Wu *et al*. [29] developed an exact confidence interval and exact joint confidence region for the parameters of the Gompertz distribution under the first-failure censored sampling plans. Applications and more survey for the Gompertz model are given by Al-Hussaini *et al*. [2].

The probability density function (*pdf*) and cumulative distribution function (*cdf*) for the Gompertz distribution are given, respectively, by

$$f(x;\lambda,\beta)=\beta\exp\left(\lambda x-\frac{\beta}{\lambda}(\exp(\lambda x)-1)t\right),\ x>0,\quad \lambda,\beta>0, \tag{1}$$

$$F(x;\lambda,\beta)=\mathbf{1}-\exp\left(-\frac{\beta}{\lambda}(\exp(\lambda\mathbf{x})-\mathbf{1})\right). \tag{2}$$

For $f(x)$ to be proper density function, both the parameters λ and β must be positive. If $0<\lambda\leq\beta$, then $(df(x)/dx)<0$, for all $x\in(0,\infty)$. So, the density function (1) is monotone decreasing on $(0,\infty)$ in which case the mode at zero. If $\lambda>\beta$, $f(x)$ increases in $(0,x_{\text{mod}})$ and then decreases in (x_{mod},∞). In this case, $x_{\text{mod}}=\left(\frac{1}{\lambda}\right)\log\left(\frac{\lambda}{\beta}\right)$. It is worth noting that when $\lambda\to 0$, Gompertz distribution will tend to an exponential distribution.

The Gompertz distribution has the following properties

(1) The expected value of X

$$E(X)=\frac{1}{\lambda}\exp\left(\frac{\beta}{\lambda}\right)\int_{\frac{\beta}{\lambda}}^{\infty}(\exp(-t)/t)dt=\frac{1}{\lambda}\exp\left(\frac{\beta}{\lambda}\right)\Gamma\left(0,\frac{\beta}{\lambda}\right), \tag{3}$$

where $\Gamma(\varpi_1,\varpi_2)$ is the incomplete gamma function define by $\Gamma(\varpi_1,\varpi_2)=\int_{\varpi_2}^{\infty}t^{\varpi_1-1}\exp(-t)dt$.

(2) The expected value of X^2

$$\begin{aligned}E(X^2) &= \frac{1}{\lambda^2}\exp\left(\frac{\beta}{\lambda}\right)\int_{\frac{\beta}{\lambda}}^{\infty}\left(\log(\frac{\lambda}{\beta}t)\right)^2\exp(-t)dt\\ &= \frac{1}{\lambda^2}\exp\left(\frac{\beta}{\lambda}\right)\left\{\gamma^2+\frac{\pi}{6}-\frac{2\beta}{\lambda}\ {}_3F_3\left[\{1,1,1\},\{2,2,2\},-\frac{\beta}{\lambda}\right]\right.\\ &\quad\left.-2\gamma\log\left(\frac{\lambda}{\beta}\right)+\left(\log\left(\frac{\lambda}{\beta}\right)\right)^2\right\},\end{aligned} \tag{4}$$

where γ is Euler constant gamma with numerical value 0.577216, and ${}_pF_q\ [\{a_1,a_2,...,a_p\},\ \{b_1,b_2,...,b_q\},z]$ is a generalized hypergeometric function has series expansion

$$ {}_pF_q\left[\{a_1,a_2,...,a_p\},\{b_1,b_2,...,b_q\},z\right]=1+\sum_{k=1}^{\infty}\left[\frac{(a_1)_k...(a_p)_k}{(b_1)_k...(b_q)_k}.\frac{z^k}{k!}\right], \qquad (5) $$

where $(a)_k = a(a+1)...(a+k-1)$.

The theoretical coefficient of variation under the Gompertz distribution is thus given by

$$ CV = g(\lambda,\beta) = \frac{\sqrt{[E(x^2)-(E(x))^2]}}{E(x)}. \qquad (6) $$

The rest of the paper is organized as follows. In Section 2, we describe the formulation of a progressive first-failure-censoring scheme. We give a brief description of Markov chain Monte Carlo (MCMC) in Section 3. MLE and parametric bootstrap confidence interval are discussed in Section 4 and 5. Section 6 describes MCMC for estimating the empirical posterior distribution of CV and its interval estimation. Section 7 contains the analysis of a real life data set to illustrate our proposed method. Simulation studies are reported in Section 8. Finally we conclude with some comments in Section 9.

2. Progressive First-Failure Censoring Plan

Censoring is very common in life tests. There are survival types of censored tests. One of the most common censored test is type II censoring. It is noted that one can use type II censoring for saving time and money. However, when the lifetimes of products are very high, the experimental time of a type II censoring life test can be still too long. A generalization of type II censoring is progressive type II censoring. Johnson [19] described a life test in which the experimenter might decide to group the test units into several sets, each as an assembly of test units, and then run all the test units simultaneously until occurrence the first failure in each group. Such a censoring scheme is called first-failure censoring. Jun *et al.* [20] discussed a sampling plan for a bearing manufacturer. The bearing test engineer decided to save test time by testing 50 bearings in sets of 10 each. The first-failure times from each group were observed. Wu *et al.* [29] and Wu and Yu [30] obtained maximum likelihood estimates (MLEs), exact confidence intervals and exact confidence regions for the parameters of the Gompertz and Burr type XII distributions based on first-failure-censored sampling, respectively. Also see Lee *et al.* [21].

