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PROJECTION METHODS FOR COMPUTING MOORE-PENROSE INVERSES OF UNBOUNDED OPERATORS

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In this article we give a characterization of the convergence of projection methods which are useful for approximating the Moore-Penrose inverse of a closed densely defined operator between Hilbert spaces. We illustrate the main theorem with an example. Also a procedure for constructing the admissible sequence of projections is discussed.

Key words : Densely defined operator; closed operator; Moore-Penrose inverse; generalized projection method.

1. Introduction

Projection methods are efficient and widely used tools to approximate the solution of a given operator equation. In this method the infinite-dimensional problem (operator equation) can be reduced to a sequence of finite-dimensional operator equations (matrix equations). Hence these matrix equations can be solved with the help of the known techniques of the finite-dimensional case.

So this method has advantages from theoretical as well as computational point of view.

Our main aim is to solve the operator equation

$$Tx = y, \tag{1}$$

where T is a densely defined, closed and unbounded operator between Hilbert spaces H_1 and H_2. In an earlier paper [12], we studied this problem with the assumption that T has a bounded inverse. In the present paper, we do not make this assumption. Consequently, we look for the least square solution of minimal norm of the equation $Tx = y$.

In this article we give a necessary and sufficient condition for the convergence of projection methods to such a least square solution of minimal norm.

To approximate the Moore-Penrose inverse of an operator by projection methods, first we should be able to approximate the given operator with the help of a pair of sequences of projections. We call such a pair admissible for the given operator (see Definition 3.2).

We can always find an admissible sequence of projections if the operator is bounded. But, since unbounded operators are defined on subspaces of Hilbert space, we have to impose some conditions on the sequence of projections. Thus it is difficult to find an admissible sequence in this case. In this article we shall prove that such a sequence of projections can be constructed by introducing some new operators (see Section 2).

The projection methods discussed in this article generalize the results of the article [12] in three directions: First, we extend the results of [12] for the usual inverse to the Moore-Penrose inverse. In this article we consider operators between different Hilbert spaces, whereas in [12] we considered operators on one separable Hilbert space. The third is that we assume that the range of the operator is just separable instead of assuming that the whole space is separable.

This paper is organized as follows: In the second section we define the convergence of generalized projection methods and give a necessary and sufficient condition for the convergence. We illustrate this method with an example. In the third section the existence of admissible sequence of projections is discussed.

2. Notations and Preliminaries

Throughout the paper we denote infinite-dimensional, complex Hilbert spaces by $H,\ H_1,\ H_2,\ H_3$, and the inner product and the corresponding norm on a Hilbert space by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ respectively.

We write $\mathcal{L}(H_1,H_2)$: for the set of all linear operators between H_1 and H_2,

and set $\mathcal{L}(H) := \mathcal{L}(H,H)$.

If $T \in \mathcal{L}(H_1,H_2)$, then the domain, null space and the range space of T are denoted by $D(T)$, $N(T)$ and $R(T)$, respectively. For $S,T \in \mathcal{L}(H_1,H_2)$ and $U \in \mathcal{L}(H_2,H_3)$, we have $D(S+T) = D(S) \cap D(T)$ and $(T+S)x = Tx + Sx$ for all $x \in D(T+S)$. The domain of the operator UT is given by $D(UT) = \{x \in D(T) : Tx \in D(U)\}$, and in this case $UTx = U(Tx)$ for all $x \in D(T)$.

We write $\mathcal{B}(H_1,H_2)$: for the space of all bounded linear operators from H_1 into H_2,

and set $\mathcal{B}(H) := \mathcal{B}(H,H)$.

The graph $G(T)$ of $T \in \mathcal{L}(H_1,H_2)$ is defined as $G(T) := \{(x,Tx) : x \in D(T)\}$. If $G(T)$ is closed in $H_1 \times H_2$, then T is a *closed operator*.

We set $\mathcal{C}(H_1,H_2) := \{T \in \mathcal{L}(H_1,H_2) :\ T \text{ is closed}\}$ and

$$\mathcal{C}(H) := \mathcal{C}(H,H).$$

Note 2.1 By the Closed Graph Theorem [13, 21.1, page 420], it follows that a closed operator $T \in \mathcal{C}(H_1,H_2)$ with $D(T) = H_1$ is bounded.

If S and T are two operators, then by $S \subseteq T$ we mean that S is the restriction of T to $D(S)$. i.e., $D(S) \subseteq D(T)$ and $Sx = Tx$, for all $x \in D(S)$. In this case, we may also write S as $T|_{D(S)}$.

