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GEOMETRICAL AND DYNAMICAL PROPERTIES OF GENERAL EULER TOP SYSTEM

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In this paper we present various geometrical and dynamical properties of the Euler top system in general form. Also the stability problem for the equilibrium states of them are discussed.

Key words : Euler top system; Poisson geometry; Lyapunov stability.

1. INTRODUCTION

The Hamilton-Poisson systems appear naturally in many areas of physical science and engineering including theoretical mechanics of fluids and plasmas, robotics and spatial dynamics ([1, 3, 7, 8]). A remarkable class of Hamilton-Poisson systems is formed by a family of differential equations on $\mathbf{R}^3$ which depends by a triple of reel parameters $(\alpha_1, \alpha_2, \alpha_3)$, called the general Euler top system. This family contains various integrable systems of classical mechanics, for instance:

the free rigid body ([5]), the rigid body dynamics on the pseudo-orthogonal group $SO(2,1)$ ([9, 4]) etc.

This paper contains five sections. In Section 2 we present a general method for construct a Poisson structure on $\mathbf{R}^3$ (Prop. 2.2). In Section 3 we discuss the general Euler top system in terms of Poisson geometry. Also, some dynamical properties of its are established (Prop. 3.1, Th. 3.1). In Section 4 we establish that the dynamics of the Euler top system and the dynamics of pendulum are linked (Prop. 4.1, Prop. 4.2). Section 5 is dedicated to study of Lyapunov stability for the equilibrium states of general Euler top system (Prop. 5.2, Prop. 5.3, Th. 5.1).

2. Hamilton-Poisson Systems on $\mathbf{R}^n$

In this section we will review some definitions and results concerning Hamilton-Poisson systems on $\mathbf{R}^n$ ([5, 8]).

Let $C^\infty(\mathbf{R}^n, \mathbf{R})$ be the ring of smooth real valued functions defined on $\mathbf{R}^n$. We shall denote a coordinate system on $\mathbf{R}^n$ with $x^i : \mathbf{R}^n \to \mathbf{R},\ i = \overline{1,n}$.

Let $\Pi(x) = (\pi^{ij}(x))_{1\leq i,j\leq n}$, where $\pi^{ij}(x) \in C^\infty(\mathbf{R}^n, \mathbf{R})$, be an $n \times n$ skew-symmetric matrix for each $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$. Consider the bracket operation $\{\cdot,\cdot\} : C^\infty(\mathbf{R}^n, \mathbf{R}) \times C^\infty(\mathbf{R}^n, \mathbf{R}) \to C^\infty(\mathbf{R}^n, \mathbf{R}),\ (f, g) \to \{f, g\},$ where $\{f, g\}$ is defined by:

$$\{f,g\}(x) = (\nabla f(x))^T \cdot \Pi(x) \cdot \nabla g(x), \tag{2.1}$$

and $\nabla f(x) = \left(\dfrac{\partial f}{\partial x_1}\ \ \dfrac{\partial f}{\partial x_2}\ \ldots\ \dfrac{\partial f}{\partial x_n}\right)^T$ is the gradient of function f.

It is easy to see that the bracket operation (2.1) is $\mathbf{R}-$bilinear and skew-symmetric.

The bracket operation $\{\cdot,\cdot\}$ given by (2.1) is a *Poisson bracket* on $\mathbf{R}^n$, if for every $f, g, h \in C^\infty(\mathbf{R}^n, \mathbf{R})$, the following relations hold:

$$\{f\cdot g, h\} = f\cdot\{g,h\} + g\cdot\{f,h\}, \quad \text{(Leibniz's rule)}$$

$$\{f,\{g,h\}\}+\{h,\{f,g\}\}+\{g,\{h,f\}\}=0, \quad \text{(Jacoby's identity)}.$$

If the bracket operation (2.1) is a Poisson bracket, then we say that the pair $(\mathbf{R}^n,\{\cdot,\cdot\})$ is a *Poisson manifold* and $\{\cdot,\cdot\}$ is a *Poisson structure* on $\mathbf{R}^n$. The matrix $\Pi(x)$ is called the *structure matrix* of the Poisson manifold $(\mathbf{R}^n,\{\cdot,\cdot\})$.

Next proposition gives us a criterion for that a skew-symmetric matrix $\Pi(x)$ to be a structure matrix of Poisson manifold $(\mathbf{R}^n,\{\cdot,\cdot\})$.

Proposition 2.1 ([6]) — A skew-symmetric matrix $\Pi=(\pi^{ij}(x))_{1\le i,j\le n}$ is a structure matrix for a Poisson bracket given by (2.1) if and only if the following relations hold:

$$\pi^{i\ell}\frac{\partial\pi^{jk}}{\partial x^\ell}+\pi^{j\ell}\frac{\partial\pi^{ki}}{\partial x^\ell}+\pi^{k\ell}\frac{\partial\pi^{ij}}{\partial x^\ell}=0, \quad i,j,k,\ell=\overline{1,n}. \tag{2.2}$$

The relations (2.2) are called the Jacobi equations associated to matrix $\Pi(x)$.

