*Indian J. Pure Appl. Math.*, **42**(5): 371-386, October 2011


# SERIES TRANSFORMATION FORMULAS OF EULER TYPE, HADAMARD PRODUCT OF SERIES, AND HARMONIC NUMBER IDENTITIES

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(*Received* 17 *August* 2010; *after final revision* 3 *June* 2011;
*accepted* 16 *August* 2011)

The Hadamard multiplication theorem for series is used to establish several Euler-type series transformation formulas. As applications we obtain a number of binomial identities involving harmonic numbers and an identity for the Laguerre polynomials. We also evaluate in a closed form certain power series with harmonic numbers.

**Key words** : Harmonic number, Stirling number, binomial transform, binomial identity, Euler transform, Hadamard theorem for series, Laguerre polynomial.

## 1. INTRODUCTION

Euler's series transformation formula is an important tool for working with power series. Given a function holomorphic in a neighborhood of the origin

$$f(t) = \sum_{k=0}^{\infty} a_k t^k, \tag{1.1}$$

Euler's series transformation formula says that

$$\frac{1}{1-t} f\left(\frac{t}{1-t}\right) = \sum_{n=0}^{\infty} t^n \left\{ \sum_{k=0}^{n} \binom{n}{k} a_k \right\} \tag{1.2}$$

for $|t|$ small enough (see [12, pp. 468-470] and also [1], [9]). Euler's formula can be used, among other things, to evaluate the binomial expression

$$c_n = \sum_{k=0}^{n} \binom{n}{k} a_k \quad (n = 0, 1, 2...), \tag{1.3}$$

which is called the *binomial transform* of the sequence $\{a_k\}$. Some examples can be found in [1] and [9]. With the substitution

$$\frac{t}{1-t} = z, \ \ t = \frac{z}{z+1}, \tag{1.4}$$

formula (1.2) takes the form

$$f(z) = \frac{1}{z+1} \sum_{n=0}^{\infty} \left(\frac{z}{z+1}\right)^n \left\{ \sum_{k=0}^{n} \binom{n}{k} a_k \right\}. \tag{1.5}$$

A more general version of this formula is (2.4) below. There is also an 'exponential' version of (1.5), namely, the transformation formula (2.8).

In this paper we want to point out a useful connection between Euler's transformation formulas and Hadamard's multiplication theorem for series (see [18, section 4.6])

$$\sum_{n=0}^{\infty} a_n b_n z^n = \frac{1}{2\pi i} \oint_L g\left(\frac{z}{\lambda}\right) f(\lambda) \frac{d\lambda}{\lambda}, \tag{1.6}$$

where $L$ is an appropriate closed curve around the origin and

$$g(t) = \sum_{k=0}^{\infty} b_k t^k \tag{1.7}$$

is a second analytic function like $f(t)$. By choosing the function $g(t)$ appropriately one can generate transformation formulas like (1.5), (2.4), (2.8), and (2.29). In this way we shall give new unified proofs of these formulas (propositions 1, 2 and 5) and also obtain two new transformation formulas for series (propositions 3 and 4).

These formulas are illustrated by a number of applications. Our main results are propositions 3, 4, and corollaries 1 and 2, all in section 2. Corollary 2, for instance, gives the evaluation in a closed form of the harmonic number series

$$\sum_{n=0}^{\infty}(H_{p+n}-H_p)\binom{n+p}{n}n^m z^n, \tag{1.8}$$

for any integer $m \geq 0$, where

$$H_\alpha = \psi(\alpha+1)+\gamma, \quad Re\ \alpha > -1, \tag{1.9}$$

are the extended harmonic numbers (here $\psi(z)$ is the digamma (or psi) function and $\gamma = -\psi(1)$ is Euler's constant [16]). When $\alpha = k \geq 0$ is an integer, then

$$H_k = 1+\frac{1}{2}+...+\frac{1}{k},\ H_0 = 0 \tag{1.10}$$

are the usual harmonic numbers.

In sections 3, 4, and 5 we present more applications – identities for the Laguerre polynomials and for the harmonic numbers.

## 2. EULER-TYPE SERIES TRANSFORMATIONS

Throughout the paper the two functions $f(t),\ g(t)$ are defined by the series (1.1) and (1.7) above. We shall need an important lemma.

