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ANNIHILATOR CONDITION ON POWER VALUES OF DERIVATIONS

Basudeb Dhara

Department of Mathematics, Belda College, Belda
Paschim Medinipur 721424, *India*
e-mail: basu_dhara@yahoo.com

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Let R be a prime ring, d, δ two derivations of R, L a noncentral Lie ideal of R and $0 \neq a \in R$. The main object in this paper is to discuss the situations $a(d(x)x - x\delta(x))^n = 0$ for all $x \in L$ and $a(d(x)x - x\delta(x)) \in Z(R)$ for all $x \in L$, where $n \geq 1$ is a fixed integer.

Key words : Prime ring, derivation, extended centroid, symmetric Martindale's quotient ring.

1. Introduction

Throughout this paper, R always denotes a prime ring with center $Z(R)$, extended centroid C and Q is its symmetric Martindale's quotient ring. For $x, y \in R$, the commutator of x, y is denoted by $[x, y]$ and defined by $[x, y] = xy - yx$. The d and δ denote the derivations of R. The standard polynomial identity s_4 in four variables is defined as $s_4(x_1, x_2, x_3, x_4) = \sum_{\sigma \in S_4} (-1)^\sigma x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}x_{\sigma(4)}$ where $(-1)^\sigma$ is $+1$ or -1 according to σ being an even or odd permutation in symmetric group S_4.

A derivation d is called Q-inner if it is inner induced by an element, say $q \in Q$ as an adjoint, that is, $d(x) = [q, x]$ for all $x \in R$. A derivation which is not Q-inner is called a Q-outer derivation.

A well known result proved by Posner [21], states that if the commutators $[d(x), x] \in Z(R)$ for all $x \in R$, then either $d = 0$ or R is commutative. Then many related generalizations of Posner's result have been obtained by a number of authors in literature. Posner's theorem was extended to Lie ideals in prime rings by Lee [18] and then by Lanski [15]. In [5], Carini and Filippis studied the situation in more generalized form considering power values. They proved that if char $(R) \neq 2$ and $[d(x), x]^n \in Z(R)$ for all $x \in L$, where L is a noncentral Lie ideal of R and $n \geq 1$ a fixed integer, then $d = 0$ or R satisfies s_4. Recently, in [23], Wang and You removed the characteristic assumption on R.

On the other hand, authors generalized Posner's theorem by considering two derivations. Brešar proved in [4] that if $d(x)x - x\delta(x) \in Z(R)$ for all $x \in R$, then either $d = \delta = 0$ or R is commutative. Later Lee and Wong [19] studied the same situation of Brešar for all x in some noncentral Lie ideal L of R and obtained that either $d = \delta = 0$ or R satisfies s_4.

There are some papers which have studied identities of derivations with annihilator conditions. In [10], Filippis proved that if char $(R) \neq 2$, $d \neq 0$ and $a \in R$ such that $a[d(x), x]^n \in Z(R)$ for all $x \in L$, where L is a noncentral Lie ideal of R and $n \geq 1$ a fixed integer, then either $a = 0$ or R satisfies s_4. In [22], Wang proved that the same conclusion holds in case char $(R) = 2$.

Recently, Argac and Filippis [1] obtained the following result:

Let R be a prime ring with char $(R) \neq 2$, L a non-central Lie ideal of R, d, δ two derivations of R, $n \geq 1$ a fixed integer. If $(d(x)x - x\delta(x))^n = 0$ for all $x \in L$, then either $d = \delta = 0$ or R satisfies the standard identity s_4 and d, δ are inner derivations, induced respectively by the elements a and b such that $a + b \in Z(R)$.

In view of all these above results, it is natural to consider the situations $a(d(x)x - x\delta(x))^n = 0$ for all $x \in L$ and $a(d(x)x - x\delta(x)) \in Z(R)$ for all $x \in L$, where $0 \neq a \in R$ and $n \geq 1$ a fixed integer. In the present paper, we shall study these two situations removing the assumption of char $(R) \neq 2$.