Note that a first-failure-censoring scheme is terminated when the first failure in each set is observed. If an experimenter desires to remove some sets of test units before observing the first failures in these sets this life test plan is called a progressive first-failure-censoring scheme which recently introduced by Wu and Kuş [28]. In this scheme, first-failure censoring scheme is combined with progressive censoring scheme. Suppose that n independent groups with k items within each group are put in a life test. R_1 groups and the group in which the first failure is observed are randomly removed from the test as soon as the first failure $X^R_{1;m,n,k}$ has occurred, R_2 groups and the group in which the second failure is observed are randomly removed from the test when the second failure $X^R_{2;m,n,k}$ has occurred, and finally R_m groups and the group in which the mth failure is observed are randomly removed from the test as soon as the mth failure $X^R_{m;m,n,k}$ has occurred. Then $X^R_{1;m,n,k} < X^R_{2;m,n,k} < ... < X^R_{m;m,n,k}$ are called progressively first-failure censored order statistics with the progressive censored scheme $R = \{R_1, R_2, ..., R_m\}$. It is clear that m is number of the first failures $(1 < m \leq n)$ and $n = m + \sum_{i=1}^{m} R_i$. If the failure times of the $(n \times k)$ items originally in the test are from a continuous population with distribution function $F(x)$ and probability density function $f(x)$, the joint probability density function for $X^R_{1;m,n,k}, X^R_{2;m,n,k}, ..., X^R_{m;m,n,k}$ is given by

$$f_{1,2,...,m}(x^R_{1;m,n,k}, x^R_{2;m,n,k}, ..., x^R_{m;m,n,k}) = Ak^m \prod_{i=1}^{m} f(x^R_{i;m,n,k}) \times [1 - F(x^R_{i;m,n,k})]^{k(Ri+1)-1}, \quad (7)$$

$$0 < x^R_{1;m,n,k} < x^R_{2;m,n,k} < ... < x^R_{m;m,n,k} < \infty,$$

where

$$A = n(n - R_1 - 1)(n - R_1 - R_2 - 2)...(n - R_1 - R_2 - ... - R_{m-1} - m + 1). \quad (8)$$

Special Cases

It is clear from (7) that the progressive first-failure censored scheme containing the following censoring schemes as special cases:

(1) The first-failure censored scheme when $R = \{0, 0, ..., 0\}$.

(2) The progressive type II censored order statistics if $k = 1$.

(3) Usually type II censored order statistics when $k = 1$ and $R = \{0, 0, ..., n - m\}$.

(4) The complete sample case when $k = 1$ and $R = \{0, 0, ..., 0\}$.

Also, It should be noted that $X^R_{1;m,n,k}, X^R_{2;m,n,k}, ..., X^R_{m;m,n,k}$ can be viewed as a progressive type II censored sample from a population with distribution function $1 - (1 - F(x))^k$. For this reason, results for progressive type II censored order statistics can be extend to progressive first-failure censored order statistics easily. Also, the progressive first-failure-censored plan has advantages in terms of reducing the test time, in which more items are used, but only m of $n \times k$ items are failures.

3. Markov Chain Monte Carlo Techniques

Markov chain Monte Carlo (MCMC) methods use computer simulation of Markov chains in the parameter space Gilks *et al*. [13] and Gamerman [9]. The Markov chains are defined in such a way that the posterior distribution in the given statistical inference problem is the asymptotic distribution. This allows to use ergodic averages to approximate the desired posterior expectations. Several standard approaches to define such Markov chains exist, including Gibbs sampling, Metropolis-Hastings and reversible jump, see for example Metropolis *et al*. [22] and Hastings [17]. Using these algorithms it is possible to implement posterior simulation in essentially any problem which allow pointwise evaluation of the prior distribution and likelihood function.

3.1 *Gibbs sampler*

The Gibbs sampling algorithm is one of the simplest Markov chain Monte Carlo algorithms. It was introduced by Geman and Geman [12]. The paper by Gelfand and Smith [11] helped to demonstrate the value of the Gibbs algorithm for a range of problems in Bayesian analysis. Gibbs sampling is a MCMC scheme where the transition kernel is formed by the full conditional distributions.

Algorithm 1 :

(1) Choose an arbitrary starting point $\theta^{(0)} = \left(\theta_1^{(0)}, ..., \theta_d^{(0)}\right)$ for which $g\left(\theta^{(0)}\right) > 0$.

(2) Obtain $\theta_1^{(t)}$ from conditional distribution $g\left(\theta_1 \mid \theta_2^{(t-1)}, \theta_3^{(t-1)}, ..., \theta_d^{(t-1)}\right)$.

(3) Obtain $\theta_2^{(t)}$ from conditional distribution $g\left(\theta_2 \mid \theta_1^{(t)}, \theta_3^{(t-1)}, ..., \theta_1^{(t-1)}\right)$.

(4) Obtain $\theta_d^{(t)}$ from conditional distribution $g\left(\theta_d \mid \theta_1^{(t)}, \theta_2^{(t)}, \theta_3^{(t)}, ..., \theta_{d-1}^{(t)}\right)$.

(5) Repeat steps $2-4$.

3.2 *The Metropolis-Hastings Algorithm*

The Metropolis-Hastings algorithm was originally introduced by Metropolis *et al.* [22]. Suppose that our goal is to draw samples from some distribution $f(\theta|x) = \nu g(\theta)$, where ν is the normalizing constant which may not be known or very difficult to compute. The Metropolis-Hastings (MH) algorithm provides a way of sampling from $f(\theta|x)$ without requiring us to know ν. Let $q\left(\theta^{(b)}|\theta^{(a)}\right)$ be an arbitrary transition kernel, that is the probability of moving or jumping from current state $\theta^{(a)}$ to $\theta^{(b)}$. This is sometimes called the proposal distribution. The following algorithm will generate a sequence of values $\theta^{(1)}, \theta^{(2)}$, ... which form a Markov chain with stationary distribution given by $f(\theta|x)$.

Algorithm 2 :

(1) Choose an arbitrary starting point $\theta^{(0)}$ for which $f\left(\theta^{(0)}|x\right) > 0$.

