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TRANSLATION INVARIANT DIFFUSIONS IN THE SPACE OF TEMPERED
DISTRIBUTIONS

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In this paper we prove existence and pathwise uniqueness for a class of stochastic differential equations (with coefficients σ_{ij}, b_i and initial condition y in the space of tempered distributions) that may be viewed as a generalisation of Ito's original equations with smooth coefficients. The solutions are characterized as the translates of a finite dimensional diffusion whose coefficients $\sigma_{ij} \star \tilde{y}, b_i \star \tilde{y}$ are assumed to be locally Lipschitz. Here \star denotes convolution and \tilde{y} is the distribution which on functions, is realised by the formula $\tilde{y}(r) := y(-r)$. The expected value of the solution satisfies a non linear evolution equation which is related to the forward Kolmogorov equation associated with the above finite dimensional diffusion.

Key words : Stochastic ordinary differential equations; Stochastic partial differential equations; non linear evolution equations; translations; diffusions; Hermite-Sobolev spaces; Monotonicity inequality.

1. INTRODUCTION

In this paper we consider a generalisation of Ito's well known equation viz.

$$dX_t = \sigma(X_t) \cdot dB_t + b(X_t) dt \quad ; \quad X_0 = x. \quad (1.1)$$

where $\sigma = (\sigma_{ij})_{1 \leq i, j \leq d}$ and $b = (b_1, \dots, b_d)$, and where σ_{ij} and b_i are functions on \mathbb{R}^d and (B_t) is a given d-dimensional Brownian motion ([7]). When σ, b are smooth functions we can use Ito's formula and the duality $\langle f, \delta_{X_t} \rangle = f(X_t)$ for the random distribution $\delta_{X_t} \in \mathcal{S}'$ to arrive at the stochastic differential equation satisfied by the \mathcal{S}' -valued process $(Y_t) \equiv (\delta_{X_t})$ viz.

$$dY_t = AY_t \cdot dB_t + LY_t dt \quad ; \quad Y_0 = y. \quad (1.2)$$

Here $A = (A_1, \dots, A_d)$, with L, A_j being non linear operators from \mathcal{S}' to \mathcal{S}' given by

$$\begin{aligned} A_j \phi &= - \sum_{i=1}^d \langle \sigma, \phi \rangle_{ij} \partial_i \phi \\ L \phi &= \frac{1}{2} \sum_{i,j=1}^d (\langle \sigma, \phi \rangle \langle \sigma, \phi \rangle^t)_{ij} \partial_{ij}^2 \phi - \sum_{i=1}^d \langle b, \phi \rangle_i \partial_i \phi \end{aligned}$$

for all $\phi \in \mathcal{S}'$ where $\langle \sigma, \phi \rangle_{ij}, \langle b, \phi \rangle_i$ is the (i,j)th entry $\langle \sigma_{ij}, \phi \rangle$ (respectively i th entry $\langle b_i, \phi \rangle$) of the matrix $\langle \sigma, \phi \rangle$ (respectively the vector $\langle b, \phi \rangle$) and the superscript t denotes transpose. The above equation (1.2) may be considered an equation for a system of diffusing particles, the initial configuration being specified by a tempered distribution y , with Ito's original equation corresponding to the position of a single diffusing particle (X_t) . For a single particle, the above equation (1.2) in \mathcal{S}' is equivalent to Ito's original equation (1.1). This can be seen by action on both sides of (1.2) by a test function $f \in \mathcal{S}$ and noting that the resulting equation is the Ito formula for $f(X_t)$, with X_t given by equation (1.1). We also note that the Ito formula for $f(X_t)$ is the point of departure for the celebrated martingale formulation of Stroock and Varadhan ([15]). For an alternate approach to constructing (finite dimensional)'symmetric' diffusions, see [2].

In Section 3, we construct solutions to the above equation in \mathcal{S}' when the coefficients σ_{ij}, b_i and the initial configuration y are allowed to be elements in \mathcal{S}' subject to the condition that the convolutions $\sigma_{ij} \star \tilde{y}, b_i \star \tilde{y}$ be locally Lipschitz functions on \mathbb{R}^d (see Theorem 3.4, Remark 3.7 below). We may take σ_{ij} and b_i to be arbitrary distributions - for example a Dirac distribution - provided y is sufficiently smooth or conversely y to be ‘arbitrary’ and σ_{ij}, b_i to be smooth. Our methods also extend to the case when the coefficients of L and A_i are non linear functions (see Remark 3.9). We use the countable Hilbertian structure of \mathcal{S}' viz. the fact that $\mathcal{S}' = \bigcup_{p \in \mathbb{R}} \mathcal{S}_p$, where the \mathcal{S}_p are Hilbert spaces, to formulate our equations ([8, 9]). The solutions $(Y_t(y))$ of equation (1.2), starting at y at time zero, are given explicitly as follows : Let $\tau_x, x \in \mathbb{R}^d$ denote the translation operators acting on \mathcal{S}' . Then $Y_t(y) = \tau_{z_t(y)}(y)$ where $(z_t(y))$ is a finite d-dimensional diffusion starting at zero and whose drift and diffusion coefficients are given by the convolutions viz. $\sigma_{ij} \star \tilde{y}$ and $b_i \star \tilde{y}$. Note that since $z_t(y)$ depends on y , the translation $\tau_{z_t(y)}(y)$ is non linear (unlike the Brownian case [13]). Existence and pathwise uniqueness of $(Y_t(y))$ are then a consequence of existence and pathwise uniqueness for $(z_t(y))$ and the Ito formula ([12]). Our proof of the above representation uses the uniqueness of solutions of equation (1.2) with random operator coefficients $\bar{A}(s, \omega), \bar{L}(s, \omega)$ (see [11, 10]) that satisfy the monotonicity inequality (see [4] and Theorem 3.5 below).

In Corollary (3.8), we show that $Y_t(y)$ has the translational invariance property viz. the solution corresponding to an initial condition which is a translate of $y \in \mathcal{S}'$ by $x \in \mathbb{R}^d$ viz. $Y_t(\tau_x(y))$ is equal to $\tau_{x+z_t(\tau_x(y))}(y)$ and $X_t(x, y) := x + z_t(\tau_x(y))$ is the solution of the SDE starting at x at $t = 0$, which is satisfied by $(z_t(y))$ for $x = 0$. Consequently, the state space \mathcal{S}_p of the flow generated by the solutions of (1.2) viz. $(Y_t(y))$ splits up into ‘irreducible’ components $C(y) := \{\tau_x(y); x \in \mathbb{R}^d\}$ on each of which $(Y_t(y))$ behaves as the finite dimensional diffusion $(X_t(x, y))$. This allows us to describe the Markov properties of $(Y_t(y))$ in terms of that of that of $(X_t(x, y))$. We do this in Section 4.

In Section 5, Theorem 5.4, we study the deterministic non linear evolution equation that results on taking expectations in equation (1.2). We represent the so-

lutions in terms of a ‘non linear convolution’. These are related to the fundamental solutions of the forward equation associated with the diffusion $X_t(x, y)$ (see Remark 5.5) We describe the latter in Theorem 5.6. This extends some results of [14] to the case when the coefficients are non smooth.

Physically, the form of the solutions viz. $Y_t = \tau_{z_t} y$ suggests a conservation law. Indeed, when the initial distribution y is given by an integrable function $y(r)$ and when the solution Y_t or equivalently z_t is defined for all $t \geq 0$ i.e. there is no explosion, then we have for all $t \geq 0$,

$$\int_{\mathbb{R}^d} Y_t(r) dr = \int_{\mathbb{R}^d} y(r) dr.$$

The form of the solution further suggests that the diffusion of a system of particles according to (1.2), starting in an initial configuration given say, by a function $y = y(r)$, in a diffusive medium represented by the coefficients σ_{ij} and b_i is equivalent to a translational flow of the same initial configuration, in a new medium - represented by coefficients $\sigma_{ij} \star \tilde{y}$ and $b_i \star \tilde{y}$ - which is the result of an interaction between the initial particles and the original diffusive medium. We refer to [1] for some related ideas.

2. PRELIMINARIES

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ be a filtered probability space satisfying the *usual conditions* viz. 1) (Ω, \mathcal{F}, P) is a complete probability space. 2) \mathcal{F}_0 contains all $A \in \mathcal{F}$, such that $P(A) = 0$, and 3) $\mathcal{F}_t = \bigcap_{s > t} \mathcal{F}_s$. On this probability space is given a standard d-dimensional \mathcal{F}_t - Brownian motion $(B_t) \equiv (B_t^1, \dots, B_t^d)$. We will denote the filtration generated by (B_t) as (\mathcal{F}_t^B) . Consider now the equation (1.1). Let $\bar{\sigma}_{ij}, \bar{b}_i$ be locally Lipschitz functions on \mathbb{R}^d for $i, j = 1, \dots, d$. Let $\bar{\sigma} := (\bar{\sigma}_{ij})$ and $(\bar{b}) := (\bar{b}_1, \dots, \bar{b}_d)$. We use the notation $\hat{\mathbb{R}}^d = \mathbb{R}^d \cup \{\infty\}$ for the one point compactification of \mathbb{R}^d .