Let Q be the symmetric Martindale's quotient ring of a prime ring R and C the center of Q, which is called extended centroid of R. Note that Q is also a prime ring with C a field. We denote by $T = Q *_C C\{X\}$, the free product over C of the

C-algebra Q and the free C-algebra $C\{X\}$, with X the countable set consisting of the noncommuting indeterminates $x_1, x_2, \ldots$. The elements of T are called generalized polynomial with coefficients in Q. Nontrivial generalized polynomial means a nonzero element of T. For more details about these objects we refer to [2, 11].

Remark 1 : Let R be a prime ring and L a noncentral Lie ideal of R. If char $(R) \neq 2$, by [3, Lemma 1] there exists a nonzero ideal I of R such that $0 \neq [I, R] \subseteq L$. If char $(R) = 2$ and $dim_C RC > 4$ i.e., char $(R) = 2$ and R does not satisfy s_4, then by [14, Theorem 13] there exists a nonzero ideal I of R such that $0 \neq [I, R] \subseteq L$. Thus if either char $(R) \neq 2$ or R does not satisfy s_4, then we may conclude that there exists a nonzero ideal I of R such that $[I, I] \subseteq L$.

Remark 2 : It is well known that each derivation of a prime ring R can be uniquely extended to a derivation of Q, and so any derivation of R can be defined on the whole of Q. We refer to [2, 17] for more details.

2. THE MAIN RESULTS

We begin with lemmas

Lemma 2.1 — Let R be a prime ring with extended centroid C and $a, p, q \in R$. If $a \neq 0$ such that $a(p[x_1, x_2]^2 - [x_1, x_2](p+q)[x_1, x_2] + [x_1, x_2]^2 q)^n = 0$ for all $x_1, x_2 \in R$, where $n \geq 1$ a fixed integer, then either R satisfies a nontrivial generalized polynomial identity (GPI) or $p, q \in C$.

PROOF : Assume that R does not satisfy any nontrivial GPI. If R is commutative, trivially R satisfies a nontrivial GPI which is a contradiction. So, R must be noncommutative. Let $T = Q *_C C\{X_1, X_2\}$, the free product of Q and $C\{X_1, X_2\}$, the free C-algebra in noncommuting indeterminates X_1 and X_2. Then, since $a(p[x_1, x_2]^2 - [x_1, x_2](p+q)[x_1, x_2] + [x_1, x_2]^2 q)^n = 0$ is a GPI for R, we see that

$$a(p[X_1, X_2]^2 - [X_1, X_2](p+q)[X_1, X_2] + [X_1, X_2]^2 q)^n = 0 \quad (1)$$

in $T = Q *_C C\{X_1, X_2\}$. If $q \notin C$, then q and 1 are linearly independent over C. Thus, (1) implies

$$a(p[X_1, X_2]^2 - [X_1, X_2](p+q)[X_1, X_2] + [X_1, X_2]^2 q)^{n-1}[X_1, X_2]^2 q = 0 \quad (2)$$

in T and then by the same argument, we obtain $a([X_1, X_2]^2 q)^n = 0$ in T implying $q = 0$, since $a \neq 0$, a contradiction. Therefore, we conclude that $q \in C$ and hence (1) reduces to

$$a((p[X_1, X_2] - [X_1, X_2]p)[X_1, X_2])^n = 0 \tag{3}$$

in T. Assuming $Y = [X_1, X_2]$, (3) becomes

$$a((pY - Yp)Y)^{n-1}(pY - Yp)Y = 0. \tag{4}$$

Let $p \notin C$. Then p and 1 are linearly C-independent and hence (4) yields $a((pY - Yp)Y)^{n-1}YpY = 0$. Thus by the same argument, we have $a(YpY)^n = 0$ implying $p = 0$, a contradiction. Therefore, $p \in C$.