*Lemma* 1 — The following representation holds

$$\sum_{n=0}^{\infty} b_n\, h(z)^n \left\{\sum_{k=0}^{n}\binom{n}{k}a_k\right\} = \frac{1}{2\pi i}\oint_L g(\, h(z)(1+\frac{1}{\lambda}))f(\lambda)\frac{d\lambda}{\lambda}, \tag{2.1}$$

where $h(z)$ is an appropriate function for which the above expression is defined and the integral is a Cauchy type integral on a closed curve around the origin, as in (1.6).

PROOF : By Cauchy's integral formula we have for $k = 0, 1, ...,$

$$a_k = \frac{1}{2\pi i}\oint_L \frac{f(\lambda)d\lambda}{\lambda^{k+1}}. \tag{2.2}$$

From here,

$$\sum_{k=0}^{n} \binom{n}{k} a_k = \frac{1}{2\pi i} \oint_L \left(1+\frac{1}{\lambda}\right)^n f(\lambda) \frac{d\lambda}{\lambda}. \tag{2.3}$$

Multiplying this identity by $b_n\, h(z)^n$ and summing for *n* we obtain (2.1). □

We shall use this lemma in the following way: by choosing $g(t)$ and $h(z)$ appropriately, we shall write the integrand in (2.1) in a form that has the same structure as the integrand in (1.6). This means the function $g(\,h(z)(1+\frac{1}{\lambda}))$ will be written in terms of function(s) in $z$ and function(s) in $z/\lambda$ only. Then we shall compare (2.1) and (1.6) in order to derive the desired representation. This technique is demonstrated in the next proposition.

*Proposition* 1 — Let $\alpha$ be a complex number. Then the following representation extending (1.5) is true

$$\sum_{n=0}^{\infty} \binom{\alpha}{n} (-1)^n a_n z^n =$$

$$(z+1)^\alpha \sum_{n=0}^{\infty} \left(\frac{z}{z+1}\right)^n \binom{\alpha}{n} (-1)^n \left\{ \sum_{k=0}^{n} \binom{n}{k} a_k \right\}. \tag{2.4}$$

PROOF : Choose

$$h(z) = \frac{z}{z+1},\ \ g(t) = (1-t)^\alpha$$

$$= \sum_{n=0}^{\infty} \binom{\alpha}{n} (-1)^n t^n,\ \ b_n = \binom{\alpha}{n} (-1)^n. \tag{2.5}$$

A simple computation shows that

$$g\left(h(z)\left(1+\frac{1}{\lambda}\right)\right) = (z+1)^{-\alpha} \left(1-\frac{z}{\lambda}\right)^\alpha, \tag{2.6}$$

and (2.1) takes the form

$$\sum_{n=0}^{\infty} b_n \left(\frac{z}{z+1}\right)^n \left\{ \sum_{k=0}^{n} \binom{n}{k} a_k \right\}$$

$$= \frac{(1+z)^{-\alpha}}{2\pi i} \oint_L \left(1 - \frac{z}{\lambda}\right)^{\alpha} f(\lambda) \frac{d\lambda}{\lambda}. \tag{2.7}$$

This representation yields (2.4) in view of (1.6).

When $\alpha = -1$ we have $\binom{-1}{n} = (-1)^n$ and (2.4) becomes (1.5).

Formula (2.4) originates from Euler and can be found in [13, p. 294].

*Proposition* 2 — The following (exponential) version of Euler's series transformation formula holds

$$\sum_{n=0}^{\infty} \frac{a_n}{n!} z^n = e^{-z} \sum_{n=0}^{\infty} \frac{z^n}{n!} \left\{ \sum_{k=0}^{n} \binom{n}{k} a_k \right\}. \tag{2.8}$$

PROOF : Take $h(z) = z,\ g(t) = e^t$. Then (2.1) becomes

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} \left\{ \sum_{k=0}^{n} \binom{n}{k} a_k \right\} = \frac{e^z}{2\pi i} \oint_L e^{\frac{z}{\lambda}} f(\lambda) \frac{d\lambda}{\lambda}, \tag{2.9}$$

and (2.8) follows from (1.6). □

This exponential transformation formula can be found in [9] with a simple direct proof.