Theorem 2.1 — *Let $\bar{\sigma}, \bar{b}, (B_t)$ be as above. Then $\exists \eta : \Omega \rightarrow (0, \infty], \eta$ an (\mathcal{F}_t^B) stopping time and an $\hat{\mathbb{R}}^d$ -valued, (\mathcal{F}_t^B) adapted process $(X_t)_{t \geq 0}$ such that*

1. For all $\omega \in \Omega$, $X_t(\omega) : [0, \eta(\omega)) \rightarrow \mathbb{R}^d$, is continuous and $X_t(\omega) = \infty$, $t \geq \eta(\omega)$
2. a.s. (P) , $\eta(\omega) < \infty$ implies $\lim_{t \uparrow \eta(\omega)} X_t(\omega) = \infty$.
3. a.s. (P) ,

$$X_t = x + \int_0^t \bar{\sigma}(X_s) \cdot dB_s + \int_0^t \bar{b}(X_s) ds \tag{2.1}$$

for $0 \leq t < \eta(\omega)$.

The solution (X_t, η) is (pathwise) unique i.e. if (X_t^1, η^1) is another solution then $P\{\eta = \eta_1, X_t = X_t^1, 0 \leq t < \eta\} = 1$.

PROOF : We refer to [5], Chapter IV, for the proofs. Theorem 2.3 for the proof of existence and Theorem 3.1 for the proof of uniqueness . □

Let $\alpha, \beta \in \mathbb{Z}_+^d := \{(x_1, \dots, x_d) : x_i \geq 0, x_i \text{ integer}\}$. Let $x^\alpha := x_1^{\alpha_1} \dots x_d^{\alpha_d}$ and $\partial^\beta := \partial_1^{\beta_1} \dots \partial_d^{\beta_d}$. For a multi index α , we use the notation $|\alpha| := \sum_{i=1}^d \alpha_i$. Let \mathcal{S} denote the space of rapidly decreasing smooth real functions on \mathbb{R}^d with the topology given by the family of semi norms $\Lambda_{\alpha, \beta}$, defined for $f \in \mathcal{S}$ and multi indices α, β by $\Lambda_{\alpha, \beta}(f) := \sup_x |x^\alpha \partial^\beta f(x)|$. Then $\{\mathcal{S}, \Lambda_{\alpha, \beta}\} : \alpha, \beta \in \mathbb{Z}_+^d$ is a Fréchet space. \mathcal{S}' will denote its continuous dual. The duality between \mathcal{S} and \mathcal{S}' will be denoted by $\langle \psi, \phi \rangle$ for $\phi \in \mathcal{S}$ and $\psi \in \mathcal{S}'$. For $x \in \mathbb{R}^d$ the translation operators $\tau_x : \mathcal{S} \rightarrow \mathcal{S}$ are defined as $\tau_x f(y) := f(y - x)$ for $f \in \mathcal{S}$ and then for $\phi \in \mathcal{S}'$ by duality : $\langle \tau_x \phi, f \rangle := \langle \phi, \tau_{-x} f \rangle$. Let $\{h_k; k \in \mathbb{Z}_+^d\}$ be the orthonormal basis in the real Hilbert space $L^2(\mathbb{R}^d, dx) \supset \mathcal{S}$ consisting of the Hermite functions (see for eg. [16]); here dx denotes Lebesgue measure. Let $\langle \cdot, \cdot \rangle_0$ be the inner product in $L^2(\mathbb{R}^d, dx)$. For $f \in \mathcal{S}$ and $p \in \mathbb{R}$ define the inner product $\langle f, g \rangle_p$ on \mathcal{S} as follows :

$$\langle f, g \rangle_p := \sum_{k=(k_1, \dots, k_d) \in \mathbb{Z}_+^d} (2|k| + d)^{2p} \langle f, h_k \rangle_0 \langle g, h_k \rangle_0$$

The corresponding norm will be denoted by $\|\cdot\|_p$. We define the Hilbert space \mathcal{S}_p as the completion of \mathcal{S} with respect to the norm $\|\cdot\|_p$. The following basic relations hold between the \mathcal{S}_p spaces (see for eg. [8, 9]): For $0 < q < p$, $\mathcal{S} \subset \mathcal{S}_p \subset \mathcal{S}_q \subset L^2 = \mathcal{S}_0 \subset \mathcal{S}_{-q} \subset \mathcal{S}_{-p} \subset \mathcal{S}'$. Further, $\mathcal{S}' = \bigcup_{p \in \mathbb{R}} \mathcal{S}_p$ and $\bigcap_{p \in \mathbb{R}} \mathcal{S}_p = \mathcal{S}$. If $\{h_k^p : k \in \mathbb{Z}_+^d\}$ denotes the orthonormal basis in \mathcal{S}_p consisting of the (suitably normalised) Hermite functions, then the dual space \mathcal{S}'_p maybe identified with \mathcal{S}_{-p} , via the basis $\{h_k^{-p} : k \in \mathbb{Z}_+^d\}$ of \mathcal{S}_{-p} . For $\phi \in \mathcal{S}$ and $\psi \in \mathcal{S}'$ the bilinear form $(\phi, \psi) \rightarrow \langle \psi, \phi \rangle$ also gives the duality between $\mathcal{S}_p (\supset \mathcal{S})$ and $\mathcal{S}_{-p} (\subset \mathcal{S}')$. It is also well known that $\partial_i : \mathcal{S}_p \rightarrow \mathcal{S}_{p-\frac{1}{2}}$ are bounded linear operators for every $p \in \mathbb{R}$ and $i = 1, \dots, d$. Suppose for $i, j = 1, \dots, d$, $\sigma_{ij}, b_i \in \mathcal{S}_{-p}$. Let $\sigma := (\sigma_{ij})_{1 \leq i, j \leq d}$ and $b := (b_1 \dots b_d)$. We can then define for $j = 1, \dots, d$ the non linear operators $A_j, L : \mathcal{S}_p \rightarrow \mathcal{S}_{p-1}$ as follows:

$$A_j \phi := - \sum_{i=1}^d \langle \sigma, \phi \rangle_{ij} \partial_i \phi$$

$$L \phi := \frac{1}{2} \sum_{i,j=1}^d (\langle \sigma, \phi \rangle \langle \sigma, \phi \rangle^t)_{ij} \partial_{ij}^2 \phi - \sum_{i=1}^d \langle b, \phi \rangle_i \partial_i \phi.$$

Here $\langle \sigma, \phi \rangle$ is the matrix whose (i, j) th entry $\langle \sigma, \phi \rangle_{ij}$ is $\langle \sigma_{ij}, \phi \rangle$, $\langle \sigma, \phi \rangle^t$ its transpose and $\langle b, \phi \rangle$ is the vector whose i th entry $\langle b, \phi \rangle_i$ is $\langle b_i, \phi \rangle$. The d -tuple of operators (A_1, \dots, A_d) will be denoted by A .

3. STOCHASTIC DIFFERENTIAL EQUATIONS IN \mathcal{S}'

We now consider a stochastic partial differential equation in \mathcal{S}' driven by the Brownian motion (B_t) and differential operators $A_i, i = 1, \dots, d$ and L defined above with given coefficients $\sigma_{ij}, b_i, i, j = 1, \dots, d$ in the space \mathcal{S}_{-p} for some fixed $p \in \mathbb{R}$ and initial condition $y \in \mathcal{S}_p$ viz.

$$dY_t = A(Y_t) \cdot dB_t + L(Y_t) dt \quad ; \quad Y_0 = y. \quad (3.1)$$

Note that if (Y_t) is an \mathcal{S}_p valued, locally bounded, (\mathcal{F}_t) adapted process then $A_i(Y_s), i = 1, \dots, d$ and $L(Y_s)$ are \mathcal{S}_{p-1} valued, adapted, locally bounded processes and hence the stochastic integrals $\int_0^t A_i(Y_s) dB_s^i$ and $\int_0^t L(Y_s) ds$ are well

defined \mathcal{S}_{p-1} valued, continuous \mathcal{F}_t adapted processes and in addition, the former processes are \mathcal{F}_t local martingales . We then have the following definition of a ‘local’ strong solution of equation (3.1).