Lemma 2.2. — Let R be a noncommutative prime ring with extended centroid C and $a, p, q \in R$. Suppose that $a \neq 0$ such that $a(p[x_1, x_2]^2 - [x_1, x_2](p + q)[x_1, x_2] + [x_1, x_2]^2 q)^n = 0$ for all $x_1, x_2 \in R$, where $n \geq 1$ is a fixed integer. Then one of the following holds

(i) $p, q \in C$, unless R satisfies s_4;

(ii) char $(R) = 2$ and R satisfies s_4;

(iii) char $(R) \neq 2$, R satisfies s_4 and $p + q \in C$.

PROOF : By assumption, R satisfies generalized polynomial identity

$$g(x_1, x_2) = a(p[x_1, x_2]^2 - [x_1, x_2](p + q)[x_1, x_2] + [x_1, x_2]^2 q)^n. \tag{5}$$

By Lemma 2.1, $p, q \in C$ which gives the conclusion (i) unless R satisfies a nontrivial GPI. Now, assume that R satisfies a nontrivial GPI. Since R and Q satisfy same generalized polynomial identity (see [7]), Q satisfies $g(x_1, x_2)$. Moreover, if C is infinite, we have $g(x_1, x_2) = 0$ for all $x_1, x_2 \in Q \otimes_C \overline{C}$, where $\overline{C}$ is the algebraic closure of C. Since both Q and $Q \otimes_C \overline{C}$ are prime and centrally closed [9], we may replace R by Q or $Q \otimes_C \overline{C}$ according to C finite or infinite. Thus we may assume that $C = Z(R)$ and R is C-algebra centrally closed, which satisfies $g(x_1, x_2) = 0$. By Martindale's theorem [20], R is then a primitive ring and hence is isomorphic to a dense ring of linear transformations of a vector space V over C.

If $\dim_C V = 2$, then $R \cong M_2(C)$, that is, R satisfies s_4. In this case using the fact $[x, y]^2 \in Z(M_2(C))$ for all $x, y \in M_2(C)$, we have that R satisfies

$$a((p + q)[x_1, x_2]^2 - [x_1, x_2](p + q)[x_1, x_2])^n = 0 \tag{6}$$

that is,

$$a([p+q,[x_1,x_2]][x_1,x_2])^n = 0. \tag{7}$$

If char $(R)=2$, we obtain conclusion (ii) and if char $(R)\neq 2$, by [8, Lemma 2.1], we obtain $p+q \in Z(R)$, which is our conclusion (iii).

Suppose next that dim$_C V \geq 3$.

We show that for any $v \in V$, v and qv are linearly C-dependent. Suppose that v and qv are linearly independent for some $v \in V$. Since dim$_C V \geq 3$, there exists $u \in V$ such that v, qv, u are linearly C-independent set of vectors. By density, there exist $x_1, x_2 \in R$ such that

$$x_1v = v, \;\; x_1qv = -qv, \;\; x_1u = 0; \;\; x_2v = 0, \;\; x_2qv = u, \;\; x_2u = v.$$

Then $[x_1,x_2]v = 0$, $[x_1,x_2]qv = u$, $[x_1,x_2]^2qv = v$, and so $0 = a(p[x_1,x_2]^2 - [x_1,x_2](p+q)[x_1,x_2] + [x_1,x_2]^2q)^n v = av$.

This implies that if $av \neq 0$, by contradiction, we can say that v and qv are linearly C-dependent. Now choose $v \in V$ such that v and qv are linearly C-independent.