In the particular case $n=3$, The Jacobi equations associated to matrix

$$\Pi(x)=\begin{pmatrix} 0 & \pi^{12}(x) & \pi^{13}(x) \\ -\pi^{12}(x) & 0 & \pi^{23}(x) \\ -\pi^{13}(x) & -\pi^{23}(x) & 0 \end{pmatrix}$$

reduces to a single equation, namely:

$$\pi^{12}\frac{\partial\pi^{13}}{\partial x^1}-\pi^{13}\frac{\partial\pi^{12}}{\partial x^1}+\pi^{12}\frac{\partial\pi^{23}}{\partial x^2}-\pi^{23}\frac{\partial\pi^{12}}{\partial x^2}+\pi^{13}\frac{\partial\pi^{23}}{\partial x^3}-\pi^{23}\frac{\partial\pi^{13}}{\partial x^3}=0. \tag{2.3}$$

Proposition 2.2 — Let $A=(A^{ij})\in\mathcal{M}_3(\mathbf{R})$ be a skew-symmetric matrix. Define the matrix $\Pi=(\pi^{ij})_{1\le i,j\le 3}$, where

$$\pi^{ij}=A^{ij}x_k, \quad k\neq i \ \text{ și } \ k\neq j.$$

Then $\Pi(x)=(\pi^{ij}(x))$ is a structure matrix for a Poisson bracket $\{\cdot,\cdot\}$ on $\mathbf{R}^3$.

PROOF : The non-nulls elements of the matrix $\Pi(x)$ are:

$\pi^{12}=-\pi^{21}=A^{12}x_3,\ \ \pi^{23}=-\pi^{32}=A^{23}x_1,\ \ \pi^{13}=-\pi^{31}=A^{13}x_2.$

The matrix Π is skew-symmetric, since $A^{ij} = -A^{ji}$. We observe that the equation (2.3) is verified. Therefore, the conditions from Proposition 2.1 are satisfied. Hence Π generates a Poisson bracket on $\mathbf{R}^3$. $\square$

If in Proposition 2.2 we consider the skew-symmetric

$$A = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}, \quad a, b, c \in \mathbf{R},$$

one obtains the structure matrix

$$\Pi_{(a,b,c)} = \begin{pmatrix} 0 & -cx_3 & bx_2 \\ cx_3 & 0 & -ax_1 \\ -bx_2 & ax_1 & 0 \end{pmatrix}, \quad a, b, c \in \mathbf{R}. \tag{2.4}$$

If in relation (2.1) we take $\Pi = \Pi_{(a,b,c)}$ with $abc \neq 0$, we say that $\Pi_{(a,b,c)}$ generates a *Poisson structure* $\{\cdot,\cdot\}$ *on* $\mathbf{R}^3$ *of* $so(3)-$*type*.

If in relation (2.1) we take the matrix Π of the form

$$\Pi_{(0,b,c)} = \begin{pmatrix} 0 & -cx_3 & bx_2 \\ cx_3 & 0 & 0 \\ -bx_2 & 0 & 0 \end{pmatrix}, \quad b, c \in \mathbf{R},\ bc \neq 0, \tag{2.5}$$

then we say that $\Pi_{(0,b,c)}$ generates a *Poisson structure* $\{\cdot,\cdot\}$ *on* $\mathbf{R}^3$ *of* $se(2)-$*type*.

An *Hamilton-Poisson system* on $\mathbf{R}^n$ is a triple $(\mathbf{R}^n, \{\cdot,\cdot\}, H)$ where $\{\cdot,\cdot\}$ is a Poisson structure on $\mathbf{R}^n$ and $H \in C^\infty(\mathbf{R}^n, \mathbf{R})$ is a function called the *Hamiltonian*. Its dynamics is described by the integral curves of the Hamiltonian vector field $X_H \in \mathcal{X}(\mathbf{R}^n)$ defined by $X_H(f) = \{H, f\}, \quad f \in C^\infty(\mathbf{R}^n, \mathbf{R})$, or locally

$$\dot{x}_i = \{x_i, H\}, \quad i = \overline{1,n}. \tag{2.6}$$

If the structure Poisson $\{\cdot,\cdot\}$ on $\mathbf{R}^n$ is generated by $\Pi(x)$, then via (2.1), the dynamics of a Hamilton-Poisson system (2.6) can be expressed in the form

$$\dot{X} = \Pi(x) \cdot \nabla H(x) \quad \text{where} \quad \dot{X} = \begin{pmatrix} \dot{x}_1 & \dot{x}_2 & \dots & \dot{x}_n \end{pmatrix}^T. \tag{2.7}$$

A *Casimir* of the configuration $(\mathbf{R}^n, \{\cdot,\cdot\})$ is a function $C \in C^\infty(\mathbf{R}^n, \mathbf{R})$ such that $\{C, f\} = 0$ for every $f \in C^\infty(\mathbf{R}^n, \mathbf{R})$ or equivalently

$$\Pi(x) \cdot \nabla C(x) = O. \tag{2.8}$$

Definition 2.1 — We say that a dynamical system on $\mathbf{R}^n$ of the form

$$\dot{x}_i(t) = f_i(x_1(t), x_2(t), ..., x_n(t)), \quad f_i \in C^\infty(\mathbf{R}^n, \mathbf{R}),\ i = \overline{1, n} \tag{2.9}$$

has an *Hamilton-Poisson realization,* if there exist a Poisson structure $\{\cdot,\cdot\}$ on $\mathbf{R}^n$ generated by the matrix Π and an Hamiltonian $H \in C^\infty(\mathbf{R}^n, \mathbf{R})$ such that the system (2.9) can be written in the form (2.7). It is denoted by $(\mathbf{R}^n, \Pi, H)$. □

In the what follows we present a concrete example of Hamilton-Poisson system.