Definition 3.1 — Let $p \in \mathbb{R}$. Let $y \in \mathcal{S}_p$, $\sigma_{ij}, b_i \in \mathcal{S}_{-p}$ and $\{B_t, \mathcal{F}_t\}$ the given standard (\mathcal{F}_t) Brownian motion. Let δ be an arbitrary state, viewed as an isolated point of $\hat{\mathcal{S}}_p := \mathcal{S}_p \cup \{\delta\}$. By an $\hat{\mathcal{S}}_p$ valued, strong, (local) solution of equation (3.1), we mean a pair $(Y_t(y), \eta)$ where $\eta : \Omega \rightarrow (0, \infty]$ is an \mathcal{F}_t^B stop time and $(Y_t(y))$ an $\hat{\mathcal{S}}_p$ valued (\mathcal{F}_t^B) adapted process such that

1. For all $\omega \in \Omega$, $Y_t(y, \omega) : [0, \eta(\omega)) \rightarrow \mathcal{S}_p$ is a continuous map and $Y_t(y, \omega) = \delta, t \geq \eta(\omega)$
2. a.s. (P) the following equation holds in \mathcal{S}_{p-1} for $0 \leq t < \eta(\omega)$,

$$Y_t(y) = y + \sum_{j=1}^d \int_0^t A_j(Y_s(y)) dB_s^j + \int_0^t L(Y_s(y)) ds. \quad (3.2)$$

Remark 3.2 : By usual stopping arguments, the \mathcal{S}_{p-1} valued stochastic integrals on the right hand side in equation (3.2) can be shown to be finite , almost surely , for $0 \leq t < \eta$. Also , all functions $f : \mathcal{S}_p \rightarrow \mathbb{R}$ are extended to functions on $\hat{\mathcal{S}}_p$ by setting $f(\delta) := 0$.

Definition 3.3 — We say that pathwise uniqueness holds for equation (3.1) iff given (B_t) , a standard Brownian motion on a probability space (Ω, \mathcal{F}, P) , and given $y \in \mathcal{S}_p, \sigma_{ij}, b_i \in \mathcal{S}_{-p}$, and any two \mathcal{S}_p valued strong solutions $(Y_t^i(y), \eta^i), i = 1, 2$ of equation (3.1), we have $P\{Y_s^1(y) = Y_s^2(y), 0 \leq s < \eta^1 \wedge \eta^2\} = 1$.

Theorem 3.4 — Let $y \in \mathcal{S}_p, \sigma_{ij}, b_i \in \mathcal{S}_{-p}$ and $\{B_t, \mathcal{F}_t\}$ the given standard (\mathcal{F}_t) Brownian motion. Suppose the \mathbb{R}^{d^2} valued function $\bar{\sigma}(x) := (\langle \sigma_{ij}, \tau_x y \rangle)$ and the \mathbb{R}^d valued function $\bar{b}(x) = (\langle b_i, \tau_x y \rangle)$ are locally Lipschitz. Then equation (3.1) has a unique strong solution.

To prove the uniqueness assertion in Theorem 3.4, we need Lemma (3.6) below. It is of independent interest since it characterises the solutions of equation (3.1). The proof of the lemma in turn depends on the so called ‘Monotonicity inequality’ which we now state. Given real numbers $\sigma_{ij}, b_i, i, j = 1, \dots, d$, let $\sigma = (\sigma_{ij})$ and σ^t the transpose of σ . We define the constant coefficient differential operators $A_{oj}, j = 1 \dots d$ and L_o as follows :

$$\begin{aligned} A_{oj}\phi &= -\sum_{i=1}^d \sigma_{ij} \partial_i \phi, \quad j = 1, \dots, d \\ L_o\phi &= \frac{1}{2} \sum_{i,j=1}^d (\sigma\sigma^t)_{ij} \partial_{ij}^2 \phi - \sum_{i=1}^d b_i \partial_i \phi. \end{aligned}$$

Theorem 3.5 — *Let $\alpha > 0$. Then there exists a constant $C = C(\alpha, p, d) > 0$ depending only on α, p and d such that for all $\phi \in \mathcal{S}_p$, and all $\sigma_{i,j}, b_i, i, j = 1, \dots, d$ with $\alpha \geq \max_{i,j} \{|\sigma_{i,j}|, |b_i|\}$, we have*

$$2\langle \phi, L_o\phi \rangle_{p-1} + \sum_{i=1}^d \|A_{oi}\phi\|_{p-1}^2 \leq C \|\phi\|_{p-1}^2.$$

PROOF : See [4].

Lemma 3.6 — Suppose equation (3.4) has an \mathcal{S}_p -valued strong solution $(Y_t(y), \eta)$. Define the continuous semi-martingale in \mathbb{R}^d as follows:

$$z_t(y) := \int_0^{t \wedge \eta} \langle \sigma, Y_s \rangle \cdot dB_s + \int_0^{t \wedge \eta} \langle b, Y_s \rangle ds.$$

Then a.s P, $Y_t(y) = \tau_{z_t(y)}(y)$, for $0 \leq t < \eta$.

PROOF : We use the notation Y_t, z_t for $Y_t(y), z_t(y)$ respectively. We apply the Ito formula in Theorem 2.3 of [12] to the process (τ_{z_t}) to get the following

equation in \mathcal{S}_{p-1} :

$$\begin{aligned} \tau_{z_t} y &= \tau_{z_0} y - \int_0^t \partial_i(\tau_{z_s} y) dz_s^i \\ &\quad + \frac{1}{2} \sum_{i,j=1}^d \int_0^t \partial_{ij}^2(\tau_{z_s} y) d\langle z^i, z^j \rangle_s \\ &= y + \int_0^t \bar{A}(s, \omega)(\tau_{z_s} y) \cdot dB_s \\ &\quad + \int_0^t \bar{L}(s, \omega)(\tau_{z_s} y) ds \end{aligned}$$

where $(\bar{A}_j(s)), j = 1 \cdots d$ and $(\bar{L}(s))$ are operator valued \mathcal{F}_t adapted processes such that for $j = 1, \dots, d$ and each (s, ω) , $\bar{A}_j(s, \omega)$ and $\bar{L}(s, \omega)$ are linear operators from \mathcal{S}_p into \mathcal{S}_{p-1} , defined as follows :

$$\bar{A}_j(s, \omega)\phi = - \sum_{i=1}^d \langle \sigma, Y_s(\omega) \rangle_{ij} \partial_i \phi, \quad j = 1, \dots, d$$

$$\bar{L}(s, \omega)\phi = \frac{1}{2} \sum_{i,j=1}^d (\langle \sigma, Y_s(\omega) \rangle \langle \sigma, Y_s(\omega) \rangle^t)_{ij} \partial_{ij}^2 \phi - \sum_{i=1}^d \langle b, Y_s(\omega) \rangle_i \partial_i \phi.$$

Note that almost surely for $0 \leq t < \eta$, (Y_t) satisfies the same equation as $(\tau_{z_t} y)$ in \mathcal{S}_{p-1} i.e.

$$Y_t = y + \int_0^t \bar{A}(s) Y_s \cdot dB_s + \int_0^t \bar{L}(s) Y_s ds.$$

Let (σ_n) be a sequence of (\mathcal{F}_t) stopping times, such that almost surely, $0 \leq \sigma_n < \eta$, $\sigma_n \uparrow \eta$ and such that for each $n \geq 1$, the processes $(\langle \sigma_{ij}, Y_{\sigma_n \wedge s} \rangle)$ and $(\langle b_i, Y_{\sigma_n \wedge s} \rangle)$ are uniformly bounded for all i, j . Define for $n \geq 1$, $X_t^n := Y_{\sigma_n \wedge t} -$

$\tau_{z_{\sigma_n \wedge t} y}$. Let for $n \geq 1, C_n$ be the constant obtained from Theorem 3.5 above applied to the operators $A_{oi} = A_i(s, \omega), i = 1, \dots, d$ and $L_o = L(s, \omega)$ for fixed (s, ω) with $s \leq \sigma_n(\omega), n \geq 1$ and

$$\alpha_n > \sup_{i,j,s,\omega} \{ \langle \sigma_{ij}, Y_s(\omega) \rangle, \langle b_i, Y_s(\omega) \rangle \}$$

where the supremum is taken over $1 \leq i, j, \leq d, s \leq \sigma_n(\omega)$ and $\omega \in \Omega$. We then have, using integration by parts,

$$\begin{aligned} \|X_{t \wedge \sigma_n}^n\|_{p-1}^2 &= \int_0^{t \wedge \sigma_n} \left(\sum_{i=1}^d \|\bar{A}_i(s) X_s^n\|_{p-1} + 2 \langle X_s^n, \bar{L}(s) X_s^n \rangle_{p-1} \right) ds + M_{t \wedge \sigma_n}^n \\ &\leq C_n \int_0^{t \wedge \sigma_n} \|X_s^n\|_{p-1}^2 ds + M_{t \wedge \sigma_n}^n \\ &\leq C_n \int_0^t \|X_{s \wedge \sigma_n}^n\|_{p-1}^2 ds + M_{t \wedge \sigma_n}^n \end{aligned}$$

where (M_t^n) is a continuous local martingale. The Gronwall inequality now implies that for every $t \geq 0$ and $n \geq 1, \Rightarrow E \|X_{t \wedge \sigma_n}^n\|_{p-1}^2 = 0$. It follows that $Y_t = \tau_{z_t}(y)$ almost surely for $0 \leq t < \eta$. \square

Remark 3.7 : For a function f we use the notation \tilde{f} to denote the function $\tilde{f}(r) := f(-r)$. For a distribution y we define the distribution \tilde{y} via duality i.e. $\langle \tilde{y}, f \rangle := \langle y, \tilde{f} \rangle \forall f \in \mathcal{S}$. We note that $\bar{\sigma}_{ij}(x) := \langle \sigma_{ij}, \tau_x y \rangle = \sigma_{ij} \star \tilde{y}(x)$ where \star denotes convolution. It is well known that when $\sigma_{ij} \in \mathcal{S}'$ and $y \in \mathcal{S}$ the convolution $\sigma_{ij} \star y$ is \mathcal{C}^∞ (see Theorem 30.2, [3]).