Then $av = 0$. Set $W = Span_C\{v, qv\}$. Since $a \neq 0$, there exists $w \in V$ such that $aw \neq 0$ and then $a(v-w) = -aw \neq 0$. By the previous argument we have that w, qw are linearly C-dependent and $(v-w), q(v-w)$ too. Thus there exist $\alpha, \beta \in C$ such that $qw = \alpha w$ and $q(v-w) = \beta(v-w)$. Then $qv = \beta(v-w) + qw = \beta(v-w) + \alpha w$ i.e., $(\alpha-\beta)w = qv - \beta v \in W$. Now $\alpha = \beta$ implies that $qv = \beta v$, a contradiction. Hence $\alpha \neq \beta$ and so $w \in W$. Again, if $u \in V$ with $au = 0$ then $a(w+u) \neq 0$. So, $w + u \in W$ forcing $u \in W$. Thus it is observed that $w \in V$ with $aw \neq 0$ implies $w \in W$ and $u \in V$ with $au = 0$ implies $u \in W$. This implies that $V = W$ i.e., dim$_C V = 2$, a contradiction.

Hence, in any case, v and qv are linearly C-dependent for all $v \in V$. Thus for each $v \in V$, $qv = \alpha_v v$ for some $\alpha_v \in C$. It is very easy to prove that α_v is independent of the choice of $v \in V$. Thus we can write $qv = \alpha v$ for all $v \in V$ and $\alpha \in C$ fixed. Now let $r \in R$, $v \in V$. Since $qv = \alpha v$,

$$[q,r]v = (qr)v - (rq)v = q(rv) - r(qv) = \alpha(rv) - r(\alpha v) = 0.$$

Thus $[q, r]v = 0$ for all $v \in V$ i.e., $[q, r]V = 0$. Since $[q, r]$ acts faithfully as a linear transformation on the vector space V, $[q, r] = 0$ for all $r \in R$. Therefore, $q \in Z(R)$.

Therefore, from (5) we have that R satisfies generalized polynomial identity

$$f(x_1, x_2) = a(p[x_1, x_2]^2 - [x_1, x_2]p[x_1, x_2])^n. \quad (8)$$

Now if v and pv are linearly C-independent for some $v \in V$, there exist $w \in V$ such that v, pv, w will be a linearly C-independent set of vectors, since $\dim_C V \geq 3$. Then again by density, there exists $x_1, x_2 \in R$ such that

$$x_1v = 0,\ \ x_1pv = v,\ \ x_1w = -v + 2pv;\ \ x_2v = pv,\ \ x_2pv = w,\ \ x_2w = 0.$$

In this case, we have $[x_1, x_2]v = v$, $[x_1, x_2]pv = -v + pv$ and hence $0 = a(p[x_1, x_2]^2 - [x_1, x_2]p[x_1, x_2])^n v = av$. Since $a \neq 0$, by the same argument as stated above, this leads a contradiction. Hence, $p \in C$.

Lemma 2.3 [16, Lemma 2] — Let R be a noncommutative simple algebra, finite-dimensional over its center Z. If $g(x_1, \ldots, x_t) \in R *_Z Z\{x_j\}$, the free product over Z, is an identity for R that is homogeneous in $\{x_1, \ldots, x_t\}$ of degree d, then for some field F and $n > 1$, $R \subseteq M_n(F)$ and $g(x_1, \ldots, x_t)$ is an identity for $M_n(F)$.

***Theorem* 2.4** — *Let R be a prime ring, d and δ two derivations of R, L a noncentral Lie ideal of R. Suppose that there exists $0 \neq a \in R$ such that $a(d(u)u - u\delta(u))^n = 0$ for all $u \in L$, where $n \geq 1$ is a fixed integer. Then one of the following holds:*

(i) $d = \delta = 0$ unless R satisfies s_4;

(ii) char $(R) = 2$ and R satisfies s_4;

(iii) char $(R) \neq 2$ and R satisfies s_4, d and δ are two inner derivations induced by p, q respectively such that $p + q \in C$.