• Let $so(2,1)$ the Lie algebra of the pseudo-orthogonal Lie group $SO(2,1)$ and its dual $(so(2,1))^*$, (see [9, 4]). The *rigid body on the Lie group* $SO(2,1)$ ([9]) is described by the following set of differential equations:

$$\dot{m}_1 = -\left(\frac{1}{I_2} + \frac{1}{I_3}\right) m_2 m_3,\ \ \dot{m}_2 = \left(\frac{1}{I_1} + \frac{1}{I_3}\right) m_1 m_3,\ \ \dot{m}_3 = \left(\frac{1}{I_1} - \frac{1}{I_2}\right) m_1 m_2, \tag{2.10}$$

I_1, I_2, I_3 being the components of the inertia tensor of the rigid body.

Proposition 2.3 ([9]) — The rigid body dynamics on the Lie group $SO(2,1)$ has the following Hamilton-Poisson realization $((so(2,1))^* \cong \mathbf{R}^3, \Pi_{rb21}, H_{rb21})$ with the Hamiltonian $H_{rb21} \in C^\infty(\mathbf{R}^3, \mathbf{R})$ and the Casimir $C_{rb21} \in C^\infty(\mathbf{R}^3, \mathbf{R})$ where:

$$\Pi_{rb21} = \begin{pmatrix} 0 & -m_3 & -m_2 \\ m_3 & 0 & m_1 \\ m_2 & -m_1 & 0 \end{pmatrix}, \tag{2.11}$$

$$H_{rb21}(m) = \frac{1}{2}\left(\frac{1}{I_1} m_1^2 + \frac{1}{I_2} m_2^2 + \frac{1}{I_3} m_3^2\right),$$
$$C_{rb21}(m) = \frac{1}{2}\left(m_1^2 + m_2^2 - m_3^2\right). \tag{2.12}$$

Remark 2.1 : The Poisson structure $\{\cdot,\cdot\}$ on $\mathbf{R}^3$ generated by the matrix Π_{rb21} given by (2.11) is in fact the minus Lie-Poisson structure on the dual $(so(2,1))^* \cong \mathbf{R}^3$ of the Lie algebra $so(2,1)$. Also, we observe that $\Pi_{rb21} = \Pi(-1,1,1)$. □

3. Poisson Geometry of General Euler Top System

The general Euler top system on $\mathbf{R}^3$ ([7]) reads:

$$\frac{dx_1}{dt} = \alpha_1 x_2(t)x_3(t),\ \ \frac{dx_2}{dt} = \alpha_2 x_3(t)x_1(t),\ \ \frac{dx_3}{dt} = \alpha_3 x_1(t)x_2(t), \tag{3.1}$$

where $\alpha_1, \alpha_2, \alpha_3 \in \mathbf{R}$ are parameters such that $\alpha_1\alpha_2\alpha_3 \neq 0$ and t is the time.

We will denote the vector of parameters by $\alpha = (\alpha_1, \alpha_2, \alpha_3)$.

If in (3.1) we take $\alpha = \left(-\frac{1}{I_2} - \frac{1}{I_3}, \frac{1}{I_1} + \frac{1}{I_3}, \frac{1}{I_1} - \frac{1}{I_2}\right)$, then we obtain the rigid body dynamics on Lie group $SO(2,1)$ given by (2.10).

Proposition 3.1 — An Hamilton-Poisson realization of the Euler top equation (3.1) is $(\mathbf{R}^3, P^\alpha, H^\alpha)$ with the Casimir $C^\alpha \in C^\infty(\mathbf{R}^3, \mathbf{R})$, where

$$P^\alpha = \begin{pmatrix} 0 & -x_3 & -\frac{\alpha_3}{\alpha_2}x_2 \\ x_3 & 0 & 0 \\ \frac{\alpha_3}{\alpha_2}x_2 & 0 & 0 \end{pmatrix}, \tag{3.2}$$

$$H^\alpha(x) = \frac{1}{2}(\alpha_2 x_1^2 - \alpha_1 x_2^2) \quad \text{and} \quad C^\alpha(x) = \frac{1}{2}(\frac{\alpha_3}{\alpha_2}x_2^2 - x_3^2). \tag{3.3}$$

Proof : We have $\dfrac{\partial H^\alpha}{\partial x_1} = \alpha_2 x_1,\ \dfrac{\partial H^\alpha}{\partial x_2} = -\alpha_1 x_2,\ \dfrac{\partial H^\alpha}{\partial x_3} = 0$. Then:

$$P^\alpha \cdot \nabla H^\alpha = \begin{pmatrix} 0 & -x_3 & -\frac{\alpha_3}{\alpha_2}x_2 \\ x_3 & 0 & 0 \\ \frac{\alpha_3}{\alpha_2}x_2 & 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha_2 x_1 \\ -\alpha_1 x_2 \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} \alpha_1 x_2 x_3 \\ \alpha_2 x_1 x_3 \\ \alpha_3 x_1 x_2 \end{pmatrix} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix}.$$