We have the following Corollary to Lemma 3.6.

Corollary 3.8 — Let $y \in \mathcal{S}_p, \sigma_{ij}, b_i \in \mathcal{S}_{-p}, i, j = 1, \dots, d$ and $x \in \mathbb{R}^d$. Suppose that the coefficients in equation (2.1) are given as $\bar{\sigma}_{ij} = \sigma_{ij} \star \tilde{y}$ and $\bar{b}_i = b_i \star \tilde{y}$. Let $(Y_t(\tau_x(y)), \eta)$ be a solution of equation (3.1) with initial condition $\tau_x(y)$. Let $(z_t(\tau_x(y)))$ be given by Lemma 3.6 with $Y_t = \tau_{z_t(\tau_x(y))}(\tau_x(y)) =$

$\tau_{x+z_t(\tau_x(y))}(y), t < \eta$. Then the process $(x + z_t(\tau_x(y)))$ solves equation (2.1) upto the random time η , with initial condition x .

PROOF : Let $X_t := x + z_t(\tau_x(y)), t < \eta$. We have a.s. for $t < \eta$,

$$\begin{aligned} X_t = x + z_t(\tau_x(y)) &= x + \int_0^t \langle \sigma, \tau_{z_s(\tau_x(y))}(\tau_x(y)) \rangle \cdot dB_s \\ &\quad + \int_0^t \langle b, \tau_{z_s(\tau_x(y))}(\tau_x(y)) \rangle ds \\ &= x + \int_0^t \langle \sigma, \tau_{x+z_s(\tau_x(y))}(y) \rangle \cdot dB_s \\ &\quad + \int_0^t \langle b, \tau_{x+z_s(\tau_x(y))}(y) \rangle ds \\ &= x + \int_0^t \bar{\sigma}(X_s) \cdot dB_s + \int_0^t \bar{b}(X_s) ds. \end{aligned}$$

Since $\bar{\sigma}(x) = \langle \sigma, \tau_x(y) \rangle$ and $\bar{b} = \langle b, \tau_x(y) \rangle$, the result follows from Theorem 2.1. □

PROOF OF THEOREM 3.4 : Assume that $\bar{\sigma}_{ij}$ and \bar{b}_i are locally Lipschitz functions on \mathbb{R}^d . Let $(z_t(y), \eta)$ be a solution of equation (2.1). $z_t := z_t(y)$. Define the S_p valued process (Y_t) as follows :

$$\begin{aligned} Y_t &:= \tau_{z_t} y, & t < \eta \\ &\delta & t \geq \eta. \end{aligned}$$

Note that $\bar{\sigma}_{ij}(z_t) = \langle \sigma_{ij}, \tau_{z_t} y \rangle = \langle \sigma_{ij}, Y_t \rangle$ and similarly $\bar{b}_i(z_t) = \langle b_i, Y_t \rangle, t < \eta$. Applying the Ito formula in Theorem 2.3 of [12] to the process $(\tau_{z_t} y)$ as in the proof of Lemma 3.6, we get , almost surely for $t < \eta$,

$$\begin{aligned}
Y_t = \tau_{z_t} y &= \tau_{z_0} y - \sum_{i=1}^d \int_0^t \partial_i \tau_{z_s} y \, dz_s^i + \frac{1}{2} \sum_{i,j=1}^d \int_0^t \partial_{ij}^2 \tau_{z_s} y \, d\langle z^i, z^j \rangle_s. \\
&= y + \sum_{j=1}^d \int_0^t A_j(Y_s) \, dB_s^j + \int_0^t L(Y_s) \, ds.
\end{aligned}$$

Hence (Y_t, η) is a solution. Suppose (Y_t^1, η^1) and (Y_t^2, η^2) are two solutions. Define the processes (z_t^i) as follows : For $0 \leq t < \eta^i$,

$$z_t^i := \int_0^{t \wedge \eta^i} \langle \sigma, Y_s^i \rangle \cdot dB_s + \int_0^{t \wedge \eta^i} \langle b, Y_s^i \rangle \, ds.$$

We take $z_t^i := \infty$ for $t > \eta^i$. Then by Lemma 3.6 $Y^i(t) = \tau_{z_t^i}(\phi)$, $t < \eta^i$. But then by Corollary 3.8, (z_t^i, η^i) , $i = 1, 2$ solves equation (2.1) with initial condition $x = 0$, upto η^i , $i = 1, 2$. The proof of uniqueness in Theorem 3.1 of [5] can be applied here (with appropriate modification to take care of the limit at $\eta^1 \wedge \eta^2$) to conclude that, almost surely, $z_t^1 = z_t^2$ for $0 \leq t < \eta^1 \wedge \eta^2$. It follows that almost surely, $Y_t^1 = Y_t^2$ for $0 \leq t < \eta^1 \wedge \eta^2$. \square

Remark 3.9 : Instead of the linear functionals $\langle \sigma_{ij}, \cdot \rangle, \langle b_i, \cdot \rangle : \mathcal{S}_p \rightarrow \mathbb{R}$ as coefficients of L and A_i , we could take more general (non linear) functions $\sigma_{ij}, b_i : \mathcal{S}_p \rightarrow \mathbb{R}$ as the coefficients in L and the A_i 's. Theorem 3.4 remains true provided $\sigma_{ij}(\tau_x y), b_i(\tau_x y)$ are locally Lipschitz functions on \mathbb{R}^d for $y \in \mathcal{S}_p$.

Consider the ordinary differential equation

$$\dot{y} = b(y); \quad y(0) = y_0$$

where $b = (b_1, \dots, b_d)$ is a smooth vector field on \mathbb{R}^d . The following corollary generalises this equation when the b_i are tempered distributions on \mathbb{R}^d .

Corollary 3.10 — Suppose $\sigma \equiv 0, b_i \in S_{-p}, y \in S_p$. Then the unique solution of the first order evolution equation viz.

$$\begin{aligned} \partial_t Y_t &= - \sum_{i=1}^d \langle b_i, Y_t \rangle \partial_i Y_t, ; Y_0 = y. \\ &= - \langle b, Y_t \rangle \cdot \nabla Y_t, ; Y_0 = y. \end{aligned}$$

is given by $\{Y_t, 0 \leq t < \eta\}$ where $Y_t := \tau_{z_t} y$ for $0 \leq t < \eta$ and $\{z_t, 0 \leq t < \eta\}$ is the unique solution of the ordinary differential equation

$$\dot{z}_t = \bar{b}(z_t), \quad z_0 = 0.$$

We can characterise the unique global solutions of equation (3.1) in terms of the unique solutions of equation (2.1) described in Theorem 2.1.

Theorem 3.11 — Let $\sigma_{ij}, b_i, \bar{\sigma}_{ij}, \bar{b}_i, y$ be as in Theorem 3.4. Then there exists a solution $(Y_t(y), \eta)$ of equation (3.1) with the property that $Y_t(y) = \tau_{z_t(y)}(y), t < \eta$, where $(z_t(y), \eta)$ is the unique solution of equation (2.1) given by Theorem 2.1. In particular, almost surely, on the set $\{\eta < \infty\}$, $\lim_{t \uparrow \eta} z_t(y) = \infty$. The solution is pathwise unique, i.e. if (Y_t^1, η^1) is another such solution, then $P\{\eta = \eta^1, Y_t(y) = Y_t^1, 0 \leq t < \eta\} = 1$.

PROOF : Let $(z_t(y), \eta)$ be the unique solution of equation (2.1) given by Theorem 2.1 and define $Y_t(y) := \tau_{z_t}(y), t < \eta$ and $Y_t(y) := \delta, t \geq \eta$. Then as in the proof of Theorem 3.4, $(Y_t(y), \eta)$ is a solution of equation (3.1). The uniqueness follows from the uniqueness of $(z_t(y), \eta)$. \square

The following proposition characterises the global solutions of equation (3.1) as processes in \mathcal{S}' . It shows that they are better behaved than their finite dimensional counterparts. It concerns the purely analytical behaviour of $\tau_x y$ as $|x| \rightarrow \infty$.

Proposition 3.12 — Let $\sigma_{i,j}, b_i, y$ be as in Theorem 3.4, with $y \neq 0$. Let $Y_t(y) = \tau_{z_t(y)} y$ be the unique solution upto time η of equation (3.1), where $(z_t(y))$ solves equation (2.1) with $x = 0$ and $\bar{\sigma}_{ij}, \bar{b}_i$ as in Corollary 3.8. Fix $\omega \in \Omega$. Then, $z_t(y, \omega) \rightarrow \infty$ as $t \rightarrow \eta(\omega)$ whenever $Y_t(y, \omega) \rightarrow 0$ weakly in \mathcal{S}' as $t \rightarrow \eta(\omega)$. Conversely, suppose one of the following two conditions is satisfied viz.