PROOF : If char $(R) = 2$ and R satisfies s_4, we obtain our conclusion (ii). So, let either char $(R) \neq 2$ or R does not satisfy s_4. Since L is a noncentral Lie ideal of R, by Remark 1, there exists a nonzero ideal I of R such that $[I, I] \subseteq L$. Hence, by our assumption we have,

$$a(d([x_1, x_2])[x_1, x_2] - [x_1, x_2]\delta([x_1, x_2]))^n = 0 \quad (9)$$

for all $x_1, x_2 \in I$. Now we divide the proof in the two cases:

Case I : Let d and δ are both Q-inner derivations of R i.e., $d(x) = [p, x]$ for all $x \in R$ and $\delta(x) = [q, x]$ for all $x \in R$, where $p, q \in Q$. Then from (9), we obtain that

$$a(p[x_1, x_2]^2 - [x_1, x_2](p + q)[x_1, x_2] + [x_1, x_2]^2 q)^n = 0 \quad (10)$$

for all $x_1, x_2 \in I$. By Chuang [7], this GPI is also satisfied by Q and hence by R. By Lemma 2.2, if R does not satisfy s_4, $p, q \in C$, that is, $d = \delta = 0$. If char $(R) \neq 2$ and R satisfies s_4, then by Lemma 2.2, $p + q \in C$.

Case II : Next assume that d and δ are not both Q-inner derivations of R, but they are C-dependent modulo inner derivations of R. Suppose $d = \lambda\delta + ad_p$, that is, $d(x) = \lambda\delta(x) + [p, x]$ for all $x \in R$, where $\lambda \in C$, $p \in Q$. Then δ cannot be an inner derivation of R. From (9), we obtain that I satisfies

$$a\bigg(\lambda\delta([x_1, x_2])[x_1, x_2] + [p, [x_1, x_2]][x_1, x_2] - [x_1, x_2]\delta([x_1, x_2])\bigg)^n = 0.$$

Since δ is not inner derivation of R, by Kharchenko's theorem [13], we have that

$$a\bigg(\lambda([u, x_2] + [x_1, v])[x_1, x_2] + [p, [x_1, x_2]][x_1, x_2] - [x_1, x_2]([u, x_2] + [x_1, v])\bigg)^n = 0 \quad (11)$$

for all $x_1, x_2, u, v \in I$. By Chuang [7], this GPI is also satisfied by Q and hence by R. In particular for $u = v = 0$, we have that R satisfies

$$a\bigg([p, [x_1, x_2]][x_1, x_2])\bigg)^n = 0.$$

By Lemma 2.2, we get that $p \in C$. Hence, we get from (11) that R satisfies

$$a\bigg(\lambda([u, x_2] + [x_1, v])[x_1, x_2] - [x_1, x_2]([u, x_2] + [x_1, v])\bigg)^n = 0. \quad (12)$$

Since by our assumption L is noncentral, R must be noncommutative. Therefore, we can choose $q \in R$ such that $q \notin C$. Replacing u with $[q, x_1]$ and v with $[q, x_2]$, we get from (12) that R satisfies

$$a\bigg(\lambda[q, [x_1, x_2]][x_1, x_2] - [x_1, x_2][q, [x_1, x_2]]\bigg)^n = 0 \quad (13)$$

which gives

$$a\left(\lambda q[x_1,x_2]^2-[x_1,x_2](\lambda q+q)[x_1,x_2]+[x_1,x_2]^2 q\right)^n=0. \tag{14}$$

If R does not satisfy s_4, then by Lemma 2.2, $q\in C$, a contradiction. Therefore, R satisfies s_4, and then char $(R)\neq 2$. In this case, by Lemma 2.2, $\lambda q+q=(\lambda+1)q\in C$. If $(\lambda+1)\neq 0$, then $(\lambda+1)$ is invertible in C. Then $(\lambda+1)q\in C$ implies $q\in C$, which is a contradiction. Thus $\lambda+1=0$, that is, $\lambda=-1$. Then from (12), we have, R satisfies

$$a\left(-([u,x_2]+[x_1,v])[x_1,x_2]-[x_1,x_2]([u,x_2]+[x_1,v])\right)^n=0. \tag{15}$$