If M is a subspace of H, then $\bar{M}$ and $M^{\perp}$ denote the closure and the orthogonal complement of M in H, respectively. If M is closed, then P_M denotes the orthogonal projection onto M.

Suppose that X_1 and X_2 are subspaces of a Hilbert space with $X_1 \cap X_2 = \{0\}$. Then we use the notation $X_1 \oplus X_2$ to denote the direct sum of X_1 and X_2, and $X_1 \oplus^{\perp} X_2$ to denote the orthogonal direct sum of X_1 and X_2; the sum is whenever $\langle x, y\rangle = 0$ for every $x \in X_1$ and $y \in X_2$.

Definition 2.2 — [2, page 308, 310]. *An operator* $T \in \mathcal{L}(H_1, H_2)$ *with domain* $D(T)$ is densely defined if $\overline{D(T)} = H_1$. The subspace $C(T) := D(T) \cap N(T)^{\perp}$ is the carrier of T. If T is densely defined, then there exists a unique adjoint T^* of T which satisfies $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all $x \in D(T)$ and $y \in D(T^*)$.

Note 2.3 If $T \in \mathcal{C}(H_1, H_2)$, then $D(T) = N(T) \oplus^{\perp} C(T)$ [2, page 340].

Proposition 2.4 — [Moore-Penrose Inverse], [2]. Let $T \in \mathcal{C}(H_1, H_2)$ be densely defined. Then there exists a unique, densely defined operator $T^{\dagger} \in \mathcal{C}(H_2, H_1)$ with domain $D(T^{\dagger}) = R(T) \oplus^{\perp} R(T)^{\perp}$ having the following properties:

(1) $TT^{\dagger}y = P_{\overline{R(T)}}\, y,$ for all $y \in D(T^{\dagger})$;

(2) $T^{\dagger}Tx = P_{N(T)^{\perp}}\, x,$ for all $x \in D(T)$;

(3) $N(T^{\dagger}) = R(T)^{\perp}$. □

The above operator $T^{\dagger}$ is called the **Moore-Penrose inverse** of T.

The following property of $T^{\dagger}$ is also well known. For every $y \in D(T^{\dagger})$, let

$$L(y) := \{x \in D(T) : ||Tx - y|| \leq ||Tu - y|| \ \forall\ u \in D(T)\}.$$

Here any $u \in L(y)$ is called a **least square solution (lss)** of the operator equation $Tx = y$. The vector $x = T^{\dagger}y$ belongs to $L(y)$ and satisfies $||T^{\dagger}y|| \leq ||x|| \ \forall\ x \in L(y)$ and is called the **least square solution of minimal norm** . A different treatment of $T^{\dagger}$ is described in [2, pages 336, 339, 341], where the authors call it "**the Maximal Tseng generalized Inverse**".

Proposition 2.5 — [2, 10]. For a densely defined $T \in \mathcal{C}(H_1, H_2)$, the following statements are equivalent:

(1) $R(T)$ is closed;

(2) $R(T^*)$ is closed;

(3) $T_0 := T|_{C(T)}$ has a bounded inverse;

(4) there exists a $k > 0$ such that $\|Tx\| \geq k\|x\|$, for all $x \in C(T)$;

(5) $T^{\dagger}$ is bounded;

(6) $R(T^*T)$ is closed;

(7) $R(TT^*)$ is closed. $\square$

In the following proposition we list some well known facts.

Proposition 2.6 — [2]. Let $T \in \mathcal{C}(H_1, H_2)$ be densely defined operator. Then:

(1) $N(T) = R(T^*)^{\perp}$;

(2) $N(T^*) = R(T)^{\perp}$;

(3) $N(T^*T) = N(T)$;

(4) $\overline{R(T^*T)} = \overline{R(T^*)}$. $\square$

Lemma 2.7 — [4, Lemma 5.1]. Let $A \in \mathcal{C}(H_1, H_2)$ be densely defined. Then:

(1) $(I + A^*A)^{-\frac{1}{2}}$ and $A(I + A^*A)^{-\frac{1}{2}}$ are bounded;

(2) $\|(I + A^*A)^{-\frac{1}{2}}\| \leq 1$ and $\|A(I + A^*A)^{-\frac{1}{2}}\| \leq 1$. $\square$