1. y is a square integrable function i.e. $y \in L^2(\mathbb{R}^d) = \mathcal{S}_0$.
2. y has compact support i.e. $y \in \mathcal{E}'$.

Then, $z_t(y, \omega) \rightarrow \infty$ as $t \rightarrow \eta(\omega)$ implies $Y_t(y, \omega) \rightarrow 0$ weakly in \mathcal{S}' .

PROOF : Let $Y_t(\omega) := Y_t(y, \omega)$ and $z_t(\omega) := z_t(y, \omega)$. Suppose first that for $\omega \in \Omega$, $Y_t(\omega) \rightarrow 0$ weakly in \mathcal{S}' and assume that $z_t(\omega) \not\rightarrow \infty$. Since the neighborhoods of ∞ in \mathbb{R}^d are complements of compact sets, if $z_t(\omega) \not\rightarrow \infty$, then there exists a ball $B(0, r)$ of radius r around zero and a sequence $t_n \uparrow \eta(\omega)$ such that $z_{t_n}(\omega) \in B(0, r)$ for all $n \geq 1$. The compactness of $B(0, r)$ implies the existence of a subsequence of (t_n) , denoted again by t_n , and $z \in B(0, r)$ such that $z_{t_n}(\omega) \rightarrow z$. The continuity of the translations and the weak convergence of $Y_{t_n}(\omega)$ to zero now forces $\tau_z(y) = 0$. This implies $y = 0$, a contradiction.

For the converse suppose first that $y \in L^2(\mathbb{R}^d)$. If $z_t(\omega) \rightarrow \infty$ as $t \rightarrow \eta(\omega)$ then for $\phi \neq 0 \in \mathcal{S}$,

$$\langle \phi, Y_t(\omega) \rangle = \int_{\{x:|x|<n\}} y(x)\phi(x - z_t(\omega)) dx + \int_{\{x:|x|\geq n\}} y(x)\phi(x - z_t(\omega)) dx$$

Since $y \in L^2(\mathbb{R}^d)$, the second integral can be made small, independent of t by choosing n large. For the first integral, we can choose t sufficiently close to $\eta(\omega)$ so that $\phi(x - z_t(\omega))$ is small, uniformly for $|x| \leq n$, proving the case when $y \in L^2(\mathbb{R}^d)$.

Suppose now that the second case holds i.e. y has compact support. Let support $y \subseteq K$ and let $N = \text{order}(y) + 2d$. Then there exist continuous functions g_α , $|\alpha| \leq N$, support $g_\alpha \subseteq V$ where V is an open set having compact closure, containing K , such that

$$y = \sum_{|\alpha| \leq N} \partial^\alpha g_\alpha.$$

See [3], Theorem 24.5, Corollary 3. Then for $\phi \neq 0 \in \mathcal{S}$,

$$\begin{aligned} \langle \phi, Y_t(\omega) \rangle &= \sum_{|\alpha| \leq N} \langle \phi, \tau_{z_t(\omega)}(\partial^\alpha g_\alpha) \rangle \\ &= \sum_{|\alpha| \leq N} (-1)^{|\alpha|} \langle \partial^\alpha \phi, \tau_{z_t(\omega)}(g_\alpha) \rangle. \end{aligned}$$

The same arguments as in the first case applied to each of the terms in the above sum will now also prove the second case. □

Remark 3.13 : It is easy to see that $z_t(y, \omega) \rightarrow \infty$ does not in general imply that $Y_t(y, \omega) \rightarrow 0$ weakly in \mathcal{S}' . For example if $d = 1$ and $y = c$, a non zero constant, then $y \in \mathcal{S}_{-p}$ for some $p > 0$. If σ, b are integrable functions with non zero integrals over \mathbb{R} then it is easy to see that $\eta = \infty$ and $z_t(y, \omega) \rightarrow \infty$ almost surely on the one hand, while on the other $Y_t(y, \omega) = c$ for all $t \geq 0$.

4. $(Y_t(y))$ AS A MARKOV PROCESS ON $\hat{\mathcal{S}}_p$.

In this section, we study the Markov properties of the solutions of equation (3.1) viz. $(Y_t(y))$. For this purpose, it is essential to obtain a version of $(Y_t(y))$ which is jointly measurable in y as well. It is of course no accident and certainly not unreasonable that the Markov properties of $Y_t(y)$ derive from that of the diffusion $(X(x, y, t))$ generated by equation (2.1). In Proposition 4.1 below, we obtain a version of $((X(x, y, t)))$ which is also jointly measurable in x and y . Let $\sigma_{ij}, b_i \in \mathcal{S}_{-p}, i, j = 1, \dots, d$ and $y \in \mathcal{S}_p$. Let $\bar{\sigma}_{ij}(x, y) = \langle \sigma_{ij}, \tau_x y \rangle$ and $\bar{b}_i(x, y) = \langle b_i, \tau_x y \rangle$ be locally Lipschitz functions on \mathbb{R}^d (earlier denoted by $\bar{\sigma}_{ij}(x), \bar{b}_i(x)$). Then by Theorem 2.1, equation (2.1) has a solution for each $x \in \mathbb{R}^d$. We will denote the Borel σ field on $\hat{\mathbb{R}}^d$ by \mathcal{B}_d .

Proposition 4.1 — Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ be a filtered probability space satisfying the usual conditions and (B_t) be a standard \mathcal{F}_t Brownian motion on it. Then, there exists a map $X : \hat{\mathbb{R}}^d \times \hat{\mathcal{S}}_p \times [0, \infty) \times \Omega \rightarrow \hat{\mathbb{R}}^d$ which is $\mathcal{B}_d \otimes \mathcal{B}(\hat{\mathcal{S}}_p) \otimes \mathcal{B}[0, \infty) \otimes \mathcal{F}$ measurable and such that for each $(x, y) \in \mathbb{R}^d \times \mathcal{S}_p$, $(X(x, y, t))$ is a solution of equation (2.1) given by Theorem 2.1.

PROOF : We note that $\bar{\sigma}(x, y)$ and $\bar{b}(x, y)$ are jointly measurable in (x, y) . Choose for each $n \geq 1$, and $i, j = 1, \dots, d$, Lipschitz functions in $x, \sigma_{ij}^n(x, y), b_i^n(x, y)$, measurable in (x, y) such that $\sigma_{ij}^n(x, y) = \bar{\sigma}_{ij}(x, y)$, $b_i^n(x, y) = \bar{b}_i(x, y)$, $|x| \leq n$. Consider equation (2.1) with $\bar{\sigma}_{ij}, \bar{b}_i$ replaced by $\sigma_{ij}^n(x), b_i^n(x)$ viz.

$$dX_t = \sigma^n(X_t, y).dB_t + b^n(X_t, y)dt \quad ; \quad X_0 = x. \quad (4.1)$$

Let for $n \geq 1, k \geq 1, (X^{n,k}(x, y, t))$ be defined iteratively (in vector form) by
$$X^{n,k}(x, y, t) := x + \int_0^t \sigma^n(X^{n,k-1}(x, y, s), y).dB_s + \int_0^t b^n(X^{n,k-1}(x, y, s), y) ds$$
 with $X^{n,0}(x, y, t) := x$ for all $t \geq 0$. It is a well known property of stochastic integrals that they are measurable with respect to a parameter when the integrands are measurable with respect to the same parameter (see [6], Theorem 17.25). Using an inductive argument and the fact that $\sigma_{ij}^n(x, y), b_i^n(x, y)$ are jointly measurable in (x, y) it follows that the integrals in the right hand side of the above equation have jointly measurable versions in (x, y, t, ω) and the same follows for $X^{n,k}(x, y, t, \omega)$. Then by the method of successive approximations, we get solutions $(X^n(x, y, t, \omega))$, of equation (4.1), where $X^n(x, y, t, \omega) = \lim_{k \rightarrow \infty} X^{n,k}(x, y, t, \omega)$, jointly measurable in (x, y, t, ω) and for each (x, y) , progressively measurable in (t, ω) (see [5], Chapter IV, Theorem 3.1). Note that for each (x, y) , $(X^n(x, y, t, \omega))$ also solves equation (2.1) upto the random time $\eta^n(x, y, \omega)$ defined as

$$\begin{aligned} \eta^n(x, y, \omega) &:= \inf\{t \geq 0 : \|X^n(x, y, t, \omega)\| \geq n\} \\ &= \inf\{t \geq 0 : \|X(x, y, t, \omega)\| \geq n\}. \end{aligned}$$

It is easy to see that $\eta^n(x, y, \omega)$ is jointly measurable in (x, y, ω) . Further, if we denote by $\eta(x, y, \omega)$ the explosion time for the solution $(X(x, y, t))$ of equation (2.1) starting at $x \in \mathbb{R}^d$, then $\eta(x, y, \omega) := \lim_{n \rightarrow \infty} \eta^n(x, y, \omega)$. As a consequence, $\eta(x, y, \omega)$ is a measurable function of $(x, y, \omega) \in \mathbb{R}^d \times \mathcal{S}_p \times \Omega$. The map X may now be defined as

$$\begin{aligned} X(x, y, t, \omega) &:= \lim_{n \rightarrow \infty} X^n(x, y, t, \omega) I_{\{t < \eta(x, y, \omega)\}}(x, y, t, \omega) \\ &\quad + \infty I_{\{t \geq \eta(x, y, \omega)\}}(x, y, t, \omega). \end{aligned}$$

Clearly for each $(x, y) \in \mathbb{R}^d \times \mathcal{S}_p$ we have, by the uniqueness of solutions, $X(x, y, t \wedge \eta^n) = X^n(x, y, t \wedge \eta^n)$. We define $X(x, y, t, \omega) = \infty$ for all (t, ω) if $x = \infty$ or $y = \delta$. \square .