In the next two examples we use the natural logarithmic function. To our knowledge, these results are new. In all expansions we assume that $|z|$ is small enough to secure convergence.

*Proposition* 3 — With $f(t)$ as in (1.1) the following representation holds

$$f(0)\log(1+z) + \sum_{n=1}^{\infty} \frac{z^n}{n} a_n = \sum_{n=1}^{\infty} \left(\frac{z}{z+1}\right)^n \frac{1}{n} \left\{ \sum_{k=0}^{n} \binom{n}{k} a_k \right\}. \tag{2.10}$$

PROOF : In formula (2.1) we put

$$h(z) = \frac{z}{z+1},\ g(t) = -\log(1-t) = \sum_{n=1}^{\infty} \frac{t^n}{n},\ \ b_n = \frac{1}{n}. \tag{2.11}$$

Then

$$g\left(h(z)\left(1 + \frac{1}{\lambda}\right)\right) = \log(1+z) - \log\left(1 - \frac{z}{\lambda}\right), \tag{2.12}$$

and the right hand side in (2.1) becomes

$$\log(1+z)\frac{1}{2\pi i}\oint_L f(\lambda)\frac{d\lambda}{\lambda} \;-\; \frac{1}{2\pi i}\oint_L \log(1-\frac{z}{\lambda})f(\lambda)\frac{d\lambda}{\lambda}. \tag{2.13}$$

Obviously, (2.10) follows from here. The first integral in (2.13) is $a_0\log(1+z)$.

We now use the series expansion [11, (7.43), p. 351-352]

$$g(t) = \frac{-\log(1-t)}{(1-t)^{p+1}} = \sum_{n=0}^{\infty}(H_{p+n}-H_p)\binom{n+p}{n}t^n, \tag{2.14}$$

where Re$p > -1$.

*Proposition* 4 — For any $p$ with Re$p > -1$ we have

$$\sum_{n=0}^{\infty}(H_{p+n}-H_p)\binom{n+p}{n}a_n z^n + \log(1+z)\sum_{n=0}^{\infty}\binom{n+p}{n}a_n z^n$$

$$= \frac{1}{(1+z)^{p+1}}\sum_{n=0}^{\infty}\left(\frac{z}{z+1}\right)^n (H_{p+n}-H_p)\binom{n+p}{n}\left\{\sum_{k=0}^{n}\binom{n}{k}a_k\right\}. \tag{2.15}$$

PROOF : With $g(t)$ as above in (2.14), and

$$h(z) = \frac{z}{z+1}, \tag{2.16}$$

a simple computation gives

$$g\left(h(z)\left(1+\frac{1}{\lambda}\right)\right) = (1+z)^{p+1}\left[\frac{\log\,(1+z)}{(1-\frac{z}{\lambda})^{p+1}} - \frac{\log\left(1-\frac{z}{\lambda}\right)}{(1-\frac{z}{\lambda})^{p+1}}\right], \tag{2.17}$$

and therefore, the right hand side of (2.1) becomes

$$(1+z)^{p+1}\left[\frac{\log(1+z)}{2\pi i}\oint_L \frac{1}{(1-\frac{z}{\lambda})^{p+1}}\frac{f(\lambda)d\lambda}{\lambda}\right.$$

$$\left. -\frac{1}{2\pi i}\oint_L \frac{\log\,(1-\frac{z}{\lambda})^{p+1}}{(1-\frac{z}{\lambda})^{p+1}}\frac{f(\lambda)d\lambda}{\lambda}\right]. \tag{2.18}$$

According to (1.6), this expression equals the left hand side of equation (2.15) multiplied by the factor $(1+z)^{p+1}$. Note that

$$\frac{1}{(1-t)^{p+1}} = \sum_{n=0}^{\infty} \binom{n+p}{n} t^n. \tag{2.19}$$

The entire identity (2.15) follows now from Lemma 1, as in this case

$$b_n = (H_{p+n} - H_p) \binom{n+p}{n}. \tag{2.20}$$

When $p = 0$, then $b_n = H_n$ and we have the corollary:

*Corollary* 1 — With $f(t)$as in (1), the following series transformation formula holds

$$\sum_{n=0}^{\infty} H_n \, a_n \, z^n + \log(1+z) f(z) = \frac{1}{1+z} \sum_{n=0}^{\infty} \left( \frac{z}{z+1} \right)^n H_n \left\{ \sum_{k=0}^{n} \binom{n}{k} a_k \right\}. \tag{2.21}$$