(2) At time t, sample a candidate point or proposal θ^* from the proposal distribution $q\left(\theta^*|\theta^{(t-1)}\right)$.

(3) Calculate the acceptance probability

$$\rho\left(\theta^{(t-1)}, \theta^*\right) = \min\left[1, \frac{f(\theta^*|x)\, q\left(\theta^{(t-1)}|\theta^*\right)}{f\left(\theta^{(t-1)}|x\right) q\left(\theta^*|\theta^{(t-1)}\right)}\right]. \tag{9}$$

(4) Generate $U \sim U(0,1)$.

(5) If $U \leq \rho\left(\theta^{(t-1)}, \theta^*\right)$ accept the proposal and set $\theta^{(t)} = \theta^*$. Otherwise, reject the proposal and set $\theta^{(t)} = \theta^{(t-1)}$.

(6) Repeat steps $2-5$.

If the proposal distribution is symmetric, so $q(\theta|\phi) = q(\phi|\theta)$ for all possible ϕ and θ then, in particular, we have $q\left(\theta^{(t-1)}|\theta^*\right) = q\left(\theta^*|\theta^{(t-1)}\right)$, so that the acceptance probability (9) is given by

$$\rho\left(\theta^{(t-1)},\theta^{*}\right)=\min\left[1,\frac{f\left(\theta^{*}|x\right)}{f\left(\theta^{(t-1)}|x\right)}\right].\tag{10}$$

4. MAXIMUM LIKELIHOOD ESTIMATION

Let $\underline{x}=(X^{R}_{1;m,n,k},X^{R}_{2;m,n,k},...,X^{R}_{m;m,n,k})$ be the progressive first-failure censored order statistics from a Gompertz distribution with censored scheme R. From (1), (2) and (3), the likelihood function is given by

$$L\left(\underline{x};\lambda,\beta\right)=Ak^{m}\beta^{m}\exp\left(\lambda\sum_{i=1}^{m}x_{i}-\frac{\beta k}{\lambda}\sum_{i=1}^{m}(R_{i}+1)\left(\exp\left(\lambda x_{i}\right)-1\right)\right),\tag{11}$$

where A as given by (8) and we used x_i instead of $X^{R}_{i;m,n,k}$. The logarithm of the likelihood function may then be written as

$$l\left(\underline{x};\lambda,\beta\right)=\log\left(Ak^{m}\right)+m\log\beta+\lambda\sum_{i=1}^{m}x_{i}-\frac{\beta k}{\lambda}\sum_{i=1}^{m}(R_{i}+1)(\exp(\lambda x_{i})-1)\cdot\tag{12}$$

Calculating the first partial derivatives of (12) with respect to β and λ and equating each to zero, we get the likelihood equations as

$$\frac{\partial l\left(\underline{x};\lambda,\beta\right)}{\partial\beta}=\frac{m}{\beta}-\frac{k}{\lambda}\sum_{i=1}^{m}(R_{i}+1)(\exp(\lambda x_{i})-1)=0,\tag{13}$$

$$\frac{\partial l\left(\underline{x};\lambda,\beta\right)}{\partial\lambda}=\sum_{i=1}^{m}x_{i}+\frac{\beta k}{\lambda^{2}}\sum_{i=1}^{m}(R_{i}+1)(\exp(\lambda x_{i})-1)-\frac{\beta k}{\lambda}\sum_{i=1}^{m}(R_{i}+1)x_{i}\exp(\lambda x_{i})=0,\tag{14}$$

hence from (13) we obtain the ML estimate of β as

$$\widehat{\beta}=\frac{m\widehat{\lambda}}{k\sum\limits_{i=1}^{m}(R_{i}+1)(\exp(\widehat{\lambda}x_{i})-1)}.\tag{15}$$

By using (15) in (14) we obtain

$$\frac{1}{m}\sum_{i=1}^{m} x_i + \frac{1}{\widehat{\lambda}} - \frac{\sum_{i=1}^{m}(R_i+1)x_i\exp(\widehat{\lambda}x_i)}{\sum_{i=1}^{m}(R_i+1)(\exp(\widehat{\lambda}x_i)-1)} = 0. \tag{16}$$

Since (16) cannot be solved analytically for $\hat{\lambda}$, some numerical methods such as Newton's method must be employed. Therefore, the ML estimate of CV is

$$\widehat{CV} = g\left(\widehat{\lambda}, \widehat{\beta}\right), \tag{17}$$

where $g\left(\widehat{\lambda}, \widehat{\beta}\right)$ as given in (6) after replacing λ and β by $\widehat{\lambda}$ and $\widehat{\beta}$, respectively.

5. Bootstrap Confidence Intervals

Bootstrap methods are widely used to improve estimators or to build confidence intervals for the parameters. Usually, they provide estimators with smaller standard errors, and confidence intervals with a coverage level closer to the nominal level than confidence intervals obtained by applying asymptotic results. For a survey of parametric and nonparametric bootstrap methods, one can refer to Davison and Hinkley [5] and Efron and Tibshirani [6]. In this section, we use the parametric bootstrap percentile method suggested by Efron and Tibshirani [6] to construct a bootstrap percentile confidence intervals for the CV. The following steps are followed to obtain progressive first-failure censoring bootstrap sample from Gompertz distribution with parameters λ and β based on simulated progressively first-failure censored data set.

Algorithm 3 :

(1) From an original data set $\underline{x} \equiv x^R_{1;m,n,k}, x^R_{2;m,n,k}, ..., x^R_{m;m,n,k}$, compute the MLEs $\widehat{\lambda}$ and $\widehat{\beta}$ from equations (15) and (16) and $\widehat{CV}$ from(17).