We define the transition probability function $\bar{P}(x, y, t, A)$ of the diffusion $(X(x, y, t))$ in the usual way : For $0 \leq t \leq \infty, x \in \mathbb{R}^d, y \in \mathcal{S}_p$ and $A \in \mathcal{B}_d$,

$$\begin{aligned} \bar{P}(x, y, t, A) &:= P\{\omega : X(x, y, t, \omega) \in A\} \\ &= P\{X(x, y, t) \in A \cap \mathbb{R}^d, t < \eta(x, y)\} + I_A(\infty)P\{t \geq \eta(x, y)\}. \end{aligned}$$

Note that $t = \infty$ is included in the definition of $\bar{P}(x, y, t, A)$ by taking $X_\infty = \infty$. We take $\bar{P}(x, y, t, A) := I_A(\infty), t \geq 0$, if $x = \infty$. We can define an induced transition probability on \mathcal{S}_p as follows : First we extend the map $\tau_x(y)$. We define $\tau_\infty(y) := \delta, y \in \hat{\mathcal{S}}_p$ and $\tau_x(\delta) := \delta$. Thus $\tau_x : \hat{\mathcal{S}}_p \rightarrow \hat{\mathcal{S}}_p, x \in \mathbb{R}^d$. For $y \in \mathcal{S}_p, 0 \leq t \leq \infty$, and $B \in \mathcal{B}(\hat{\mathcal{S}}_p)$ define

$$P(y, t, B) := \bar{P}(0, y, t, \tau_x^{-1}(y)(B))$$

where $\tau_x^{-1}(y)(B) = \{z : \tau_z(y) \in B\}$. We take $P(y, t, B) := I_B(\delta), t \geq 0$, if $y = \delta$ and define $Y_t(y) = \delta$ if $t = \infty$ or $y = \delta$. We note that because $X(0, y, t) = z_t(y), t < \eta(0, y)$ we have

$$\begin{aligned} P(y, t, B) &= P\{X(0, y, t) \in \tau_x^{-1}(y)(B)\} \\ &= P\{X(0, y, t) \in \tau_x^{-1}(y)(B), t < \eta(0, y)\} \\ &\quad + P\{t \geq \eta(0, y)\}I_{\tau_x^{-1}(y)(B)}(\infty) \\ &= P\{\tau_{z_t(y)}(y) \in B, t < \eta(0, y)\} + P\{t \geq \eta(0, y)\}I_B(\delta) \\ &= P\{Y_t(y) \in B\} \end{aligned}$$

The strong Markov property for $(Y_t(y))$ is now a simple consequence of that for the process $(X(x, y, t))$.

Proposition 4.2 — Let $y \in \mathcal{S}_p$ and let T be an \mathcal{F}_t stopping time. Then for $0 \leq s \leq \infty$, and $B \in \mathcal{B}(\hat{\mathcal{S}}_p)$, we have, almost surely,

$$P\{Y_{s+T}(y) \in B | \mathcal{F}_T\} = P(s, Y_T(y), B).$$

PROOF : Since the result holds trivially at $s = \infty$, we assume $s < \infty$. We have, using the strong Markov property of the process $(X(x, y, t))$

$$\begin{aligned}
P\{Y_{s+T}(y) \in B | \mathcal{F}_T\} &= P\{\tau_{z_{s+T}(y)}(y) \in B | \mathcal{F}_T\} \\
&= P\{z_{s+T}(y) \in \tau^{-1}(y)(B) | \mathcal{F}_T\} \\
&= P\{X(0, y, s+T) \in \tau^{-1}(y)(B) | \mathcal{F}_T\} \\
&= \bar{P}(X(0, y, T), y, s, \tau^{-1}(y)(B)).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\bar{P}(x, y, s, \tau^{-1}(y)(B)) &= P\{X(x, y, s) \in \tau^{-1}(y)(B)\} \\
&= P\{x + z_s(\tau_x(y)) \in \tau^{-1}(y)(B), s < \eta(x, y)\} \\
&\quad + P\{s \geq \eta(x, y)\} I_{\tau^{-1}(y)(B)}(\infty) \\
&= P\{\tau_{x+z_s(\tau_x(y))}(y) \in B, s < \eta(x, y)\} \\
&\quad + P\{s \geq \eta(x, y)\} I_B(\delta) \\
&= P\{\tau_{z_s(\tau_x(y))}(\tau_x(y)) \in B, s < \eta(x, y)\} \\
&\quad + P\{s \geq \eta(x, y)\} I_B(\delta) \\
&= P(\tau_x(y), s, B)
\end{aligned}$$

where in the last equality we have made use of the observation made preceding the statement of the proposition and the fact that $Y_s(\tau_x(y)) = \tau_{z_s(\tau_x(y))}(\tau_x(y))$. Hence

$$\begin{aligned}
P\{Y_{s+T}(y) \in B | \mathcal{F}_T\} &= \bar{P}(X(0, y, T), y, s, \tau^{-1}(y)(B)) \\
&= P(\tau_x(y), s, B) |_{x=X(0, y, T)} \\
&= P(\tau_x(y), s, B) |_{x=z_T(y)} \\
&= P(Y_T(y), s, B).
\end{aligned}$$

This completes the proof of the Proposition. \square

Let $\mathbb{B}_p = \{f : \hat{\mathcal{S}}_p \rightarrow \mathbb{R}; f \text{ bounded and measurable, } f(\delta) = 0\}$. Let $\|\cdot\|_{p, \infty}$ denote the norm on \mathbb{B}_p given by $\|f\|_{p, \infty} := \sup_{y \in \hat{\mathcal{S}}_p} |f(y)|$. Then $(\mathbb{B}_p, \|\cdot\|_{p, \infty})$ is a

Banach space. Let $(T_t)_{0 \leq t < \infty}$ denote the linear operators $T_t : \mathbb{B}_p \rightarrow \mathbb{B}_p$ given by $T_t f(y) := E(f(Y_t(y)))$, for $y \in \mathcal{S}_p$ and $f \in \mathbb{B}_p$.

Corollary 4.3 — $(T_t)_{t \geq 0}$ is a semi-group of linear operators on \mathbb{B}_p with $T_0 = Identity$, $T_t 1 = 1$ and $T_t f \geq 0$ whenever $f \geq 0$.

Let $(\mathcal{L}, Dom(\mathcal{L}))$ denote the infinitesimal generator of (T_t) . Recall that $f \in \mathbb{B}_p$ belongs to $Dom(\mathcal{L})$ iff the limit of $\frac{1}{t}(T_t f - f)$ as t tends to zero exists in \mathbb{B}_p and further $\mathcal{L}(f) := \lim_{t \rightarrow 0} \frac{1}{t}(T_t f - f)$. We shall denote by (\bar{T}_t^y) and $\bar{\mathcal{L}}^y$ the semi-group and infinitesimal generator, respectively on \mathbb{B}_d and $Dom(\bar{\mathcal{L}}^y) \subset \mathbb{B}_d$, associated with the diffusion $(X(x, y, t))$ generated by $(\bar{\sigma}(x, y))$ and $\bar{b}(x, y)$. Here \mathbb{B}_d is the Banach space of bounded measurable functions on $\hat{\mathbb{R}}^d$ endowed with the supremum norm.

For $y \in \mathcal{S}_p$, let $C(y) \subset \mathcal{S}_p$ be defined by $C(y) := \{y' \in \mathcal{S}_p : y' = \tau_x(y), x \in \mathbb{R}^d\}$. $\hat{C}(y) := C(y) \cup \{\delta\}$. Then observe that $\mathcal{S}_p = \bigcup_{y \in \mathcal{S}_p} C(y)$ and for $y_1 \neq y_2$ either $C(y_1) \cap C(y_2) = \phi$ or $C(y_1) = C(y_2)$. We shall consider $C(y)$ as a measurable space with the σ -field induced by $\mathcal{B}(\hat{\mathcal{S}}_p)$. Let

$$\mathbb{B}_p(y) := \{f : \hat{C}(y) \rightarrow \mathbb{R}; f \text{ bounded and measurable, } f(\delta) = 0\}.$$

Since $P\{Y_t(y) \in C(y') \text{ for some } t \geq 0\} = 0$ if $C(y) \cap C(y') = \phi$, we have (the restriction) $T_t|_{\mathbb{B}_p(y)} : \mathbb{B}_p(y) \rightarrow \mathbb{B}_p(y)$ is a semi-group on $(\mathbb{B}_p(y), \|\cdot\|_{p, \infty})$, for every $y \in \mathcal{S}_p$. We shall continue to denote the restrictions of T_t and \mathcal{L} to $\mathbb{B}_p(y)$ and $\mathbb{B}_p(y) \cap Dom(\mathcal{L})$ respectively by (T_t) and \mathcal{L} or by T_t^y and \mathcal{L}^y if there is a risk of confusion.