***Theorem* 2.8** — [1, 3]. *Let $\{H_k\}$, $k = 1, 2, 3, \ldots$ be closed subspaces of H, and set $P_k = P_{H_k}$. Suppose that $\{P_k\}$ is a monotone, sequence of orthogonal projections. Then the strong limit $P = \lim\limits_{k\to\infty} P_{H_k}$ exists, and P is the projection onto $\cap_k H_k$ in the case where P_k is non-increasing and onto $\overline{\cup_k H_k}$ in the case where $\{P_k\}$ is non-decreasing.* $\square$

***Theorem* 2.9** — *(Uniform boundedness principle). [5, Theorem 14.3, page 83]. Let X be a Banach space, and let Y be a normed linear space. Suppose that $\mathcal{F}$ is a subset of $\mathcal{B}(X, Y)$ with the property that, for each $x \in X$, we have $\sup\limits_{A\in\mathcal{F}} ||Ax|| < \infty$. Then $\sup\limits_{A\in\mathcal{F}} ||A|| < \infty$.*

3. Generalized Projection Methods

3.1 *The Method and a Characterization* : Let $T \in \mathcal{C}(H_1, H_2)$ be densely defined with closed and separable range. Let $\{Y_n\}$ be an increasing sequence of subspaces of $R(T)$ such that $\overline{\cup_{n=1}^{\infty} Y_n} = R(T)$ and $\{X_n\}$ an increasing sequence of subspaces of $N(T)^{\perp}$ such that $\overline{\cup_{n=1}^{\infty} X_n} = N(T)^{\perp}$. Let $P_n : H_2 \rightarrow H_2$ and $Q_n : H_1 \rightarrow H_1$ be bounded projections with $R(P_n) = Y_n$ and $R(Q_n) = X_n$ for each natural number n.

Let $n \in \mathbb{N}$. Set $T_n := P_n T Q_n$ and $\widehat{T_n} := T_n|_{X_n}$. Our aim is to approximate the least square solution of minimal norm of equation (1). To do this, we find the least square solution of minimal norm x_n of the finite system of equations

$$\widehat{T_n} x = P_n y \tag{2}$$

and expect that $x_n = \widehat{T_n}^{\dagger} P_n y \rightarrow x = T^{\dagger} y$ for every $y \in H_2$. This is the idea of the projection methods. The operators $\widehat{T_n}$ are known as sections. If these sections are finite dimensional, these are known as finite sections and the method of approximating the solution with the help of these finite sections is called a finite section method.

We now give a formal definition of the convergence of this generalized projection method.

Definition 3.2 — Suppose that $T \in \mathcal{C}(H_1, H_2)$ is densely defined and has a closed range. Let P_n and Q_n be bounded projections on H_2 and H_1, respectively, with $\dim R(P_n) = n = \dim R(Q_n)$ such that:

(1) $P_n y \rightarrow P_{R(T)} y$ for all $y \in H_2$;

(2) $Q_n x \rightarrow P_{N(T)^{\perp}} x$ for all $x \in H_1$;

(3) $Q_n u \in N(\widehat{T_n})^{\perp}$ for all $u \in C(T)$;

(4) $Q_n x \in D(T)$ for all $x \in D(T)$;

(5) $T Q_n x \rightarrow T x$ for all $x \in D(T)$.

Then the sequence of pairs $\{P_n, Q_n\}$ is said to be **admissible** for T. **The generalized projection method for** T is said to **converge** with respect to $\{P_n, Q_n\}$ if, for each $y \in H_2$, $\widehat{T_n}^{\dagger} P_n y \rightarrow T^{\dagger} y$.

Remark 2.3 : The condition that $Q_n x \in D(T)$ for all $x \in D(T)$ is equivalent to the condition that $R(Q_n) \subseteq D(T)$ for all n [12, Proposition 2.7].

The following theorem provides a criterion for the convergence of the generalized projection method. This is analogous to [12, Theorem 3.1], which was proved for closed operators with bounded inverse.

***Theorem* 3.4** — *Let $T \in \mathcal{C}(H_1, H_2)$ be densely defined with closed and separable range. Let $\{P_n, Q_n\}$ be an admissible sequence for T. Then the generalized projection method for T is convergent with respect to the pair $\{P_n, Q_n\}$ if and only if the operators $\widehat{T}_n^\dagger$ are uniformly bounded.*

Proof : Assume that the generalized projection method for T is convergent, so that $\widehat{T}_n^\dagger P_n y \to T^\dagger y$ for all $y \in H_2$. Hence, by Theorem 2.9, $m := \sup\limits_n ||\widehat{T}_n^\dagger P_n|| < \infty$. Next, let $z \in R(P_n)$. Then $P_n z = z$. Hence

$$||\widehat{T}_n^\dagger z|| = ||\widehat{T}_n^\dagger P_n P_n z|| \leq m||P_n z|| = m||z||.$$

Hence $||\widehat{T}_n^\dagger|| \leq m$.