In our next corollary we present one interesting particular case of (2.15). Let $m \geq 0$ be an integer. Taking $a_k = (-1)^k k^m$ and changing $z$ to $-z$ in (2.15) we obtain the identity

$$\sum_{n=0}^{\infty} (H_{p+n} - H_p) \binom{n+p}{n} n^m z^n + \log(1-z) \sum_{n=0}^{\infty} \binom{n+p}{n} n^m z^n = \frac{1}{(1-z)^{p+1}} \sum_{n=0}^{\infty} \left( \frac{z}{1-z} \right)^n (H_{p+n} - H_p) \binom{n+p}{n} (-1)^n \left\{ \sum_{k=0}^{n} \binom{n}{k} (-1)^k k^m \right\}. \tag{2.22}$$

We shall use now the representation

$$\sum_{n=0}^{\infty} \binom{n+p}{n} n^m z^n = \frac{1}{(1-z)^{p+1}} \omega_{m,p+1} \left( \frac{z}{1-z} \right) \tag{2.23}$$

from [2]. Here $\omega_{m,p+1}$ are the generalized geometric polynomials

$$\omega_{m,p+1}(x) = \frac{1}{\Gamma(p+1)} \sum_{k=0}^{m} S(m,k)\,\Gamma(p+k+1)\,x^k, \tag{2.24}$$

introduced in [2], and $S(m,k)$ are the Stirling numbers of the second kind [7], [11]. Also, in the right hand side of (2.22) we shall use the well known representation of the numbers $S(m,k)$

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} k^m = n!S(m,n), \tag{2.25}$$

to obtain the following.

*Corollary* 2 — For every Re$p > -1$ and for every integer $m \geq 0$, with the polynomials $\omega_{m,p+1}$ defined in (2.24), we have the following harmonic number summation

$$\sum_{n=0}^{\infty} (H_{p+n} - H_p) \binom{n+p}{n} n^m z^n$$
$$= \frac{1}{(1-z)^{p+1}} \left\{ -\log(1-z)\,\omega_{m,p+1}\left(\frac{z}{1-z}\right) \right.$$
$$\left. + \sum_{n=0}^{m} \left(\frac{z}{1-z}\right)^n (H_{p+n} - H_p) \binom{n+p}{n} n!S(m,n) \right\}. \tag{2.26}$$

Notice that the sum on the right hand side is now finite, as $S(m,n) = 0$ for $n > m$. When $p = 0$, (2.26) turns into the shorter summation formula

$$\sum_{n=0}^{\infty} H_n\, n^m z^n = \frac{1}{1-z} \left\{ -\log(1-z)\,\omega_m\left(\frac{z}{1-z}\right) + \sum_{n=0}^{m} \left(\frac{z}{1-z}\right)^n H_n\, n!S(m,n) \right\}, \tag{2.27}$$

where,

$$\omega_m(x) = \omega_{m,1}(x) = \sum_{k=0}^{m} S(m,k)\,k!x^k \tag{2.28}$$

are the geometric polynomials as defined in [2].

The representations (2.26) and (2.27) can also be derived directly from the summation formula (3.23) in [2]. This was done by Dil and Kurt in the recent paper [8].

For completeness, at the end of this section we obtain from (1.6) another interesting series transformation formula, also attributed to Euler.

*Proposition* 5 — Given two analytic functions $f(t)$ and $g(t)$ as in (1.1) and (1.7), the following representation is true.

$$\sum_{n=0}^{\infty} a_n b_n t^n = \sum_{n=0}^{\infty} \frac{g^{(n)}(-t)}{n!} t^n \left\{ \sum_{k=0}^{n} \binom{n}{k} a_k \right\}. \tag{2.29}$$

This transformation formula can be found in a modified form in [15, Chapter 6, Problem 19, p. 245].