Let $F : \mathbb{R}^{d+n} \rightarrow \mathbb{R}$, $F \in C^\infty(\mathbb{R}^{d+n})$ such that $support(F) \subset K \times \mathbb{R}^n$ for some compact $K \subset \mathbb{R}^d$. Let $\varphi_i \in \mathcal{S}, i = 1, \dots, d$. Fix $y \in \mathcal{S}_p$. Define $f \equiv f^y : \mathbb{R}^d \times \mathcal{S}_p \rightarrow \mathbb{R}$ as follows : If $y' = \tau_x(y)$ define

$$f^y(x, y') := F(x, \langle \varphi_1, \tau_x(y) \rangle, \dots, \langle \varphi_n, \tau_x(y) \rangle).$$

If $y' \notin C(y)$ we define $f^y(x, y') = 0$. Define $\bar{f}^y(x) := f^y(x, \tau_x(y)) : \mathbb{R}^d \rightarrow \mathbb{R}$ and $\bar{f}^y(\infty) = 0$. Let $\bar{\sigma}^t(x, y)$ denote the transpose of the matrix given by

$\bar{\sigma}(x, y) := (\bar{\sigma}_{ij}(x, y))$. We will denote by \bar{L}^y the second order differential operator in the variable x given as

$$\bar{L}^y := \sum_{i,j=1}^d (\bar{\sigma}(x, y) \bar{\sigma}^t(x, y))_{ij} \partial_{ij}^2 + \sum_{i=1}^d \bar{b}_i(x, y) \partial_i.$$

Proposition 4.4 — Let $y \in \mathcal{S}_p$. Then $\bar{f}^y(x) \in C_K^\infty(\mathbb{R}^d) \subset \text{Dom}(\bar{L})$. Consequently, for $x \in \mathbb{R}^d$, $f^y(x, \cdot) \in \text{Dom}(\mathcal{L})$ and

$$\mathcal{L}f^y(x, \tau_x(y)) = \mathcal{L}^y f^y(x, \tau_x(y)) = \bar{L}^y \bar{f}^y(x) = \bar{L}^y \bar{f}^y(x).$$

$\mathcal{L}f^y(x, y') = 0$ if $y' \notin C(y)$.

PROOF : It is clear that $\bar{f}^y \in C_K^\infty$. Further, using the compactness of K and Ito's formula it can be shown that $C_K^\infty \subset \text{Dom}(\bar{L})$. It is easily seen from the definitions that $T_t f^y(x, \tau_x(y)) = \bar{T}_t^y \bar{f}^y(x)$ and $T_t f^y(x, y') = 0, y' \notin C(y)$. In particular,

$$\frac{1}{t} \{T_t f^y(x, y') - f^y(x, y')\} = I_{\{\tau_x(y): x \in \mathbb{R}^d\}}(y') \frac{1}{t} \{\bar{T}_t^y \bar{f}^y(x) - \bar{f}^y(x)\}.$$

The result follows. □

Remark 4.5 : Let $F(x, z) := g(x)z$, where $g \in C_K^\infty, K = B(0, r)$, the ball of radius r centred at 0, and $g(x) = 1, x \in B(0, r_1)$ for some $r_1 < r$. Then for $y \in \mathcal{S}_p$ $f^y(x, y') = g(x)\langle \phi, y' \rangle, \phi \in \mathcal{S}, y' \in C(y); f^y(x, y') = 0, y' \notin C(y)$. It is easy to see that

$$\mathcal{L}f^y(0, y) = \bar{L}^y \bar{f}^y(0) = \langle \phi, L(y) \rangle$$

where L is the operator in equation (3.1).

Remark 4.6 : Assume $y \in \mathcal{S}_p$ is such that $\tau_x y \neq y$ for any $x \neq 0$. Let $j : \hat{C}(y) \rightarrow \hat{\mathbb{R}}^d$ be defined by $j(\tau_x(y)) := x$. Then j is one-one and onto and we provide $\hat{C}(y)$ with a topology and corresponding Borel structure \mathcal{B}_y that makes $j : (\hat{C}(y), \mathcal{B}_y) \rightarrow (\hat{\mathbb{R}}^d, \mathcal{B}_d)$ a Borel isomorphism with inverse $\tau_\cdot(y) : \hat{\mathbb{R}}^d \rightarrow \hat{C}(y)$. We can extend j as a map $j : \mathbb{B}_p(y) \rightarrow \mathbb{B}_d$ such that $\bar{f} \in \mathbb{B}_d$ iff $f \in \mathbb{B}_p(y)$, where

$f = \bar{f} \circ j$. This extends to semi groups viz. $T_t^y = \bar{T}_t^y \circ j$. In other words, the Markov process $(Y_t(y)) \equiv (\tau_{z_t(y)}(y))$ on the state space $\hat{C}(y)$ is ‘isomorphic’ to the Markov process $(X(x, y, t))$ on $\hat{\mathbb{R}}^d$. Note however that the topology on $\hat{C}(y)$ is that of $\hat{\mathbb{R}}^d$ and is different from the one induced from \hat{S}_p . Proposition 3.12 is a reflection of the same phenomenon.

5. A NON LINEAR EVOLUTION EQUATION

In this section we derive a non linear evolution equation associated with the operator L with the initial condition $y \in S_p$ viz.

$$\partial_t \psi(t, y) = \psi(t, L(y)) \quad ; \quad \psi(0, y) = y. \tag{5.1}$$

We construct solutions of (5.1) via what maybe called non linear convolutions, that we define below. This is also closely related to the notion of stochastic representation of solutions to evolution equations of the type (5.1). See [13, 14]. The solutions of equation (5.1) are also related to the solutions of the forward equation for the diffusion $(X(x, y, t))$ (see below, equation (5.2), Remark 5.5.) We recall from [13] that $\tau_x : S_p \rightarrow S_p$ are bounded linear operators for all $p \in \mathbb{R}$.

Definition 5.1 — Let $p \in \mathbb{R}$ and let $q \leq p$. Suppose $h : S_p \rightarrow S_q$ and $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be Borel measurable maps. For $y \in S_p$, the convolution $h(y) \circ f$ is defined to be the element of S_q given by the Bochner integral $h(y) \circ f := \int_{\mathbb{R}^d} h(\tau_x y) f(x) dx$ provided the integral exists i.e. provided $\int_{\mathbb{R}^d} \|h(\tau_x(y))\|_q |f(x)| dx < \infty$. More generally, let μ be a finite measure on the Borel sigma field of \mathbb{R}^d and $h(y)$ be as above. The convolution $h(y) \circ \mu$ is defined as $h(y) \circ \mu := \int_{\mathbb{R}^d} h(\tau_x y) \mu(dx)$ provided $\int_{\mathbb{R}^d} \|h(\tau_x(y))\|_q \mu(dx) < \infty$.

Remark 5.2 : Our notation is a compromise between two contrasting interpretations of the above definition. We could interpret the above definition as an extension of the notion of convolution of a functions $h : \mathbb{R}^d \rightarrow \mathbb{R}$ and a finite measure μ on \mathbb{R}^d , to that of convolution of μ and a map $h : S_p \rightarrow S_q$. The notation then would be $h \circ \mu(y)$, where $h \circ \mu : S_p \rightarrow S_q$. The definition also affords

an interpretation as an extension of the notion of convolution of a tempered distribution y and a measure μ via the map h . We may view it then as a non linear convolution between y and μ where the nonlinearity arises because of the map h . An appropriate notation then could be $y \circ_h f$ or $y \star_h f$. In any case, our definition of convolution reduces to the usual convolution between two distributions y and μ , if we take $q = p$ and $h(y) = y$ in Definition 5.1. We will use the notation $y \star \mu$ for the usual convolution between the distribution y and the measure μ . We further note that when h is non linear as in Definition 5.1, $h(y) \circ \mu$ is, in general, different from the ordinary convolution $h(y) \star \mu$, between the distribution $h(y)$ and the measure μ . In our application of the notion of convolution to construct solutions of equation (5.1) however, there is an additional source of non-linearity viz. μ would also depend on y .