(2) Use $\widehat{\lambda}$ and $\widehat{\beta}$ to generate a bootstrap sample $\underline{x}^*$ with the same values of R_i, $(i = 1, 2, ..., m)$ using the algorithm of Balakrishnan and Sandhu [3].

(3) As in step 1 based on $\underline{x}^*$ compute the bootstrap sample estimates of $\widehat{\lambda}$, $\widehat{\beta}$ and $\widehat{CV}$ say $\widehat{\lambda}^*$, $\widehat{\beta}^*$and $\widehat{CV}^*$.

(4) Repeat steps 2-3 N times representing N bootstrap MLEs of $\widehat{CV}$ based on N different bootstrap samples.

(5) Arrange all $\widehat{CV}^{*j}$ in an ascending order to obtain bootstrap sample ($\widehat{CV}^{*[1]}$, $\widehat{CV}^{*[2]}, ..., \widehat{CV}^{*[N]}$).

(6) Let $G(z) = P(\widehat{CV}^{*} \leqslant z)$ be cumulative distribution function of $\widehat{CV}^{*}$. Define $\widehat{CV}^{*}_{boot} = G^{-1}(z)$ for given z. The approximate bootstrap $100(1-\gamma)\%$ confidence interval of $\widehat{CV}^{*}$ given by

$$\left[\widehat{CV}^{*}_{boot}(\frac{\gamma}{2}),\ \widehat{CV}^{*}_{boot}(1-\frac{\gamma}{2})\right]. \tag{18}$$

6. Bayes Estimations Using MCMC

Under the assumption that both of the parameters λ and β are unknown, we may consider the joint prior density as a product of a conditional density of β for given λ (which is taken to be the conjugate gamma prior when λ is known) and inverted gamma density for λ. So that the joint prior density of λ and β can be written as $\pi(\lambda, \beta) = \pi_1(\beta|\lambda)\pi_2(\lambda)$, where

$$\pi_1(\beta|\lambda) = \frac{b^a}{\Gamma(a)\lambda^a}\beta^{a-1}\exp\left(-\beta\frac{b}{\lambda}\right), \quad \beta > 0,\ a,\ b > 0, \tag{19}$$

and

$$\pi_2(\lambda) = \frac{d^c}{\Gamma(c)\lambda^{c+1}}\exp\left(-\frac{d}{\lambda}\right), \quad \lambda > 0,\ c,\ d > 0. \tag{20}$$

Multiplying $\pi_1(\beta|\lambda)$ by $\pi_2(\lambda)$ we obtain the joint prior density of λ and β as

$$\pi(\lambda, \beta) = \frac{b^a d^c \beta^{a-1}}{\Gamma(a)\Gamma(c)\lambda^{a+c+1}}\exp\left(\frac{-1}{\lambda}(d+\beta b)\right),\ (\lambda,\ \beta > 0). \tag{21}$$

Using the joint prior distribution of λ and β, the joint posterior density function of λ and β given the data, denoted by $\pi^*(\lambda, \beta)$, can be written as

$$\pi^*(\lambda, \beta) = \frac{L(\text{data}|\lambda, \beta) \times \pi(\lambda, \beta)}{\int_0^\infty \int_0^\infty L(\text{data}|\lambda, \beta) \times \pi(\lambda, \beta) d\lambda d\beta}, \tag{22}$$

therefore, the Bayes estimate of any function of λ and β say $g(\lambda,\beta)$, under squared error loss function is

$$\begin{aligned}\widehat{g}(\lambda,\beta) &= E_{\lambda,\beta|\text{data}}(g(\lambda,\beta)) \\ &= \frac{\int_0^\infty\int_0^\infty g(\lambda,\beta)L(data|\lambda,\beta)\times\pi(\lambda,\beta)d\lambda d\beta}{\int_0^\infty\int_0^\infty L(data|\lambda,\beta)\times\pi(\lambda,\beta)d\lambda d\beta}. \qquad (23)\end{aligned}$$

Generally, the ratio of two integrals given by (23) can not be obtained in a closed form. In this case, we use the MCMC method to generate samples from the posterior distributions and then compute the Bayes estimator of $g(\lambda,\beta)$ under the squared errors loss (SEL) function.

MCMC Approach

In this subsection we consider the MCMC method to generate samples from the posterior distributions and then compute the Bayes estimates of CV of the Gompertz distribution under the squared errors loss function. A wide variety of MCMC schemes are available, and it can be difficult to choose among them. An important sub-class of MCMC methods are Gibbs sampling and more general Metropolis-within-Gibbs samplers. The advantage of using the MCMC method over the MLE method is that we can always obtain a reasonable interval estimate of the parameters by constructing the probability intervals based on the empirical posterior distribution. This is often unavailable in maximum likelihood estimation. Indeed, the MCMC samples may be used to completely summarize the posterior uncertainty about the parameters λ and β, through a kernel estimate of the posterior distribution. This is also true of any function of the parameters, CV in particular. Suppose we wish to give point and interval estimates for CV.

The joint posterior density function of λ and β can be written as

$$\begin{aligned}\pi^*(\lambda,\beta) \quad \propto \quad & \frac{\beta^{m+a-1}}{\lambda^{a+c+1}} \\ & \exp\left(\lambda\sum_{i=1}^m x_i - \frac{1}{\lambda}\left[d+\beta b+\beta k\sum_{i=1}^m (R_i+1)\left(\exp\left(\lambda x_i\right)-1\right)\right]\right), \qquad (24)\end{aligned}$$

from (24) it is clear that the posterior density function of β given λ is

$$\pi_1^*(\beta|\lambda) \propto \beta^{m+a-1}\exp\left(-\frac{\beta}{\lambda}\left[b+k\sum_{i=1}^m (R_i+1)\left(\exp\left(\lambda x_i\right)-1\right)\right]\right). \qquad (25)$$

Therefore, the posterior density function of β given λ, is gamma with parameters as $m+a$ and $\frac{1}{\lambda}(b+k\sum_{i=1}^{m}(R_i+1)\left(\exp\left(\lambda x_i\right)-1\right))$.