Conversely, suppose that $\sup\limits_n ||\widehat{T}_n^\dagger|| := M < \infty$. Let $y \in H_2$ and $X_n = \widehat{T}_n^\dagger$, consider

$$\begin{aligned}
||x_n - Q_n T^\dagger y|| &= ||x_n - P_{N(\widehat{T}_n)^\perp} Q_n T^\dagger y|| \quad \text{(by condition (3) of Definition 3.2)} \\
&= ||\widehat{T}_n^\dagger P_n y - \widehat{T}_n^\dagger \widehat{T}_n Q_n T^\dagger y|| \\
&\leq ||\widehat{T}_n^\dagger|| ||P_n y - \widehat{T}_n Q_n T^\dagger y|| \\
&\leq M||P_n y - \widehat{T}_n Q_n T^\dagger y|| \\
&= M||P_n y - T_n T^\dagger y|| \quad \text{(since } Q_n^2 = Q_n) \\
&\to M||P_{R(T)} y - P_{R(T)} y|| = 0.
\end{aligned}$$

Now $||x_n - T^\dagger y|| \leq ||x_n - Q_n T^\dagger y|| + ||Q_n T^\dagger y - T^\dagger y|| \to 0$ as $n \to \infty$. The result follows. ■

Example 3.5 — Take H to be the real space $L^2[0,\pi]$ of real-valued functions, and define

$$\mathcal{AC}[0,\pi] := \{\phi \in H : \phi \text{ is absolutely continuous}\}$$

and $H' = \{\phi \in \mathcal{AC}[0,\pi] : \phi' \in H\}$. Let $L = \frac{d}{dt}$ with $D(L) = \{\phi \in H' : \phi(0) = \phi(\pi) = 0\}$.

Using the fundamental theorem of integral calculus, it can be shown that $L \in \mathcal{C}(H)$. Since the functions $\phi_n(t) := \sqrt{\frac{2}{\pi}} \sin nt, t \in [0,\pi]$ $(n \in \mathbb{N})$, form an orthonormal basis of H, L is densely defined.

Note that $R(L) = \{y \in H : \int_0^\pi y(t)dt = 0\} = \{1\}^\perp$ is closed. It can be shown that $L^* = -\frac{d}{dt}$ with $D(L^*) = \mathcal{AC}[0,\pi]$. Let $\psi_n := \sqrt{\frac{2}{\pi}} \cos nt, t \in [0,\pi]$, $n \in \mathbb{N}$. Then $\{\psi_n : n \in \mathbb{N}\}$ forms an orthonormal basis for $R(L)$.

Now, define $X_n := \text{span}\{\phi_1, \phi_2, \ldots, \phi_n\}$ and $Y_n := LX_n = \text{span}\{\psi_1, \psi_2, \ldots \psi_n\}$. Let $P_n, Q_n : H \to H$ be orthogonal projections such that $R(P_n) = Y_n$ and $R(Q_n) = X_n$. That is $P_n y = \sum_{j=1}^n \langle y, \psi_j\rangle \psi_j,\ y \in H$ and $Q_n x = \sum_{i=1}^n \langle x, \phi_i\rangle \phi_i,\ x \in H$.

Let $L_n = P_n L Q_n$ and $\hat{L}_n = L_n|_{X_n}$. For each $x \in X_n$ we have

$$\begin{aligned}\hat{L}_n x = P_n L Q_n x = P_n L x = P_n L\Big(\sum_{j=1}^n \langle x, \phi_j\rangle \phi_j\Big) &= P_n\Big(\sum_{j=1}^n \langle x, \phi_j\rangle L\phi_j\Big) \\ = P_n\Big(\sum_{j=1}^n \langle x, \phi_j\rangle j\psi_j\Big) &= \sum_{j=1}^n \langle x, \phi_j\rangle j P_n(\psi_j) \\ = \sum_{j=1}^n j\langle x, \phi_j\rangle \psi_j.\end{aligned}$$

Hence, for any $x \in D(T)$, $LQ_n x = \sum_{j=1}^n j\langle x, \phi_j\rangle \psi_j \to Lx$. Further, $\{P_n, Q_n\}$ satisfies the conditions of Definition 3.2. Hence $\{P_n, Q_n\}$ is admissible for L.