Hence $\dot{X}(t) = P^\alpha(x)\nabla H^\alpha(x)$, where $\dot{X}(t) = (\dot{x}_1(t), \dot{x}_2(t), \dot{x}_3(t))^T$ and (3.1) is an Hamilton-Poisson system. Also, C^α is a Casimir, since:

$$P^\alpha(x) \cdot \nabla C^\alpha(x) = \begin{pmatrix} 0 & -x_3 & -\frac{\alpha_3}{\alpha_2}x_2 \\ x_3 & 0 & 0 \\ \frac{\alpha_3}{\alpha_2}x_2 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \dfrac{\alpha_3}{\alpha_2}x_2 \\ -x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \quad \square$$

Remark 3.1 : We have $P^\alpha = \Pi_{(0,-\frac{\alpha_3}{\alpha_2},1)}$, see (2.5). Hence the Poisson geometry of the Euler top system (3.1) is generated by a matrix of $se(2)-$type. □

Proposition 3.2 — The functions H^α and C^α given by (3.3), are constants of the motion (first integrals) for the dynamics (3.1).

PROOF : Indeed, $\dfrac{dH^\alpha}{dt} = \alpha_2 x_1 \dot{x}_1 - \alpha_1 x_2 \dot{x}_2 = \alpha_2 x_1(\alpha_1 x_2 x_3) - \alpha_1 x_2(\alpha_2 x_1 x_3)$ $= 0$ and $\dfrac{dC^\alpha}{dt} = \dfrac{\alpha_3}{\alpha_2} x_2 \dot{x}_2 - x_3 \dot{x}_3 = \dfrac{\alpha_3}{\alpha_2} x_2(\alpha_2 x_1 x_3) - x_2(\alpha_3 x_1 x_2) = 0.$ □

Remark 3.2 : Since H^α and C^α are first integrals for (3.1), it follows that the trajectories of motion of Euler top system (3.1) are intersections of the surfaces:

$$\frac{1}{2}(\alpha_2 x_1^2 - \alpha_1 x_2^2) = \text{constant} \quad \text{and} \quad \frac{1}{2}(\frac{\alpha_3}{\alpha_2}\mathrm{x}_2^2 - \mathrm{x}_3^2) = \text{constant}. \qquad \square$$

Corollary 3.1 — An Hamilton-Poisson realization for the dynamics (2.10) is $(\mathbf{R}^3, P^1_{rb21}, H^1_{rb21})$ with the Casimir $C^1_{rb21} \in C^\infty(\mathbf{R}^3, \mathbf{R})$, where

$$P^1_{rb21} = \begin{pmatrix} 0 & -m_3 & \dfrac{(I_1 - I_2)I_3}{(I_1 + I_3)I_2} m_2 \\ \\ m_3 & 0 & 0 \\ \\ -\dfrac{(I_1 - I_2)I_3}{(I_1 + I_3)I_2} m_2 & 0 & 0 \end{pmatrix},$$

$$H^1_{rb21}(m) = \frac{1}{2}\left((\frac{1}{I_1} + \frac{1}{I_3}) m_1^2 + (\frac{1}{I_2} + \frac{1}{I_3}) m_2^2 \right), \tag{3.4}$$

$$C^1_{rb21}(m) = \frac{1}{2}\left(\frac{(I_1 - I_2)I_3}{(I_1 + I_3)I_2} m_2^2 + m_3^2 \right). \tag{3.5}$$

PROOF : The above assertion follows from Proposition 3.1, by replacement of parameters $\alpha_i,\ i = 1, 2, 3$ with the corresponding values. □

Corollary 3.2 — The rigid body dynamics on the Lie group $SO(2,1)$ given by (2.10) have two Hamilton-Poisson realizations, namely: the first is generated by matrix Π_{rb21} (see Proposition 2.4) of $so(3)-$type and the second generated by matrix P^1_{rb21} (see Corollary 3.1) of tip $se(2)-$type. □

Define the functions $C^\alpha_{ab}, H^\alpha_{cd} \in C^\infty(\mathbf{R}^3, \mathbf{R})$ given by:

$$C^\alpha_{ab} = aC^\alpha + bH^\alpha, \quad H^\alpha_{cd} = cC^\alpha + dH^\alpha, \quad a, b, c, d \in \mathbf{R}, \tag{3.6}$$

i.e.

$$\begin{cases} C^\alpha_{ab}(x) = \dfrac{1}{2}\left(b\alpha_2 x_1^2 + (a\dfrac{\alpha_3}{\alpha_2} - b\alpha_1) x_2^2 - a x_3^2 \right) \\ \\ H^\alpha_{cd}(x) = \dfrac{1}{2}\left(d\alpha_2 x_1^2 + (c\dfrac{\alpha_3}{\alpha_2} - d\alpha_1) x_2^2 - c x_3^2 \right) \end{cases} \tag{3.7}$$

***Theorem* 3.1** — *The Euler top system (3.1) has an infinite number of Hamilton-Poisson realizations. More precisely,* $(\mathbf{R}^3, \{\cdot, \cdot\}^\alpha_{ab}, H^\alpha_{cd})$, *where:*

$$\{f, g\}^\alpha_{ab} = -\nabla C^\alpha_{ab} \cdot (\nabla f \times \nabla g), \quad (\forall) f, g \in C^\infty(\mathbf{R}^3, \mathbf{R}) \tag{3.8}$$

and $a, b, c, d \in \mathbf{R}$ *such that* $ad - bc = 1$.