The posterior density function of λ given β can be written as

$$\pi_2^*(\lambda|\beta) \propto \frac{1}{\lambda^{a+c+1}} \exp\left(\lambda\sum_{i=1}^{m}x_i - \frac{1}{\lambda}\left[d+\beta b+\beta k\sum_{i=1}^{m}(R_i+1)\left(\exp\left(\lambda x_i\right)-1\right)\right]\right). \quad (26)$$

Now, to ensure that we can employ the Gibbs sampling scheme, we need to carefully check if the marginal conditional posteriors distribution $\pi_2^*(\lambda|\beta)$ is log-concave. The second-order partial derivatives of $\ln\pi_2^*(\lambda|\beta)$ with respect to λ is

$$\frac{\partial^2 \ln\pi_2^*(\lambda|\beta)}{\partial\lambda^2} = \frac{-1}{\lambda^3}[2(d+\beta b)-\lambda(a+c+1) + \sum_{i=1}^{m}(R_i+1)g_1(\lambda x_i)], \quad (27)$$

where

$$g_1(\lambda x_i) = ((\lambda x_i-1)^2\exp(\lambda x_i)+\exp(\lambda x_i)-2).$$

It can be shown that the function $g(\lambda x_i)$ has limiting 0 as $\lambda x_i \longrightarrow 0$ and limiting ∞ as $\lambda x_i \longrightarrow \infty$, then for all $m>1$ the summation is greater than 0, hence $\pi_2^*(\lambda|\beta)$ depends on the choice of the hyperparameters $(a,\, b,\, c,\, d)$ and (λ, β). Then $\pi_2^*(\lambda|\beta)$ is not necessary log-concave. We, therefore, employ the Metropolis-within-Gibbs method instead to sample λ. The choice of the hyperparameters $(a,\, b,\, c,\, d)$ and the censoring scheme R are which make (26) close to the proposal distribution and obviously more convergence of the MCMC iteration. We propose the following MCMC algorithm to draw samples from the posterior density functions; and in turn compute the Bayes estimates and also, construct the corresponding credible intervals.

Algorithm 4 :

(1) $\lambda_0=\widehat{\lambda}$, $M=burn-in$.

(2) Generate β_1 from gamma distribution $\pi_1^*(\beta|\lambda)$.

(3) Generate λ_1 from $\pi_2^*(\lambda|\beta)$ using (MH) algorithm in subsection (3.2).

(4) Compute $CV_1 = g(\lambda_1, \beta_1)$.

(5) Repeat steps 2-4 N times we obtain CV_1, CV_2,. . . , CV_N.

(6) Obtain the Bayes estimate of CV with respect to the SEL function as

$$\widehat{E}(CV|data) = \frac{1}{N-M} \sum_{i=M+1}^{N} CV_i. \tag{28}$$

(7) To compute the credible intervals of CV, order CV_{M+1}, CV_{M+2},. . . ,CV_N as $CV_{(1)}$, $CV_{(2)}$,. . . ,$CV_{(N-M)}$. Then the $100(1-\gamma)\%$ symmetric credible interval is

$$\left(CV_{((N-M)\gamma/2)},\ CV_{((N-M)(1-\gamma/2))}\right). \tag{29}$$

7. ILLUSTRATIVE EXAMPLE

To illustrate the application of our proposed method, we chose the real data set from Hoel [18], concerning the time (in days) at death of 39 irradiated mices. These life times in days are: 40, 42, 51, 62, 163, 179, 206, 222, $228, 249$, $252, 282$, 324, 333, 341, 366, 385, $407, 420, 431, 441, 461$, 462, 482, 517, 517, $524, 564$, 567, 586, 619, 620, 621, 622, $647, 651$, 686, 761, 763. Elandt and Johnson [7] fitted a two parameter Gompertz distribution to the data. The point estimate of CV using complete data set is given by $\widehat{CV}_{samp} = \dfrac{s}{\bar{x}} = 0.4937$.

The data are randomly grouped into 13 sets and listed in ascending order in Table 1. Now, we consider the following four cases.

Case I : Progressive first-fauilure censoring data.

For the first row in Table 1 with ($m = 8, n = 13, k = 3$) and the censoring scheme $R = \{0, 2, 0, 0, 1, 0, 0, 2\}$, we obtain the following progressive first-fauilure censoring data: $40, 42, 62, 206, 222, 228, 333, 420$.

Case II : First-fauilure censoring data.

The values of the first row in Table 1: $40, 42, 51, 62, 179, 206, 222, 228, 324, 333, 420, 441, 462$. are the first-fauilure censoring data. In this case ($m = n = 13,\ k = 3$) and $R = \{13^0\}$.

Case III : Progressive type II censoring data.

Using the algorithm described in Balakrishnan and Sandhu [3], we generate the following progressive type II censoring data: $40, 42, 51, 62, 163, 179, 206, 222, 228, 252, 282, 333, 341, 366, 420$, $431, 441, 462, 482, 517$. In this case ($m = 20, n = 39, k = 1$) and the censoring scheme $R = \{0, 0, 0, 3, 0, 0, 0, 3, 0, 0, 3, 0, 0, 0, 5, 0, 0, 0, 0, 5\}$

Case IV : The complete data set, with ($m = n = 39, k = 1$) and $R =\{39^0\}$.