Using the formula $\hat{L}_n^\dagger = (\hat{L}_n^* \hat{L}_n)^{-1} \hat{L}_n^*$, we see that $\hat{L}_n^\dagger y = \displaystyle\sum_{j=1}^n \frac{1}{j}\langle y, \psi_j\rangle \phi_j$. It can easily be verified that $||(\hat{L}_n)^\dagger|| \leq 1$ for all $n \in \mathbb{N}$ and that

$$\hat{L}_n^\dagger P_n y \to \sum_{n=1}^\infty \frac{1}{n}\langle y, \psi_n\rangle \phi_n = L^\dagger y$$

for each $y \in H$. The expression that we have obtained for $L^\dagger y$ is equivalent to the following formula: $(L^\dagger y)(s) = \int_0^s y(t)dt - \frac{s}{\pi}\int_0^\pi y(t)dt,\ 0 \le s \le \pi$ (for details see [9]).

4. Existence of Admissible Sequence of Projections

We have observed in the previous section that the generalized projection methods depend on the given operator and the admissible sequences of projections $\{P_n, Q_n\}$. In the case of a bounded operator, it is easy to find such an admissible sequence. We can choose any pair of sequences $\{P_n, Q_n\}$ satisfying the conditions

(1) $P_n y \to P_{R(T)}\, y$ for every $y \in H_2$;

(2) $Q_n x \to P_{N(T)^\perp}$ for every $x \in H_1$.

Then, in view of the continuity of T, the assumptions (3), (4) and (5) of Definition 3.2 are satisfied. Hence a natural question one can ask is whether it is possible to find an admissible sequence $\{P_n, Q_n\}$ for a given densely defined operator with closed and separable range. In this section we answer this question affirmatively.

Proposition 4.1 — [6, 7, 8, 14]. Let $T \in \mathcal{C}(H_1, H_2)$ be densely defined. Then:

(1) $(I + T^*T)^{-1} \in \mathcal{B}(H_1)$, $(I + TT^*)^{-1} \in \mathcal{B}(H_2)$;

(2) if $g \in C[0,1]$, then $g\big((I + T^*T)^{-1}\big)T^* \subseteq T^* g\big((I + TT^*)^{-1}\big)$ and $g\big((I + TT^*)^{-1}\big)T \subseteq Tg\big(I + TT^*)^{-1}\big)$. In particular, $||T(I + T^*T)^{-1}|| \le \dfrac{1}{2}$ and $||T^*(I + TT^*)^{-1}|| \le \dfrac{1}{2}$. $\square$

Lemma 4.2 — Let $T \in \mathcal{C}(H_1, H_2)$ be densely defined. Let $C := T(I + T^*T)^{-\frac{1}{2}}$ and $D := T^*(I + TT^*)^{-\frac{1}{2}}$. Then:

(1) $R(C^*C) = R(T^*T)$, $R(CC^*) = R(TT^*)$ and $R(D^*D) = R(TT^*)$, $R(DD^*) = R(T^*T)$;

(2) $N(C) = N(T)$ and $N(D) = N(T^*)$;

(3) if $R(T)$ is closed, then $R(C) = R(T)$ and $R(D) = R(T^*)$. Conse-

quently, $R(C)$ and $R(D)$ are also closed.

PROOF : Note that $C^* = T^*(I+TT^*)^{-\frac{1}{2}}$ and $CC^* = TT^*(I+TT^*)^{-1}$. By [8, Section 2], $R(CC^*) = R(TT^*)$. A similar argument holds for $R(C^*C) = R(T^*T)$. The proofs of the equalities $R(D^*D) = R(TT^*)$ and $R(DD^*) = R(T^*T)$ follow by the observation that $D = C^*$.

The statement (2) follows from Proposition 2.6. We have $\overline{R(C)} = \overline{R(T)}$. Hence, if $R(T)$ is closed, then $R(C)$ is also closed and $R(C) = R(T)$. Again $R(T)$ is closed implies that $R(T^*)$ is closed. Hence $R(D) = R(T^*)$. □

Lemma 4.3 — Let $T \in \mathcal{C}(H_1, H_2)$ be densely defined. Let $Y_n \subseteq R(T)$ be such that:

(a) $Y_n \subseteq Y_{n+1}$ for each $n \in \mathbb{N}$;

(b) $\dim Y_n = n$;

(c) $\overline{\cup_{n=1}^{\infty} Y_n} = \overline{R(T)}$.