PROOF : The relation (3.8) can be written in the equivalent form:

$$\{f, g\}_{ab}^{\alpha} = -\det \begin{pmatrix} \dfrac{\partial C_{ab}^{\alpha}}{\partial x_1} & \dfrac{\partial C_{ab}^{\alpha}}{\partial x_2} & \dfrac{\partial C_{ab}^{\alpha}}{\partial x_3} \\ \dfrac{\partial f}{\partial x_1} & \dfrac{\partial f}{\partial x_2} & \dfrac{\partial f}{\partial x_3} \\ \dfrac{\partial g}{\partial x_1} & \dfrac{\partial g}{\partial x_2} & \dfrac{\partial g}{\partial x_3} \end{pmatrix}. \tag{3.9}$$

It is easy to verify that the operation $\{\cdot, \cdot\}_{ab}^{\alpha}$ given by (3.8) is a Poisson bracket on $C^{\infty}(\mathbf{R}^3, \mathbf{R})$. We have $\dfrac{\partial H_{cd}^{\alpha}}{\partial x_1} = d\alpha_2 x_1$, $\dfrac{\partial H_{cd}^{\alpha}}{\partial x_2} = (c\dfrac{\alpha_3}{\alpha_2} - d\alpha_1)x_2$, $\dfrac{\partial H_{cd}^{\alpha}}{\partial x_3} = -cx_3$, $\dfrac{\partial C_{ab}^{\alpha}}{\partial x_1} = b\alpha_2 x_1$, $\dfrac{\partial C_{ab}^{\alpha}}{\partial x_2} = (a\dfrac{\alpha_3}{\alpha_2} - b\alpha_1)x_2$, $\dfrac{\partial C_{ab}^{\alpha}}{\partial x_3} = -ax_3$. Then:

$$\{x_1, H_{cd}^{\alpha}\}_{ab}^{\alpha} = -\det \begin{pmatrix} b\alpha_2 x_1 & (a\dfrac{\alpha_3}{\alpha_2} - b\alpha_1)x_2 & -ax_3 \\ 1 & 0 & 0 \\ d\alpha_2 x_1 & (c\dfrac{\alpha_3}{\alpha_2} - d\alpha_1)x_2 & -cx_3 \end{pmatrix} = (ad-bc)\alpha_1 x_2 x_3 = \dot{x}_1.$$

Similarly, $\{x_2, H_{cd}^{\alpha}\}_{ab}^{\alpha} = \alpha_2 x_1 x_3 = \dot{x}_2$ and $\{x_3, H_{cd}^{\alpha}\}_{ab}^{\alpha} = \alpha_3 x_1 x_2 = \dot{x}_3$.

Therefore, $(\mathbf{R}^3, \{\cdot, \cdot\}_{ab}^{\alpha}, H_{cd}^{\alpha})$ is an Hamilton-Poisson realization for (3.1).

The Poisson structure on $\mathbf{R}^3$ given by (3.8) is generated by the matrix

$$P_{ab}^{\alpha} = \begin{pmatrix} 0 & ax_3 & (a\dfrac{\alpha_3}{\alpha_2} - b\alpha_1)x_2 \\ -ax_3 & 0 & -b\alpha_2 x_1 \\ -(a\dfrac{\alpha_3}{\alpha_2} - b\alpha_1)x_2 & b\alpha_2 x_1 & 0 \end{pmatrix},$$

and C_{ab}^{α} is a Casimir for the configuration $(\mathbf{R}^3, \{\cdot, \cdot\}_{ab}^{\alpha})$. □

Remark 3.3 : If in P_{ab}^{α} we take $a = -1$ and $b = 0$ (hence $c = 0, d = -1$) we obtains P^{α} which generates the Poisson structure on $\mathbf{R}^3$ given in Proposition 3.1. □

Remark 3.4 : From Theorem 3.1 follows that the equations (3.1) are invariant, so the trajectories of motion of the Euler top system remain the same when the Hamiltonian and Casimir are replaced by $SL(2,\mathbf{R})-$combinations of H^α and C^α. □

4. The Euler Top System and Mathematical Pendulum

In this section we shall prove that in certain restrictions on α_i, the motion of Euler top system reduces to motion on surfaces described by conservation laws.

Using the fact that H^α and C^α given by (3.3) are first integrals (see Proposition 3.2) one easily prove that the Euler top system has the following first integrals:

$$H_0^\alpha(x) = \frac{1}{2}\left(x_1^2 - \frac{\alpha_1}{\alpha_2}x_2^2\right) \quad \text{and} \quad C_0^\alpha(x) = \frac{1}{2}\left(-\frac{\alpha_3}{\alpha_2}x_2^2 + x_3^2\right). \tag{4.1}$$