For each case mentioned above, we run the chain for $11,000$ times and discard the first 1000 values as 'burn-in'. When the non-informative prior distribution is used, the joint posterior distribution of the parameters is proportional to the likelihooh function. The Bayes point estimates and 95% credible intervals for CV are computed. We also performed a simple bootstrap procedure to generate the sampling distribution of CV based on the observed data in all cases. The point estimates of CV using the maximum likelihood method and bootstrap method as well as 95% bootstrap confidence interval are presented in Table 2. If we adopt the Bayesian approach, we have the results in Table 3. The descriptive statistics as well as the 95% credible interval for CV based on the MCMC generated sample are also given in Table 3. As we can see from the histograms of the posterior distributions of CV in Figures (1-4).

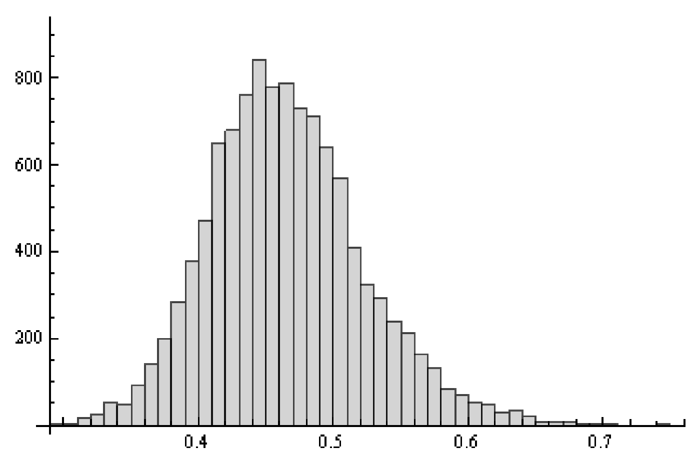

FIG. 1: Histogram of posterior distribution of CV (Case I)

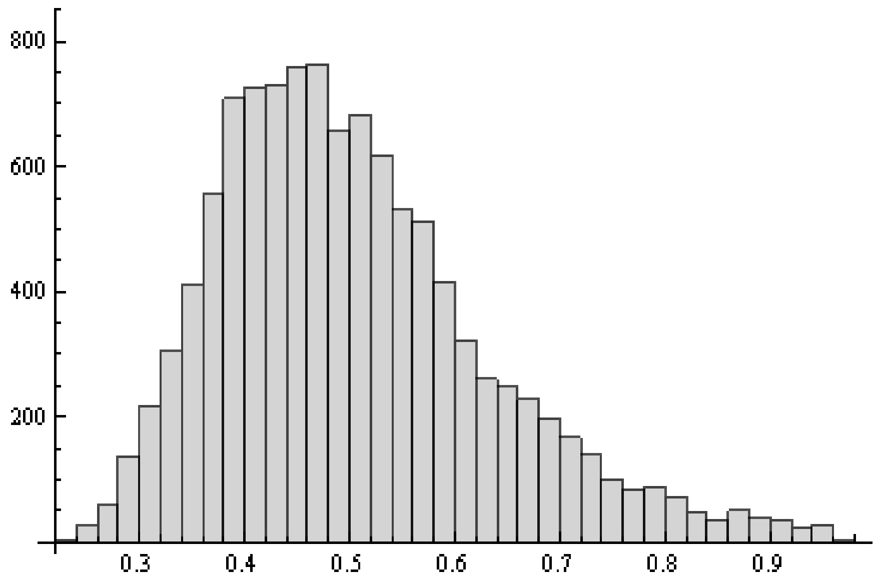

FIG. 2: Histogram of posterior distribution of CV (Case II)

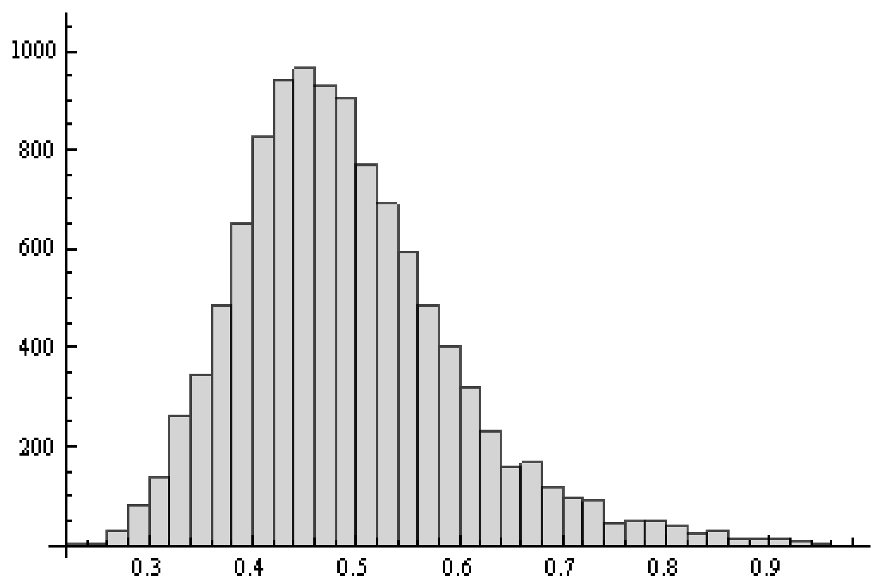

FIG. 3: Histogram of posterior distribution of CV (Case III)