Replacing $u=x_1$ and $v=x_2$ and using char $(R)\neq 2$, we have that R satisfies

$$a[x_1,x_2]^{2n}=0. \tag{16}$$

For any two fixed $x,y\in R$, set $w=[x,y]^{2n}$. Then $aw=0$. From (16), we can write $a[u,wva]^{2n}=0$ for all $u,v\in R$. Since $aw=0$, it reduces to $a(uwva)^{2n}=0$. This can be written as $(wvau)^{2n+1}=0$ for all $u,v\in R$. By Levitzki's lemma [12, Lemma 1.1], $wvau=0$ for all $u,v\in R$. Since R is prime, either $a=0$ or $w=0$. Since $a\neq 0$, then $w=[x,y]^{2n}=0$ for all $x,y\in R$. This is a polynomial identity for R and so R is finite dimensional simple algebra. Since L is noncentral, R must be noncommutative. Then by Lemma 2.3, $R\subseteq M_k(F)$ and $M_k(F)$ satisfies $[x,y]^{2n}=0$ for $k>1$. This leads contradiction for $x=e_{12}$, $y=e_{21}$.

The situation when $\delta=\lambda d+ad_q$ is similar.

Next assume that d and δ are C-independent modulo inner derivations of R. Since neither d nor δ is Q-inner, by Kharchenko's theorem [13], we have from (9) that

$$a(([u_1,x_2]+[x_1,u_2])[x_1,x_2]-[x_1,x_2]([v_1,x_2]+[x_1,v_2]))^n=0 \tag{17}$$

for all $x_1,x_2,u_1,u_2\in I$. Since I, R and Q satisfy same generalized polynomial identities (see [7]), this GPI is also satisfied by R. Now assuming $v_1=v_2=0$ and replacing u_1 with $[q,x_1]$ and u_2 with $[q,x_2]$ for some $q\notin C$ in (17), we obtain that R satisfies

$$a([q,[x_1,x_2]][x_1,x_2])^n=0. \tag{18}$$

By Lemma 2.2, this gives $q \in C$, unless char $(R) = 2$ and R satisfies s_4, which is a contradiction. Hence the theorem is proved.

Now we begin to prove our next theorem.

From the proof of theorem 2.4, following lemma is straightforward.

Lemma 2.5. — Let R be a prime ring, d and δ two derivations of R, I a nonzero ideal of R. Suppose that there exists $0 \neq a \in R$ such that $a(d([x,y])[x,y] - [x,y]\delta([x,y])) = 0$ for all $x, y \in I$. Then $d = \delta = 0$ unless R satisfies s_4.

Lemma 2.6 — Let $R = M_k(F)$ be the ring of all $k \times k$ matrices over a field F and $k \geq 3$. Let a be an invertible element in R. If for some $p, q \in R$, $([p,[x,y]][x,y] - [x,y][q,[x,y]]) \in F.a^{-1}$ for all $x, y \in R$, then $p, q \in F \cdot I_k$.

PROOF : Let $p = (p_{ij})_{k\times k}$ and $q = (q_{ij})_{k\times k}$. By assumption, for every $x, y \in R$, $([p,[x,y]][x,y] - [x,y][q,[x,y]])$ is zero or invertible. We choose $x = e_{ii}, y = e_{ij}$ for $i \neq j$. Then $[x,y] = e_{ij}$ and so $([p,[x,y]][x,y] - [x,y][q,[x,y]]) = -(p+q)_{ji}e_{ij}$ which is not invertible. Therefore, $(p+q)_{ji}e_{ij} = 0$ implying $(p+q)_{ji} = 0$ for any $i \neq j$. This shows that $p+q$ is diagonal. Let $p + q = \sum\limits_{i=1}^{k} \alpha_{ii}e_{ii}$. For any F-automorphism θ of R, $(p+q)^\theta$ enjoys the same property as $p+q$ does, namely, $([p^\theta,[x,y]][x,y] - [x,y][q^\theta,[x,y]])$ is zero or invertible for every $x, y \in R$. Hence $(p+q)^\theta$ must be diagonal. For each $j \neq 1$, we have $(1+e_{1j})(p+q)(1-e_{1j}) = \sum\limits_{i=1}^{k} \alpha_{ii}e_{ii} + (\alpha_{jj} - \alpha_{11})e_{1j}$ diagonal. Therefore, $\alpha_{jj} = \alpha_{11}$ and so $p + q \in F.I_k$.