PROOF : Multiplying both sides in equation (2.3) by

$$\frac{g^{(n)}(-t)}{n!} t^n \tag{2.30}$$

and summing for $n$ we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{g^{(n)}(-t)}{n!} t^n \left\{ \sum_{k=0}^{n} \binom{n}{k} a_k \right\} &= \frac{1}{2\pi i} \oint_L \sum_{n=0}^{\infty} \frac{g^{(n)}(-t)}{n!} \left( \frac{t}{\lambda} + t \right)^n f(\lambda) \frac{d\lambda}{\lambda} \\ &= \frac{1}{2\pi i} \oint_L g\left( \frac{t}{\lambda} \right) f(\lambda) \frac{d\lambda}{\lambda}, \end{aligned} \tag{2.31}$$

(recognizing the Taylor expansion of $g\left(\frac{t}{\lambda}\right)$ centered at $-t$ inside the first integral). Now (2.29) follows from (1.6). □

## 3. AN IDENTITY FOR THE LAGUERRE POLYNOMIALS

To illustrate how the above formulas can be used we shall provide some examples. The first application involves the Laguerre polynomials $L_n(x) = \frac{e^x}{n!} \frac{d^n}{dx^n}(e^{-x} x^n)$, see [16].

*Corollary* 3 — For the Laguerre polynomials $L_n(x)$, $n = 1, 2, ...$, we have

$$\int_0^x \frac{L_n(t) - 1}{t} \, dt = \sum_{k=1}^{n} \frac{L_k(x) - 1}{k}. \tag{3.1}$$

PROOF : Take an arbitrary $x$ and set in (2.10)

$$f(t)=\sum_{k=0}^{\infty}\frac{(-xt)^k}{k!}=e^{-xt}, \quad a_k=\frac{(-x)^k}{k!}. \tag{3.2}$$

Thus

$$\log(1+z)\ +\ \sum_{n=1}^{\infty}\frac{z^n(-x)^n}{n!n}\ =\sum_{n=1}^{\infty}\left(\frac{z}{z+1}\right)^n\frac{1}{n}\,L_n(x), \tag{3.3}$$

because of the well-known identity [16, p. 153],

$$\sum_{k=0}^{n}\binom{n}{k}\frac{(-x)^k}{k!}\ =\ L_n(x). \tag{3.4}$$

With the substitution (1.4) equation (3.3) can be written in the form

$$-\log(1-t)\ +\ \sum_{n=1}^{\infty}\left(\frac{t}{1-t}\right)^n\frac{(-x)^n}{n!n}\ =\sum_{n=1}^{\infty}\frac{t^n}{n}\,L_n(x), \tag{3.5}$$

and we divide now both sides by $1-t$ to obtain

$$\frac{-\log(1-t)}{1-t}\ +\ \frac{1}{1-t}\sum_{n=1}^{\infty}\left(\frac{t}{1-t}\right)^n\frac{(-x)^n}{n!n}\ =\sum_{n=1}^{\infty}t^n\sum_{k=1}^{n}\frac{L_k(x)}{k}\ , \tag{3.6}$$

(for the last equality we use property (5.6) below). From (3.4), dividing by $x$ and integrating we find

$$\sum_{k=1}^{n}\binom{n}{k}\frac{(-x)^k}{k!k}\ =\ \int_0^x\frac{L_n(t)-1}{t}\,dt, \tag{3.7}$$

and therefore, from (1.2)

$$\frac{1}{1-t}\sum_{n=1}^{\infty}\left(\frac{t}{1-t}\right)^n\frac{(-x)^n}{n!n}\ =\sum_{n=1}^{\infty}t^n\int_0^x\frac{L_n(t)-1}{t}\,dt\ . \tag{3.8}$$

At the same time

$$\frac{-\log(1-t)}{1-t}\ =\sum_{n=1}^{\infty}H_n\,t^n\ . \tag{3.9}$$

Substituting (3.8) and (3.9) in (3.6) and comparing coefficients we arrive at (3.1).

## 4. A SYMMETRIC IDENTITY FOR HARMONIC NUMBERS

First we obtain an equivalent version of the representation (2.14).