Proposition 4.1 — We assume that $\alpha_1\alpha_2 < 0$. The solution of the Euler top system (3.1), restricted to the constant level surface defined by:

$$x_1^2 - \frac{\alpha_1}{\alpha_2}x_2^2 = 2H = constant,\ H = H_0^\alpha > 0 \tag{4.2}$$

is:

$$\begin{cases} x_1(t) &= \sqrt{2H}\cdot\cos\dfrac{\theta(t)}{2} \\ x_2(t) &= \sqrt{2H}\sqrt{-\dfrac{\alpha_2}{\alpha_1}}\cdot\sin\dfrac{\theta(t)}{2} \\ x_3(t) &= \dfrac{1}{2\alpha_2}\sqrt{-\dfrac{\alpha_2}{\alpha_1}}\cdot\dot{\theta}(t), \end{cases} \tag{4.3}$$

where $\theta(t)$ is a solution of the pendulum equation:

$$\ddot{\theta}(t) = 2H\alpha_2\alpha_3\cdot\sin\theta(t). \tag{4.4}$$

Proof : By a direct computation, it is easy to see that

$$(i) \qquad x_1(t) = \sqrt{2H} \cdot \cos\frac{\theta(t)}{2}, \quad x_2(t) = \sqrt{2H}\sqrt{-\frac{\alpha_2}{\alpha_1}} \cdot \sin\frac{\theta(t)}{2}$$

are solutions of the first equation from (4.1).

We have $x_1(t)x_2(t) = H\sqrt{-\frac{\alpha_2}{\alpha_1}} \cdot \sin\theta(t)$ and using the third equation of Euler top system (3.1) we obtains:

$$(ii) \qquad \dot{x}_3(t) = H\sqrt{-\frac{\alpha_2}{\alpha_1}}\alpha_3 \cdot \sin\theta(t).$$

By deriving of the second relation of (i) with respect to t, we have:

$$\dot{x}_2(t) = \sqrt{2H}\sqrt{-\frac{\alpha_2}{\alpha_1}} \cdot \frac{\dot{\theta}(t)}{2} \cdot \cos\frac{\theta(t)}{2}$$

and using the first relation of (i), we obtains:

$$(iii) \qquad \dot{x}_2(t) = \frac{1}{2}\sqrt{-\frac{\alpha_2}{\alpha_1}} \cdot \dot{\theta}(t)x_1(t).$$

From (iii) and $\dot{x}_2(t) = \alpha_2 x_1 x_3$, we deduce:

$$(iv) \qquad x_3(t) = \frac{1}{2\alpha_2}\sqrt{-\frac{\alpha_2}{\alpha_1}} \cdot \dot{\theta}(t).$$

Therefore, the relations (4.3) are verified.

From (iv) follows:

$$(v) \qquad \dot{\theta}(t) = 2\alpha_2\sqrt{-\frac{\alpha_1}{\alpha_2}} \cdot x_3(t).$$

Differentiating again (v) and using (ii), it follows $\ddot{\theta}(t) = 2H\alpha_2\alpha_3 \cdot \sin\theta(t)$, i.e. the relation (4.4) holds. □

Proposition 4.2 — We suppose that $\alpha_2\alpha_3 < 0$. The solution of the Euler top system (3.1), restricted to the constant level surface defined by:

$$-\frac{\alpha_3}{\alpha_2}x_2^2 + x_3^2 = 2K = constant, \; K = C_0^\alpha > 0 \qquad (4.5)$$

is:

$$\begin{cases} x_1(t) &= -\dfrac{1}{2\alpha_2}\sqrt{-\dfrac{\alpha_2}{\alpha_3}}\cdot\dot{\theta}(t) \\ x_2(t) &= \sqrt{2K}\sqrt{-\dfrac{\alpha_2}{\alpha_3}}\cdot\cos\dfrac{\theta(t)}{2} \\ x_3(t) &= \sqrt{2K}\cdot\sin\dfrac{\theta(t)}{2}, \end{cases} \tag{4.6}$$

where $\theta(t)$ is a solution of the pendulum equation:

$$\ddot{\theta}(t) = -2K\alpha_1\alpha_2\cdot\sin\theta(t). \tag{4.7}$$

PROOF : Apply the same procedure as in the demonstration of Proposition 4.1. □

5. STABILITY PROBLEM FOR EULER TOP DYNAMICS

For the Euler top system (3.1) we introduce the following notations:

$$f_1(x) = \alpha_1 x_2 x_3, \quad f_2(x) = \alpha_2 x_1 x_3, \quad f_3(x) = \alpha_3 x_1 x_2. \tag{5.1}$$

Proposition 5.1 — The equilibrium states of the system (3.1) are the points $e_0 = (0,0,0),\ e_1^m = (m,0,0), e_2^m = (0,m,0)$ and $e_3^m = (0,0,m)$ for all $m \in \mathbf{R}^*$.

PROOF : The equilibrium states are the solutions of the system $f_i(x) = 0,\ i = \overline{1,3}$ where $f_i,\ i = \overline{1,3}$ are given by (5.1). □

Let $A(x)$ be the matrix of the linear part of the system (3.1), i.e.

$$A(x) = \begin{pmatrix} 0 & \alpha_1 x_3 & \alpha_1 x_2 \\ \alpha_2 x_3 & 0 & \alpha_2 x_1 \\ \alpha_3 x_2 & \alpha_3 x_1 & 0 \end{pmatrix}.$$

Proposition 5.2 — (i) The equilibrium states $e_i^m,\ m \in \mathbf{R}^*$ for $i = \overline{1,3}$ are spectrally stable if $\frac{\alpha_1\alpha_2\alpha_3}{\alpha_i} < 0$ and unstable if $\frac{\alpha_1\alpha_2\alpha_3}{\alpha_i} > 0$.

(ii) The equilibrium state $e_0 = (0,0,0)$ is spectrally stable.