Let $Z_n := (I + TT^*)^{-1} Y_n$ and $X_n := T^* Z_n = T^*(I + TT^*)^{-1} Y_n$. Then:

(1) $X_n \subseteq X_{n+1} \cdots \subseteq N(T)^{\perp}$, $\dim X_n = n$;

(2) $\overline{\cup_{n=1}^{\infty} Z_n} = \overline{R(T)}$;

(3) $\overline{\cup_{n=1}^{\infty} X_n} = \overline{R(T^*)}$;

(4) $\overline{\cup_{n=1}^{\infty} T X_n} = \overline{R(T)}$.

PROOF : By the definition of X_n, we have $X_n \subseteq C(T) \subseteq N(T)^{\perp} = R(T^*)$ and $X_n \subseteq X_{n+1}$ for all $n \in \mathbb{N}$. Since the operator $T^*(I + TT^*)^{-1}|_{\overline{R(T)}}$ is injective, $\dim X_n = n = \dim Y_n$.

For a proof of (2), we make use of the following observation:

$$\overline{(I + TT^*)^{-1}(\overline{R(T)})} = \overline{R(T)}.$$

It can be proved easily that $(I + TT^*)^{-1}(N(TT^*)) = N(TT^*)$. By the Projection Theorem [13, 21.1, page 420], $H_2 = N(TT^*) \oplus^{\perp} N(TT^*)^{\perp}$. That is $H_2 = N(TT^*) \oplus^{\perp} \overline{R(TT^*)}$. But

$$(I + TT^*)^{-1}(H_2) = D(TT^*) = N(TT^*) \oplus^{\perp} C(TT^*).$$

Hence

$$\begin{aligned}(I+TT^*)^{-1}H_2 &= (I+TT^*)^{-1}\big(N(TT^*)\oplus^{\perp}\overline{R(TT^*)}\big)\\ &= N(TT^*)\oplus^{\perp}(I+TT^*)^{-1}(\overline{R(TT^*)}).\end{aligned}$$

From this we can conclude that $(I+TT^*)^{-1}(\overline{R(TT^*)}) = C(TT^*)$ and as $\overline{C(TT^*)} = N(TT^*)^{\perp}$, we have $\overline{(I+TT^*)^{-1}(\overline{R(TT^*)})} = \overline{R(TT^*)}$. Hence $\overline{(I+TT^*)^{-1}(\overline{R(T)})} = \overline{R(T)}$, by Proposition (2.6). Thus

$$\begin{aligned}\overline{R(T)} = \overline{(I+TT^*)^{-1}(\overline{R(T)})} &= \overline{(I+TT^*)^{-1}(\overline{\cup_{n=1}^{\infty}Y_n})}\\ &= \overline{\cup_{n=1}^{\infty}(I+TT^*)^{-1}Y_n}\\ &= \overline{\cup_{n=1}^{\infty}Z_n}.\end{aligned}$$

This proves (2).

It is clear that $\overline{\cup_{n=1}^{\infty}}X_n \subseteq \overline{R(T^*)} = N(T)^{\perp}$.

Suppose that $\overline{\cup_{n=1}^{\infty}}X_n \subsetneq N(T)^{\perp}$. Then there exists $z_0 \in N(T)^{\perp}$ with $z_0 \neq 0$ such that $z_0 \in (\overline{\cup_{n=1}^{\infty}}X_n)^{\perp}$. That is

$$\langle z_0, T^*(I+TT^*)^{-1}y\rangle = 0 \quad \text{for all} \quad y \in R(T).$$

By the continuity of $T^*(I+TT^*)^{-1}$, this holds for all $y \in \overline{R(T)}$.

We claim that this holds for all $y \in H_2$. Let $y \in H_2$. Then $y = u + v$ for some $u \in \overline{R(T)}$ and $v \in R(T)^{\perp} = N(T^*) \subseteq D(T^*)$. Hence by Proposition 4.1, $T^*(I+TT^*)^{-1}v = (I+T^*T)^{-1}T^*v = 0$. Hence

$$\langle z_0, T^*(I+TT^*)^{-1}y\rangle = \langle z_0, T^*(I+TT^*)^{-1}u\rangle = 0.$$

This proves the claim.