*Corollary* 4 — For any complex $\alpha$ and $|t| < 1$ we have

$$\frac{-\log(1-t)}{(1-t)^{\alpha+1}} = \sum_{n=0}^{\infty} t^n \left\{ \binom{n+\alpha}{n} H_n - \sum_{k=1}^{n} \binom{n}{k}\binom{\alpha}{k} H_k \right\}. \tag{4.1}$$

PROOF : We set in (2.21)

$$f(z) = (1+z)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} z^n, \quad a_k = \binom{\alpha}{n} \tag{4.2}$$

to get

$$\sum_{n=0}^{\infty} \binom{\alpha}{n} H_n z^n + (1+z)^\alpha \log(1+z)$$
$$= \frac{1}{1+z} \sum_{n=0}^{\infty} \left(\frac{z}{z+1}\right)^n H_n \binom{n+\alpha}{n}, \tag{4.3}$$

by using on the RHS the well-known Vandermonde identity [10],

$$\sum_{k=0}^{n} \binom{n}{k}\binom{\alpha}{k} = \binom{n+\alpha}{n}. \tag{4.4}$$

With the substitution (1.4) we can write (4.3) in the form

$$\frac{1}{1-t} \sum_{n=0}^{\infty} \left(\frac{t}{1-t}\right)^n \binom{\alpha}{n} H_n - \frac{\log(1-t)}{(1-t)^{\alpha+1}}$$
$$= \sum_{n=0}^{\infty} H_n \binom{n+\alpha}{n} t^n. \tag{4.5}$$

Next, for the sum on the left hand side we have, according to (1.2),

$$\frac{1}{1-t} \sum_{n=0}^{\infty} \left(\frac{t}{1-t}\right)^n \binom{\alpha}{n} H_n = \sum_{n=0}^{\infty} t^n \left\{ \sum_{k=0}^{n} \binom{n}{k}\binom{\alpha}{k} H_k \right\}. \tag{4.6}$$

Substituting this in (4.5) yields the representation (4.1).

*Corollary* 5 — For every integer $n \geq 0$ and every complex number $\alpha$ with $Re\ \alpha > -1$ we have

$$\sum_{k=0}^{n} \binom{n}{k}\binom{\alpha}{k} H_k = \binom{n+\alpha}{n}(H_\alpha + H_n - H_{\alpha+n}). \tag{4.7}$$

PROOF : The identity follows immediately by comparing the representations (4.1) and (2.14) (with $\alpha = p$). □

Two similar identities with integer indices can be found on p.355 in [11], namely, (7.63) and (7.64). Identity (A10) in [4] is of the same nature. With $\alpha = n$ in (4.7) we obtain

$$\sum_{k=0}^{n} \binom{n}{k}^2 H_k = \binom{2n}{n}(2H_n - H_{2n}) \tag{4.8}$$

which is (A11) in [4]. See also [3] for more general results extending (4.8).

## 5. MORE HARMONIC NUMBER IDENTITIES

*Corollary* 6 — For every complex $\alpha$ and every positive integer $n$ we have

$$\sum_{k=0}^{n} \binom{\alpha+1}{n-k}\binom{\alpha+k}{k}(-1)^{n-k} H_k \;=\; \frac{1}{n} \;+\; \binom{\alpha}{n}\frac{(-1)^{n-1}}{n}, \tag{5.1}$$

$$\sum_{k=0}^{n} \binom{\alpha}{n-k}\binom{\alpha+k}{k}(-1)^{n-k} H_k \;=\; H_n + \sum_{k=1}^{n}\binom{\alpha}{k}\frac{(-1)^{k-1}}{k}. \tag{5.2}$$

In particular, when $\alpha = n$, these identities become correspondingly,

$$\sum_{k=0}^{n} \binom{n+1}{n-k}\binom{n+k}{k}(-1)^{n-k} H_k \;=\; \frac{1+(-1)^{n-1}}{n}, \tag{5.3}$$

$$\sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}(-1)^{n-k} H_k \;=\; 2H_n. \tag{5.4}$$

The identities (5.3) and (5.4) are similar to the identities (3.123) and (3.122) in [10]. Identity (5.4) was obtained by Prodinger in [14] together with several other identities of this kind. See also [6, pp.62-64].

Many interesting identities of this nature (for harmonic number and binomial coefficients) can be found in [5], [6] and [17]. The recent paper [6] is very informative and highly recommended.

For the proof of Corollary 6 we need a simple lemma.