PROOF : (i) *Case* $i = 1$. We have $A(e_1^m) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \alpha_2 m \\ 0 & \alpha_3 m & 0 \end{pmatrix}$. Its characteristic polynomial is $p_{A(e_1^m)}(\lambda) = \det \begin{pmatrix} -\lambda & 0 & 0 \\ 0 & -\lambda & \alpha_2 m \\ 0 & \alpha_3 m & -\lambda \end{pmatrix} = -\lambda(\lambda^2 - \alpha_2\alpha_3 m^2)$.

Then the characteristic roots of $A(e_1^m)$ are $\lambda_1 = 0$ and $\lambda_{2,3} = \pm m\sqrt{\alpha_2\alpha_3}$, if $\alpha_2\alpha_3 > 0$ şi $\lambda_{2,3} = \pm im\sqrt{-\alpha_2\alpha_3}$, if $\alpha_2\alpha_3 < 0$.

By Lyapunov's Theorem follows that e_1^m is spectrally stable for $\alpha_2\alpha_3 < 0$ and unstable for $\alpha_2\alpha_3 > 0$.

Case $i = 2$. The characteristic polynomial of $A(e_2^m) = \begin{pmatrix} 0 & 0 & \alpha_1 m \\ 0 & 0 & 0 \\ \alpha_3 m & 0 & 0 \end{pmatrix}$ is $p_{A(e_2^m)}(\lambda) = -\lambda(\lambda^2 - \alpha_1\alpha_3 m^2)$ with characteristic roots $\lambda_1 = 0$, $\lambda_{2,3} = \pm m\sqrt{\alpha_1\alpha_3}$.

Applying now similar arguments as in the case $i = 1$, one obtains the required results.

Case $i = 3$. The characteristic roots of $p_{A(e_3^m)}(\lambda) = -\lambda(\lambda^2 - \alpha_1\alpha_2 m^2)$ are $\lambda_1 = 0$, $\lambda_{2,3} = \pm m\sqrt{\alpha_1\alpha_2}$, if $\alpha_1\alpha_2 > 0$ and $\lambda_{2,3} = \pm im\sqrt{-\alpha_1\alpha_2}$, if $\alpha_1\alpha_2 < 0$.

Similarly, we conclude that the assertions hold in this case.

(ii) It is easy to see that e_0 is spectrally stable. □

Let us we discuss the nonlinear stability of the equilibrium states e_0 and $e_i^m,\ m \in$

$\mathbf{R}^*$ (if $\frac{\alpha_1\alpha_2\alpha_3}{\alpha_i} < 0$) for $i = \overline{1,3}$.

Proposition 5.3 — If $\frac{\alpha_1\alpha_2\alpha_3}{\alpha_i} < 0$ for $i = \overline{1,3}$, then the equilibrium state e_0 of the dynamics (3.1) is nonlinear stable.

PROOF : We suppose that $\alpha_1\alpha_2 < 0$. Consider the function

$$L^\alpha(x_1,x_2,x_3) = \frac{1}{2}(\alpha_2 x_1^2 - \alpha_1 x_2^2).$$

For $L^\alpha \in C^\infty(\mathbf{R}^3,\mathbf{R})$ we have successively:

(1) $L^\alpha(e_0) = L^\alpha(0,0,0) = 0.$

(2) If $\alpha_1 < 0$ and $\alpha_2 > 0$ (resp. $\alpha_1 > 0$ and $\alpha_2 < 0$) we have $L^\alpha(x_1,x_2,x_3) > 0$ (resp. $L^\alpha(x_1,x_2,x_3) < 0$) for all $x \in \mathbf{R}^3 \setminus \{e_0\}$.

(3) The derivative of L^α with respect to t along the trajectories of (3.1) is zero. Indeed, $\dfrac{dL^\alpha}{dt} = \alpha_2 x_1\dot{x}_1 - \alpha_1 x_2\dot{x}_2 = \alpha_2 x_1(\alpha_1 x_2 x_3) - \alpha_1 x_2(\alpha_2 x_1 x_3) = 0$. Therefore L^α is a Lyapunov function. Via Lyapunov's theorem ([2]), e_0 is nonlinear stable.

In hypothesis $\alpha_1\alpha_3 < 0$ (resp. $\alpha_2\alpha_3 < 0$) we can also verify that the function

$$L_1^\alpha(x_1,x_2,x_3) = \frac{1}{2}(\alpha_3 x_1^2 - \alpha_1 x_3^2) \quad \text{(resp. } L_2^\alpha(x_1,x_2,x_3) = \frac{1}{2}(\alpha_3 x_2^2 - \alpha_2 x_3^2))$$

is a Lyapunov function. □

***Theorem* 5.1** — *If* $\frac{\alpha_1\alpha_2\alpha_3}{\alpha_i} < 0$ *for* $i = \overline{1,3}$*, then* e_i^m, $m \in \mathbf{R}^*$ *is nonlinear stable.*

PROOF : For the cases $i = 1$ and $i = 3$ we shall make the proof using Arnold's method ([1]).