Thus our assumption reduces to, for every $x, y \in R$, $([p,[x,y]][x,y] + [x,y][p,[x,y]])$ is zero or invertible. Now since $k \geq 3$, we may choose $x = e_{ij}, y = e_{js} - e_{si}$ for i, j, s distinct integers. Then we have $([p,[x,y]][x,y] + [x,y][p,[x,y]]) = pe_{ij} - e_{ij}p$ is zero or invertible. Since rank of $pe_{ij} - e_{ij}p$ is ≤ 2, it can not be invertible and so $pe_{ij} - e_{ij}p = [p, e_{ij}] = 0$. Since $\{e_{ij} | i \neq j\}$ generates R as an F-algebra, $[p, R] = 0$, that is $p \in F \cdot I_k$.

Example : Let $R = M_2(D)$ be the complete 2×2 matrix ring over D, where D is any commutative domain of characteristic 2, and d, δ be two nonzero derivations of R. Choose

$$L = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix}; \quad a, b \in D \right\}$$

a noncentral Lie ideal of R. Then for $0 \neq a \in Z(R)$ and $\delta = -d$, we have $a(d(u)u - u\delta(u)) \in Z(R)$ for all $u \in L$.

***Theorem* 2.7** — *Let R be a prime ring with center $Z(R)$, d and δ two derivations of R and L a noncentral Lie ideal of R. Suppose that there exists $0 \neq a \in R$ such that $a(d(u)u - u\delta(u)) \in Z(R)$ for all $u \in L$. Then $d = \delta = 0$ unless R satisfies s_4.*

PROOF : Suppose that R does not satisfy s_4. Then by remark 1, we have $0 \neq [I, I] \subseteq L$, where I is a nonzero ideal of R. By our assumption, we have

$$a(d([x_1, x_2])[x_1, x_2] - [x_1, x_2]\delta([x_1, x_2])) \in Z(R) \tag{19}$$

for all $x_1, x_2 \in I$. If for all $x_1, x_2 \in I$, $a(d([x_1, x_2])[x_1, x_2]-[x_1, x_2]\delta([x_1, x_2])) = 0$, by Lemma 2.5, we conclude $d = \delta = 0$, as desired. So, let there exist $r_1, r_2 \in I$ such that $a(d([r_1, r_2])[r_1, r_2]-[r_1, r_2]\delta([r_1, r_2])) \neq 0$. Then $a(d([x_1, x_2])[x_1, x_2]-[x_1, x_2]\delta([x_1, x_2]))$ is a central differential identity for I. It follows from [6, Theorem 1] that R is a prime PI-ring and so $RC(= Q)$ is a finite-dimensional central simple C-algebra by Posner's theorem for prime PI-ring. Now divide the proof in the following two cases:

Case I : Let d and δ are both Q-inner derivations of R induced by $p, q \in Q$ respectively. Then

$$[a([p, [x_1, x_2]][x_1, x_2] - [x_1, x_2][q, [x_1, x_2]]), x_3] = 0 \tag{20}$$

for all $x_1, x_2 \in I$ and so for all $x_1, x_2 \in Q$, since I and Q satisfy same GPI [7]. Since $a([p, [r_1, r_2]][r_1, r_2] - [r_1, r_2][q, [r_1, r_2]]) \neq 0$ for some $r_1, r_2 \in I$, (20) is a nontrivial GPI for Q. Since Q is a finite-dimensional central simple C-algebra, it follows from Lemma 2.3 that there exists a suitable field F such that $Q \subseteq M_k(F)$, the ring of all $k \times k$ matrices over F, and moreover $M_k(F)$ satisfies (20). Since R does not satisfy s_4, $k \geq 3$. Therefore, we have