Next, since $\overline{C(T)} = N(T)^{\perp}$ [11, Lemma 3.3], there exists a sequence $\{z_n\} \subseteq C(T)$ such that $z_n \to z_0$. Hence for all $y \in H_2$,

$$\begin{aligned}0 = \langle z_0, T^*(I+TT^*)^{-1}y\rangle &= \lim_{n\to\infty}\langle z_n, T^*(I+TT^*)^{-1}y\rangle\\ &= \lim_{n\to\infty}\langle Tz_n, (I+TT^*)^{-1}y\rangle\\ &= \lim_{n\to\infty}\langle (I+TT^*)^{-1}Tz_n, y\rangle\\ &= \lim_{n\to\infty}\langle T(I+T^*T)^{-1}z_n, y\rangle.\end{aligned}$$

This shows that $T(I+T^*T)^{-1}z_n \xrightarrow{w} 0$ (weakly). But, since $T(I+T^*T)^{-1}$ is bounded, we have $T(I+T^*T)^{-1}z_0 = 0$. That is $(I+T^*T)^{-1}z_0 \in N(T)$. Let $y = (I+T^*T)^{-1}z_0$. Then $Ty = 0$. Hence $z_0 = (I+T^*T)y = y \in N(T)$. Thus $z_0 \in N(T) \cap N(T)^{\perp} = \{0\}$. Hence $z_0 = 0$, a contradiction to our assumption. This proves (3).

Using a similar argument we can prove (4). □

***Theorem* 4.4** — *Let $T \in \mathcal{C}(H_1, H_2)$ be densely defined with closed and separable range. Then there exists a sequence $\{P_n, Q_n\}$ of projections with finite dimensional ranges which is admissible for T.*

PROOF : Choose a sequence $\{X_n\}$ of subspaces of $C(T)$ such that:

(a) $X_n \subseteq X_{n+1}$ for each n;

(b) $\dim X_n = n$;

(c) $\overline{\cup_{n=1}^{\infty} X_n} = N(T)^{\perp}$.

Let $Y_n := TX_n$. Then

(1) $Y_n \subseteq Y_{n+1}$ for each n;

(2) $\dim Y_n = n$;

(3) $\overline{\cup_{n=1}^{\infty} Y_n} = R(T)$.

Since $T|_{C(T)} : C(T) \to R(T)$ is bijective, we have $\dim Y_n = \dim X_n = n$. It is clear by the linearity of T that $Y_n \subseteq Y_{n+1}$ for each n. Also $\overline{\cup_{n=1}^{\infty} Y_n} \subseteq R(T)$. We prove that these two subspaces are equal. If $\overline{\cup_{n=1}^{\infty} Y_n} \subset R(T)$, then there exists $0 \neq z \in R(T)$ such that $z \in (\cup_{n=1}^{\infty} Y_n)^{\perp}$. Hence, for all $x \in X_n$, we have $\langle z, Tx \rangle = 0$. The map $(I+T^*T)^{-\frac{1}{2}} : H \to D(T)$ is a bijective map. With the help of Proposition 4.1, it can be shown that $(I+T^*T)^{-\frac{1}{2}}(N(T)) = N(T)$ and that $(I+T^*T)^{-\frac{1}{2}}(N(T)^{\perp}) = C(T)$. Since $X_n \subseteq C(T)$, there exists subspaces $Z_n \subseteq N(T)^{\perp}$ such that $X_n = (I+T^*T)^{-\frac{1}{2}}(Z_n)$, $Z_n \subseteq Z_{n+1}$, $\text{dim} Z_n = \text{dim} X_n$ and $\overline{\cup_n Z_n} = N(T)^{\perp}$. Hence

$$\langle z, Tx \rangle = 0, \text{ for all } x \in \cup_n X_n \Rightarrow \langle z, T(I+T^*T)^{-\frac{1}{2}} y \rangle = 0, \text{ for all } y \in \cup_n Z_n.$$

Since $T(I+T^*T)^{-\frac{1}{2}}$ is bounded by Lemma 2.7, $\langle z, T(I+TT^*)^{-\frac{1}{2}} y \rangle = 0$ for all $y \in N(T)^{\perp}$. Using the projection theorem and statement (2) of

Proposition 4.2, we can show that $\langle z, T(I+T^*T)^{-\frac{1}{2}}y\rangle = 0$ for all $y \in H_1$, and hence conclude $z \in R(T)^{\perp}$. Thus $z = 0$, which is a contradiction.

Let $P_n : H_2 \to H_2$ and $Q_n : H_1 \to H_1$ be sequence of orthogonal projections such that $R(P_n) = Y_n$ and $R(Q_n) = X_n$. Let $T_n := P_n T Q_n$ and $\widehat{T}_n := T_n|_{X_n}$. That is $\widehat{T}_n \in \mathcal{B}(X_n, Y_n)$.

We claim that $\{P_n, Q_n\}$ is admissible for T.

Since $\{P_n\}$ and $\{Q_n\}$ are increasing projections, by Theorem 2.8, it follows that

(1) $P_n y \to P_{R(T)}\, y$ for all $y \in H_2$;

(2) $Q_n x \to P_{N(T)^{\perp}}\, x$ for all $x \in H_1$.