Case $i = 1$. We suppose that $\alpha_2\alpha_3 < 0$. Let the function $F_{1,\lambda}^\alpha \in C^\infty(\mathbf{R}^3,\mathbf{R})$, $\lambda \in \mathbf{R}$, given by:

$$F_{1,\lambda}^\alpha(x_1,x_2,x_3) = C^\alpha(x_1,x_2,x_3) - \lambda H^\alpha(x_1,x_2,x_3), \quad \text{i. e.}$$

$$F^{\alpha}_{1,\lambda}(x_1,x_2,x_3) = \frac{1}{2}\left(\frac{\alpha_3}{\alpha_2}x_2^2 - x_3^2\right) - \frac{\lambda}{2}\left(\alpha_2 x_1^2 - \alpha_1 x_2^2\right).$$

Then we have successively:

(1) $\nabla F^{\alpha}_{1,\lambda}(e_1^m) = 0$ if and only if $\lambda = 0$;

(2) $W := ker\ dH^{\alpha}(e_1^m) = span_{\mathbf{R}}\left((0,\ 1,\ 0)^T, (0,\ 0,\ 1)^T\right)$;

(3) For all $w \in W$, i.e. $w = (0,c,d)^T,\ c,d \in \mathbf{R}$, we have:

$$w^T \cdot \nabla^2 F^{\alpha}_{1,0}(e_1^m) \cdot w = \frac{\alpha_3}{\alpha_2} \cdot c^2 - d^2$$

and so $\nabla^2 F^{\alpha}_{1,0}(e_1^m)\Big|_{W\times W}$ is negative definite since $\alpha_2\alpha_3 < 0$.

Therefore via Arnold's method we conclude that e_1^m is nonlinear stable.

Case $i = 3$. We suppose $\alpha_1\alpha_2 < 0$. Let $F^{\alpha}_{3,\lambda} \in C^{\infty}(\mathbf{R}^3, \mathbf{R})$, $\lambda \in \mathbf{R}$ given by:

$$F^{\alpha}_{3,\lambda}(x_1,x_2,x_3) = H^{\alpha}(x_1,x_2,x_3) - \lambda C^{\alpha}(x_1,x_2,x_3), \quad \text{i.e.}$$
$$F^{\alpha}_{3,\lambda}(x_1,x_2,x_3) = \frac{1}{2}\left(\alpha_2 x_1^2 - \alpha_1 x_2^2\right) - \frac{\lambda}{2}\left(\frac{\alpha_3}{\alpha_2}x_2^2 - x_3^2\right).$$

Then we have successively:

(1) $\nabla F^{\alpha}_{3,\lambda}(e_3^m) = 0$ if and only if $\lambda = 0$;

(2) $W := ker\ dC^{\alpha}(e_3^m) = span_{\mathbf{R}}\left((1,\ 0,\ 0)^T, (0,\ 1,\ 0)^T\right)$;

(3) For all $v \in W$, i.e. $v = (a,b,0)^T,\ a,b \in \mathbf{R}$, we have:

$$v^T \cdot \nabla^2 F^{\alpha}_{3,0}(e_3^m) \cdot v = \alpha_2 a^2 - \alpha_1 b^2$$

and so $\nabla^2 F^{\alpha}_{3,0}(e_3^m)\Big|_{W\times W}$ is positive definite if $\alpha_1 < 0,\ \alpha_2 > 0$ and negative definite if $\alpha_1 > 0,\ \alpha_2 < 0$.

Therefore via Arnold's method we conclude that e_3^m is nonlinear stable.

Case $i = 2$. We suppose $\alpha_1\alpha_3 < 0$. We shall make the proof using Lyapunov's theorem ([2]). Let be the function $L^{\alpha} : \mathbf{R}^3 \to \mathbf{R}$ given by:

$$L^{\alpha}(x_1,x_2,x_3)=\left(\frac{\alpha_3}{\alpha_2}x_2^2-x_3^2-\frac{\alpha_3}{\alpha_2}m^2\right)^2-\frac{\alpha_3}{\alpha_1}x_1^2+x_3^2.$$

For the function L^{α} we have successively:

(1) $L^{\alpha}\in C^{\infty}(\mathbf{R}^3,\mathbf{R})$ and $L^{\alpha}(0,m,0)=0$;

(2) $L^{\alpha}(x_1,x_2,x_3)>0$, for all $x\in\mathbf{R}^3,\ x\neq e_2^m$, since $\alpha_1\alpha_3<0$;

(3) The derivative of L^{α} with respect to t along the trajectories of the dynamics (3.1) is zero. For this we verify that $\dfrac{dL^{\alpha}}{dt}=\dfrac{\partial L^{\alpha}}{\partial x_1}\dot{x}_1+\dfrac{\partial L^{\alpha}}{\partial x_2}\dot{x}_2+\dfrac{\partial L^{\alpha}}{\partial x_3}\dot{x}_3=0.$ Therefore L^{α} is a Lyapunov function and imply that e_2^m is nonlinear stable. □

Corollary 5.1 — The equilibrium states $e_0,e_1^m,e_2^m,e_3^m,m\in\mathbf{R}^*$ of the rigid body dynamics on Lie group $SO(2,1)$ given by (2.10)] have the following behavior:

(i) e_0,e_1^m and e_3^m are nonlinear stable; (ii) e_2^m are unstable.

PROOF : The assertions follows immediately from Theorem 5.1. □

CONCLUSION

In this paper we have presented the geometric and dynamical properties of a family of Hamilton-Poisson systems on $\mathbf{R}^3$, called general Euler top system. The denomination used is justified by the fact that the Euler equations of free rigid body belongs to respective family. □

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