$$a([p, [x_1, x_2]][x_1, x_2] - [x_1, x_2][q, [x_1, x_2]]) \in Z(M_k(F))$$

for all $x_1, x_2 \in M_k(F)$. If for all $r_1, r_2 \in M_k(F)$, $a([p, [r_1, r_2]][r_1, r_2] - [r_1, r_2]$ $[q, [r_1, r_2]]) = 0$, then by Lemma 2.2, $p, q \in Z(M_k(F))$, that is, $d = \delta = 0$, as desired. So, let there exist $r_1, r_2 \in M_k(F)$ such that $a([p, [r_1, r_2]][r_1, r_2] - [r_1, r_2][q, [r_1, r_2]]) \neq 0$. Then a is invertible and so $([p, [r_1, r_2]][r_1, r_2] - [r_1, r_2]$ $[q, [r_1, r_2]]) \in F.a^{-1}$ for all $r_1, r_2 \in M_k(F)$. By Lemma 2.6, $p, q \in Z(M_k(F))$ implying $d = \delta = 0$.

Case II : Let d and δ are not both Q-inner derivations of R.

Assume that d and δ are C-dependent modulo inner derivations of R, say $d = \lambda\delta + ad_p$, where $\lambda \in C$, $p \in Q$ and $ad_p(x) = [p, x]$ for all $x \in R$. Then δ can not be Q-inner derivation of R. From (19), we obtain that

$$a(\lambda\delta([x_1, x_2])[x_1, x_2] + [p, [x_1, x_2]][x_1, x_2] - [x_1, x_2]\delta([x_1, x_2])) \in Z(R) \quad (21)$$

for all $x_1, x_2 \in I$. Since δ is not Q-inner derivation of R, by Kharchenko's theorem [13],

$$\begin{aligned} a(\lambda([u, x_2] + [x_1, v])[x_1, x_2] + [p, [x_1, x_2]][x_1, x_2] \\ -[x_1, x_2]([u, x_2] + [x_1, v])) \in Z(R) \end{aligned} \quad (22)$$

for all $x_1, x_2, u, v \in I$. Since L is noncentral, R must be noncommutative. Therefore, we may choose $b \in R$ such that $b \notin C$. Then replacing u with $[b, x_1]$ and v with $[b, x_2]$ in (22), we obtain that for all $x_1, x_2, x_3 \in I$

$$a(\lambda[b, [x_1, x_2]][x_1, x_2] + [p, [x_1, x_2]][x_1, x_2] - [x_1, x_2][b, [x_1, x_2]]) \in Z(R) \quad (23)$$

which gives

$$[a([\lambda b + p, [x_1, x_2]][x_1, x_2] - [x_1, x_2][b, [x_1, x_2]]), x_3] = 0. \quad (24)$$

This is similar to (20). Then by the same argument as above, $b \in C$, a contradiction.

The situation when $\delta = \lambda d + ad_q$ is similar.

Next assume that d and δ are C-independent modulo inner derivations of R. Since neither d nor δ is inner, by Kharchenko's theorem [13], we have from (19) that I satisfies

$$a(([u_1, x_2] + [x_1, u_2])[x_1, x_2] - [x_1, x_2]([v_1, x_2] + [x_1, v_2])) \in Z(R). \quad (25)$$

Then assuming $v_1 = v_2 = 0$ and replacing u_1 with $[b, x_1]$ and u_2 with $[b, x_2]$ for some $b \notin C$ in (25), we obtain that for all $x_1, x_2, x_3 \in I$

$$[a[b, [x_1, x_2]][x_1, x_2], x_3] = 0. \quad (26)$$

Then by the same argument as given in case-1, $b \in C$, a contradiction.

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