*Lemma* 2 — For any power series as in (1.1) we have

$$(1+\lambda\, t)^{\alpha} \sum_{n=0}^{\infty} a_n t^n = \sum_{n=0}^{\infty} t^n \left\{ \sum_{k=0}^{n} \binom{\alpha}{n-k} a_k \lambda^{n-k} \right\}, \tag{5.5}$$

for every complex $\alpha, \lambda$ with $|\lambda\, t|$ small enough. In particular,

$$\frac{1}{1-t} \sum_{n=0}^{\infty} a_n t^n = \sum_{n=0}^{\infty} t^n \left\{ \sum_{k=0}^{n} a_k \right\}. \tag{5.6}$$

The proof is left to the reader. After expanding $(1+\lambda\, t)^{\alpha}$ one can either change the order of summation or use the Cauchy rule for multiplication of power series.

*Proof of the corollary* — We put $a_k = (-1)^{k-1} H_k$ in (2.4) to obtain

$$\sum_{n=0}^{\infty} \binom{\alpha}{n} H_n z^n = (z+1)^{\alpha} \sum_{n=1}^{\infty} \left( \frac{z}{z+1} \right)^n \binom{\alpha}{n} \frac{(-1)^{n-1}}{n}, \tag{5.7}$$

in view of the well-known binomial transform

$$\sum_{k=1}^{n} \binom{n}{k} (-1)^{k-1} H_k = \frac{1}{n}. \tag{5.8}$$

At the same time, from (4.3),

$$\sum_{n=0}^{\infty} \binom{\alpha}{n} H_n z^n + (1+z)^{\alpha} \log(1+z)$$

$$= \frac{1}{1+z} \sum_{n=0}^{\infty} \left( \frac{z}{z+1} \right)^n H_n \binom{\alpha+n}{n}, \tag{5.9}$$

so that replacing the first sum here by the right hand side of (5.7) yields

$$(z+1)^{\alpha}\sum_{n=1}^{\infty}\left(\frac{z}{z+1}\right)^{n}\binom{\alpha}{n}\frac{(-1)^{n-1}}{n}+(1+z)^{\alpha}\log(1+z)$$
$$=\frac{1}{1+z}\sum_{n=0}^{\infty}\left(\frac{z}{z+1}\right)^{n}H_n\binom{\alpha+n}{n}. \tag{5.10}$$

The substitution (1.4) brings (5.10) to the form

$$\sum_{n=1}^{\infty}t^n\binom{\alpha}{n}\frac{(-1)^{n-1}}{n}-\log(1-t)$$
$$=(1-t)^{\alpha+1}\sum_{n=0}^{\infty}t^nH_n\binom{\alpha+n}{n}, \tag{5.11}$$

and according to (5.6) this can be written as

$$\sum_{n=1}^{\infty}t^n\binom{\alpha}{n}\frac{(-1)^{n-1}}{n}-\log(1-t)$$
$$=\sum_{n=0}^{\infty}t^n\left\{\sum_{k=0}^{n}\binom{\alpha+1}{n-k}\binom{\alpha+k}{k}(-1)^{n-k}H_k\right\}. \tag{5.12}$$

Now using the expansion of the logarithm

$$-\log(1-t)=\sum_{n=1}^{\infty}\frac{t^n}{n}, \tag{5.13}$$

and comparing coefficients in (5.12) we arrive at (5.1). In order to obtain (5.2) we write (5.11) in an equivalent form, dividing it by $1-t$,

$$\frac{1}{1-t}\sum_{n=1}^{\infty}t^n\binom{\alpha}{n}\frac{(-1)^{n-1}}{n}-\frac{\log(1-t)}{1-t}=(1-t)^{\alpha}\sum_{n=0}^{\infty}t^nH_n\binom{\alpha+n}{n}. \tag{5.14}$$

According to Lemma 2 and (3.9) this becomes

$$\sum_{n=0}^{\infty}t^n\left\{\sum_{k=1}^{n}\binom{\alpha}{k}\frac{(-1)^{k-1}}{k}\right\}+\sum_{n=1}^{\infty}H_nt^n$$
$$=\sum_{n=0}^{\infty}t^n\left\{\sum_{k=0}^{n}\binom{\alpha}{n-k}\binom{\alpha+k}{k}(-1)^{n-k}H_k\right\}, \tag{5.15}$$

and yields (5.2) after comparing coefficients. □

The author is very grateful to the referee for a number of valuable and competent remarks.

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