Next, we prove that

$$Q_n x \in N(\widehat{T}_n)^{\perp} \quad \text{for every } x \in C(T). \tag{3}$$

First we observe that $N(\widehat{T}_n) = N(TQ_n|_{X_n})$. Let $x \in N(TQ_n|_{X_n})$. Then $TQ_n x = 0$. Hence $P_n T Q_n x = 0$. That is $N(TQ_n) \subseteq N(\widehat{T}_n)$.

For the reverse inclusion, let $x \in N(\widehat{T}_n)$. Then

$$\begin{aligned} P_n T Q_n x &= P_n T x = 0 \\ &\Rightarrow Tx \in N(P_n) = Y_n^{\perp} \\ &\Rightarrow Tx \in N(P_n) = Y_n^{\perp} \cap Y_n \quad \text{since } Y_n = TX_n \\ &\Rightarrow Tx = 0 \\ &\Rightarrow Q_n x \in N(T) \\ &\Rightarrow x \in N(TQ_n). \end{aligned}$$

Now we prove (3). Let $y \in N(\widehat{T}_n)$ and $x \in C(T)$. Since $x \in R(T^*) = R(T^*(I+TT^*)^{-\frac{1}{2}})$, there exists $w \in H_2$ such that $x = T^*(I+TT^*)^{-\frac{1}{2}}w$. Thus

$$\begin{aligned} \langle x, Q_n y\rangle = \langle Q_n x, y\rangle &= \langle T^*(I+TT^*)^{-\frac{1}{2}}w, Q_n y\rangle \\ &= \langle (I+TT^*)^{-\frac{1}{2}}w, TQ_n y\rangle \\ &= 0 \quad \text{since } N(TQ_n|X_n) = N(\widehat{T}_n). \end{aligned}$$

Hence for all $x \in C(T)$, $Q_n x \in N(\widehat{T}_n)^{\perp}$.

Also for for every $x \in D(T)$, $Q_n x \in X_n \subseteq D(T)$. Hence condition (4) is satisfied.

Next we prove that

$$TQ_n x \to Tx \quad \text{for all } x \in D(T). \tag{4}$$

If $x \in D(T)$, then $Q_n x \in X_n \subseteq N(T)^\perp = R(T^*) = R(T^*(I+TT^*)^{-\frac{1}{2}})$. Therefore $Q_n x = T^*(I+TT^*)^{-\frac{1}{2}} w_n$, for some $w_n \in N(T^*)^\perp$. Hence $TQ_n x = TT^*(I+TT^*)^{-\frac{1}{2}} w_n$. As $Q_n x \to P_{N(T)^\perp} x$ and $R(T^*(I+TT^*)^{-\frac{1}{2}})$ is closed (by Lemma 4.2), there exists $k > 0$ such that

$$||T^*(I+TT^*)^{-\frac{1}{2}} w_n|| \geq k\, ||w_n||. \tag{5}$$

By Proposition 2.5(4) and Lemma 4.2(2), the left hand side of the inequality (5) is the n^{th} term in a convergent sequence, it follows that the sequence $\{w_n\}$ is Cauchy, and hence convergent.

Assume that $w_n \to w$. As $T^*(I+TT^*)^{-\frac{1}{2}}$ is bounded, $Q_n x = T^*(I+TT^*)^{-\frac{1}{2}} w_n \to T^*(I+TT^*)^{-\frac{1}{2}} w = P_{N(T)^\perp} x$. Now $TQ_n x = TT^*(I+TT^*)^{-\frac{1}{2}} w_n \to TT^*(I+TT^*)^{-\frac{1}{2}} w = TP_{N(T)^\perp} x$. Since $x \in D(T)$, $x = u + v$, where $u \in N(T)$ and $v \in C(T)$. Therefore $TP_{N(T)^\perp} x = Tv = T(u+v) = Tx$. Hence $TQ_n x \to Tx$ for all $x \in D(T)$.

Now for any $x \in D(T)$,

$$\begin{aligned} ||P_n T Q_n - Tx|| &\leq ||P_n T Q_n x - P_n T x|| + ||P_n T x - Tx|| \\ &\leq ||P_n||\, ||TQ_n x - Tx|| + ||P_n T x - Tx|| \\ &\leq ||TQ_n x - Tx|| + ||P_n T x - Tx|| \to 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

Hence $\{P_n, Q_n\}$ is admissible for T. □

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