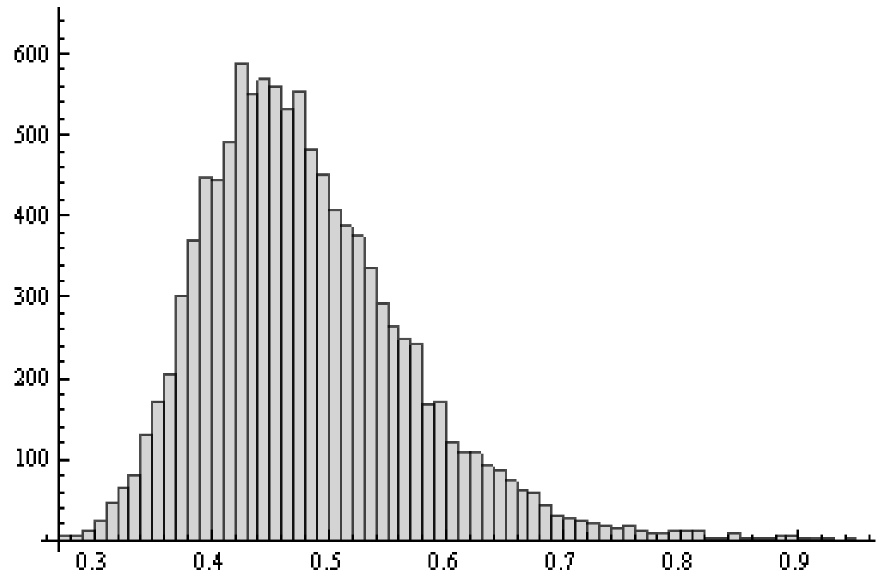

FIG. 4: Histogram of posterior distribution of CV (Case IV)

TABLE 1: Randomly grouped sets using data from Hoel [22]

206	333	42	462	40	179	420	51	324	228	222	62	441
586	407	163	686	252	524	567	619	482	385	341	249	517
651	620	282	763	461	564	622	621	647	431	761	366	517

TABLE 2: MLE and Bootstrap results of CV

Cases	$\widehat{CV}_{MLE}$	$\widehat{CV}_{Boot}$	95 boot. CI	Leng.
I	0.5317	0.4688	(0.2672,0.8062)	0.5391
II	0.4811	0.4560	(0.2843,0.6781)	0.3938
III	0.4817	0.4616	(0.3213,0.6332)	0.3120
IV	0.4639	0.4543	(0.3544,0.5538)	0.1994

TABLE 3: MCMC method results of CV

Cases	Mean	Median	Mode	SD	Skew.	95% CI	Length
I	0.5056	0.4864	0.4479	0.1298	0.8173	(0.3060,0.8233)	0.5173
II	0.4918	0.4773	0.4483	0.1065	0.8734	(0.3222,0.7521)	0.4300
III	0.4794	0.4662	0.4396	0.0888	0.9739	(0.3415,0.6862)	0.3447
IV	0.466	0.4621	0.4541	0.058	0.4337	(0.3627,0.5943)	0.2316

8. Simulation Study

In this section we report some numerical experiments performed to evaluate the behavior of the proposed methods for different sample sizes, different censoring schemes, different parameter values for λ and β, and informative and non-informative priors. All of the computations were performed by (mathematica 7.0) using a Pentium IV processor. The samples were generated by using the algorithm described in Balakrishnan and Sandhu [3]. We used different sample sizes ($n \times k$), different group sizes (k), and different sampling schemes (i.e., different R_i values).

TABLE 4 : Average values, MSEs, the coverege percentages and average credible interval lengths of the estimates of CV when $\lambda = 0.01$ and $\beta = 0.01$.

k	n	m	C.S.	Means	MSE	C.P.	Length
			non-informative prior				
1	30	15	$(15,14^0)$	0.6983	0.0827	0.976	0.3839
			(15^1)	0.6673	0.0898	0.978	0.4122
			$(14^0,15)$	0.6319	0.1065	0.976	0.4655
	30	20	$(10,19^0)$	0.7083	0.0771	0.978	0.3546
			$(5^0,10^1,5^0)$	0.6937	0.0748	0.972	0.3569
			$(19^0,10)$	0.6686	0.0891	0.970	0.4179
	30	30	(30^0)	0.7030	0.0700	0.972	0.3181
	50	30	$(20,29^0)$	0.7149	0.0747	0.956	0.3123
			(1,1,0,...,1,1,0)	0.7010	0.0736	0.978	0.3330
			$(29^0,20)$	0.6792	0.0777	0.978	0.3949
	50	50	(50^0)	0.7104	0.0606	0.970	0.2651
5	30	15	$(15,14^0)$	0.6233	0.1152	0.966	0.4363
			(15^1)	0.5667	0.1610	0.906	0.4491
			$(14^0,15)$	0.5058	0.2080	0.890	0.4828
	30	20	$(10,19^0)$	0.6311	0.1095	0.946	0.4140
			$(5^0,10^1,5^0)$	0.6186	0.1171	0.944	0.4113
			$(19^0,10)$	0.5570	0.1651	0.912	0.4619
	30	30	(30^0)	0.6554	0.0916	0.952	0.3878
	50	30	$(20,29^0)$	0.6569	0.0864	0.974	0.3883
			(1,1,0,...,1,1,0)	0.6251	0.1101	0.954	0.4011
			$(29^0,20)$	0.5636	0.1556	0.926	0.4500
	50	50	(50^0)	0.6802	0.0746	0.966	0.3517

We used two sets of parameter values $(\lambda,\ \beta) = (0.01, 0.01)$ and $(\lambda,\ \beta) = (0.5, 0.02)$. First, we used the non-informative priors for the parameters, in this case the joint posterior distribution of the parameters is proportional to the likelihooh function. We also used informative prior $a = 5$, $b = 5$, $c = 5$ and $d = 0.05$, when $(\lambda,\ \beta) = (0.01, 0.01)$. The hyperparameter values are selected to satisfy the prior means $E(\lambda) = \dfrac{d}{c-1} \cong \lambda$ and $E(\beta) = a\dfrac{\lambda}{b} \cong \beta$. In all cases, we used the squared error loss function to compute the Bayes estimates. We computed the Bayes estimates and 95% credible intervals based on $11,000$ MCMC samples (discard the first 1000 samples as 'burn-in'). We report the average Bayes estimates of CV, mean squared errors (MSEs), coverage percentages (C.P.) average confidence interval lengths based on 500 replications.

The results for $(\lambda,\ \beta) = (0.01, 0.01)$ are reported in Table 4 for non-informative prior, and in Table 5 for informative prior Table 6 reports the results for $(\lambda,\ \beta) = (0.5, 0.02)$ with non-informative prior.

9. Concluding Remarks

In field of reliability studies nonnegative and non-gaussian data are commonly encountered, and it is not easy to obtain an interval estimate for the CV using the theoretical sampling distribution of CV. Therefore, it is hard to make inferences about the CV in many applications. We not aware of any discussion concerning inference on CV of non-normal distribution based on censored data. In this paper we offer a simulation-based Bayesian approach. MCMC method is proposed for finding a point estimate as well as an interval estimate for the CV, based on a progressive first failure-censored data. This sampling plan is quite useful to practitioners, because they provide savings in resources and in total test time. Results from simulation studies illustrate that the performance of our proposed method is acceptable. The proposed method works routinely well for censored samples. From Tables (4-6), in terms of both MSEs and lengths of the credible intervals, it is clear that the MSEs, and average credible interval lengths of Bayes estimators are smallest for the censoring schemes $(n-m, 0, \cdots, 0)$, which corresponding to the removal in first stage. When the effective sample proportion m/n increases, the MSEs and average credible interval lengths are almost decrease in all cases. The MSEs for the estimates of CV for the progressive first-failure censoring ($k = 5$) are almost similar to those for progressive type II censoring ($k = 1$).

TABLE 5 : Average values, MSEs, the coverege percentages and average credible interval lengths of the estimates of CV when $\lambda = 0.01$ and $\beta = 0.01$.

k	n	m	C.S.	Means	MSE	C.P.	Length
Informative prior ($a = 5$, $b = 5$, $c = 5$, $d = 0.05$)							
1	30	15	$(15,14^0)$	0.6948	0.0198	1.000	0.1721
			(15^1)	0.6738	0.0862	0.992	0.4106
			$(14^0,15)$	0.6946	0.0160	1.000	0.1706
	30	20	$(10,19^0)$	0.7017	0.0780	0.980	0.3570
			$(5^0,10^1,5^0)$	0.6924	0.0769	0.974	0.3541
			$(19^0,10)$	0.6692	0.0862	0.984	0.4206
	30	30	(30^0)	0.6855	0.0451	0.994	0.2376
	50	30	$(20,29^0)$	0.7078	0.0691	0.974	0.3162
			(1,1,0,...,1,1,0)	0.6839	0.0457	0.994	0.2635
			$(29^0,20)$	0.6689	0.0843	0.966	0.3943
	50	50	(50^0)	0.6917	0.0382	0.990	0.2042
5	30	15	$(15,14^0)$	0.6956	0.0157	1.000	0.1661
			(15^1)	0.5937	0.1188	0.988	0.3870
			$(14^0,15)$	0.5578	0.1492	0.998	0.4085
	30	20	$(10,19^0)$	0.6378	0.0799	0.988	0.3518
			$(5^0,10^1,5^0)$	0.6351	0.0828	0.990	0.3571
			$(19^0,10)$	0.5878	0.1215	1.000	0.3947
	30	30	(30^0)	0.6536	0.0662	0.988	0.3361
	50	30	$(20,29^0)$	0.6500	0.0694	1.000	0.3358
			(1,1,0,...,1,1,0)	0.6270	0.0913	0.986	0.3570
			$(29^0,20)$	0.6965	0.0123	1.000	0.1588
	50	50	(50^0)	0.6680	0.0573	0.984	0.3131

TABLE 6 : Average values, MSEs, the coverege percentages and average credible interval lengths of the estimates of CV when $\lambda = 0.5$ and $\beta = 0.02$.

k	n	m	C.S.	Means	MSE	C.P.	Length
non-informative prior							
1	30	15	$(15,14^0)$	0.4090	0.0725	0.956	0.2876
			(15^1)	0.4142	0.0811	0.954	0.3083
			$(14^0,15)$	0.4230	0.0892	0.968	0.3685
	30	20	$(10,19^0)$	0.4081	0.0670	0.956	0.2549
			$(5^0,10^1,5^0)$	0.3969	0.0575	0.960	0.2410
			$(19^0,10)$	0.4087	0.0780	0.946	0.2963

TABLE 6 (continued)

	30	30	(30^0)	0.3925	0.0552	0.950	0.2071
	50	30	$(20,29^0)$	0.3941	0.0522	0.950	0.2010
			(1,1,0,...,1,1,0)	0.3948	0.0521	0.958	0.2007
			$(29^0,20)$	0.3993	0.0643	0.940	0.2389
	50	50	(50^0)	0.3875	0.0417	0.938	0.1591
5	30	15	$(15,14^0)$	0.4031	0.0718	0.968	0.2991
			(15^1)	0.4054	0.0742	0.964	0.3339
			$(14^0,15)$	0.3957	0.0755	0.970	0.3789
	30	20	$(10,19^0)$	0.4030	0.0674	0.956	0.2634
			$(5^0,10^1,5^0)$	0.3958	0.0634	0.942	0.2537
			$(19^0,10)$	0.4087	0.0779	0.958	0.3372
	30	30	(30^0)	0.3926	0.0493	0.968	0.2048
	50	30	$(20,29^0)$	0.3966	0.0555	0.930	0.2051
			(1,1,0,...,1,1,0)	0.3917	0.0543	0.948	0.2145
			$(29^0,20)$	0.4000	0.0687	0.956	0.2865
	50	50	(50^0)	0.3855	0.0377	0.954	0.1497

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