Definition 5.3 — Let $y \in \mathcal{S}_p$. We say that a continuous map $\psi(\cdot, y) : [0, \infty) \rightarrow \mathcal{S}_p$, is a solution by convolution of the initial value problem (5.1) iff there exists kernels $\mu(t, dx)$ on $[0, \infty) \times \mathbb{R}^d$ such that

1. $\int_{\mathbb{R}^d} \|\tau_x(y)\|_p \mu(t, dx) < \infty, t \geq 0$, and $\int_{\mathbb{R}^d} \|\phi(x)\tau_x(y)\|_p \mu(t, dx) < \infty$, for all $t \geq 0$, and for all ϕ , where $\phi(x) = (\langle \sigma, \tau_x(y) \rangle \langle \sigma, \tau_x(y) \rangle^t)_{ij}, i, j = 1, \dots, d$, or $\phi(x) = \langle b_i, \tau_x(y) \rangle, i = 1, \dots, d$. In particular if $L : \mathcal{S}_p \rightarrow \mathcal{S}_{p-1}$ is as in equation (3.1), then the convolution $L(y) \circ \mu(t, \cdot)$ exists in \mathcal{S}_{p-1} for all $t \geq 0$.

2. $\psi(t, y) = y \circ \mu_t, t \geq 0$

3. $\psi(t, y)$ is continuously differentiable for $t \in (0, \infty)$ and we have

$$\partial_t \psi(t, y) = \partial_t (y \circ \mu_t) = L(y) \circ \mu(t, \cdot) \quad ; \quad \psi(0, y) = y \circ \mu_0 = y.$$

Recall from Section 4, the transition probability measure for the solutions $(Y_t(y))$ of equation (3.1) viz. $P(y, t, B)$ and the transition probability measure for the process $(z_t(y))$ given by Corollary 3.8 viz. $\bar{P}(0, y, t, A)$. We recall that $P(y, t, B) = \bar{P}(0, y, t, \tau^{-1}(y)(B))$. The following theorem constructs the solutions of the initial value problem equation (5.1) via a stochastic representation.

Theorem 5.4 — Let σ_{ij}, b_i, y be as in Theorem 3.4 and let $\{(Y_t(y)), \eta\}$ be the unique S_p valued solution of equation (3.1) given by Theorem 3.11. Let $\bar{\sigma}_{ij}(x, y) := \langle \sigma_{ij}, \tau_x(y) \rangle, \bar{b}_i(x, y) := \langle b_i, \tau_x(y) \rangle, i, j = 1, \dots, d$ and suppose that for fixed y , these are bounded and continuous functions of x . Let $(z_t(y))$ be the unique solution of equation (2.1), as in Theorem 3.11. Then $\psi(t, y) := E(Y_t(y)), t \geq 0$ defines an S_p valued continuous map that solves the initial value problem (5.1) by convolution.

PROOF : We note that under the assumptions on $\bar{\sigma}_{ij}(\cdot, y), \bar{b}_i(\cdot, y), (z_t(y))$ has moments of all orders and further for all $t \geq 0$,

$$\sup_{s \leq t} E|z_s(y)|^k < C(t, y, k) < \infty \quad k = 1, 2, \dots .$$

In particular, $\eta = \infty, a.s.$ Note that $\psi(t, y)$ is well defined : for $z \in \mathbb{R}^d$,

$$\|\tau_z y\|_p \leq \|y\|_p P(|z|)$$

where $P(x)$ is a polynomial in $x \in \mathbb{R}$ of degree $2|p| + 1$ (see [13]). It follows from our assumption on the moments of (z_t) that

$$E\|Y_t\|_p \leq \|y\|_p EP(|z_t|) < \infty$$

for every $t \geq 0$. In particular $\psi(t, y) := EY_t(y)$ exists as a Bochner integral in S_p . Further it is clear that $\psi(t, y) = y \circ \bar{P}(0, y, t, \cdot)$. The theorem is proved by taking expectations in equation (3.2) satisfied by $(Y_t(y))$. We first show that this is indeed a legitimate operation.

Using the fact that the moments of $z_s := z_s(y)$ are finite, we have for each $t \geq 0$,

$$\begin{aligned} E\left\| \int_0^t L(\tau_{z_s} y) ds \right\|_{p-1} &\leq E \int_0^t \|L(\tau_{z_s} y)\|_{p-1} ds \\ &\leq C' E \int_0^t \{ \|\tau_{z_s} y\|_p^3 + \|\tau_{z_s} y\|_p^2 \} ds \\ &\leq CE \int_0^t \{ P(|z_s|)^3 + P(|z_s|)^2 \} ds \\ &< \infty. \end{aligned}$$

where $C' = C'(d, \|\sigma_{i,j}\|_{-p}, \|b_i\|_{-p}, i, j = 1 \cdots d)$ and $C = C(d, \|\sigma_{i,j}\|_{-p}, \|b_i\|_{-p}, i, j = 1 \cdots d, \|y\|_p)$ are positive constants depending on the indicated quantities. A similar calculation verifies that for each $t \geq 0$

$$\begin{aligned} E \left\| \sum_{j=1}^d \int_0^t A_j(\tau_{z_s} y) dB_s^j \right\|_{p-1}^2 &\leq \sum_{j=1}^d E \int_0^t \|A_j(\tau_{z_s} y)\|_{p-1}^2 ds \\ &\leq CE \int_0^t \{P(|z_s|)^4\} ds \\ &< \infty \end{aligned}$$

We can thus take expectations in equation (3.2) to get

$$\begin{aligned} \psi(t, y) = EY_t &= y + E \int_0^t L(\tau_{z_s} y) ds \\ &= y + \int_0^t \int_{\mathbb{R}^d} L(\tau_x y) \bar{P}(0, y, s, dx) ds \\ &= y + \int_0^t L(y) \circ \bar{P}(0, y, s, \cdot) ds \\ &= y + \int_0^t E(L(Y_s(y))) ds \end{aligned}$$

That $\psi(t, y)$ is continuously differentiable and satisfies equation (5.1) now follows from the above equation and the continuity of $E(L(Y_s(y)))$. This completes the proof of Theorem 5.4. \square

For $\phi, y \in \mathcal{S}'$ such that the products $\bar{\sigma}_{ij}(\cdot, y)\phi$, $\bar{b}_i(\cdot, y)\phi$, $i, j = 1, \cdots, d$ are tempered distributions in the variable x , we define the operator $\bar{L}^{*,y}$ as follows :

$$\bar{L}^{*,y} \phi := \frac{1}{2} \sum_{i,j=1}^d \partial_{ij}^2 ((\bar{\sigma}(\cdot, y) \bar{\sigma}^t(\cdot, y))_{ij} \phi) - \sum_{i=1}^d \partial_i (\bar{b}_i(\cdot, y) \phi).$$

We note that $\bar{L}^{*,y}$ is the formal adjoint of the second order differential operator \bar{L} associated with the diffusion with coefficients $\bar{\sigma}_{ij}, \bar{b}_i, i, j = 1, \cdots, d$. The initial

value problem (5.1) is closely connected with solutions of the forward Kolmogorov equation for \bar{L} viz.

$$\partial_t \psi_t = \bar{L}^{*,y} \psi_t \quad ; \quad \psi_0 = \psi \tag{5.2}$$

When ψ is a distribution with compact support and $\bar{\sigma}_{ij}, \bar{b}_i$ are smooth, solutions to the above equation may be obtained by convolution with the transition probability measure $\bar{P}(0, y, t, \cdot)$ (see [14], Theorem 4.5.). We extend that result in Theorem 5.6 when the coefficients are only bounded and continuous.

Remark 5.5 : To see the connection between solutions of equation (5.1) and solutions of equation (5.2), consider the following integrated version of equation (5.2) with $\psi_t = \bar{P}(0, y, t, \cdot)$ viz.

$$\bar{P}(0, y, t, \cdot) = \delta_0 + \int_0^t \bar{L}^* \bar{P}(0, y, s, \cdot) ds \tag{5.3}$$

By convolving with $y \in \mathcal{S}_p$ and using the relation

$$y \star \bar{L}^{*,y} \bar{P}(0, y, t) = L(y) \circ \bar{P}(0, y, t, \cdot)$$

we get ,

$$y \circ \bar{P}(0, y, t, \cdot) = y + \int_0^t L(y) \circ \bar{P}(0, y, s, \cdot) ds$$

which is equivalent to equation (5.1). On the other hand we can Fourier transform the above to get back (5.3): Suppose y has compact support. For a tempered distribution ϕ , the Fourier transform of ϕ is denoted by $\hat{\phi}$. For each $\xi \in \mathbb{R}^d$,

$$\begin{aligned} \hat{y}(\xi) \bar{P}(0, y, t, \cdot) \hat{y}(\xi) &= \hat{y}(\xi) + \int_0^t (L(y) \circ \bar{P}(0, y, s, \cdot)) \hat{y}(\xi) ds \\ &= \hat{y}(\xi) + \int_0^t (\bar{L}^* \bar{P}(0, y, s, \cdot)) \hat{y}(\xi) ds \end{aligned}$$

Since y has compact support, $\hat{y}(\xi)$, is by the Paley-Wiener theorem an entire function and hence we may cancel off $\hat{y}(\xi)$ in the above equation to get, almost surely with respect to Lebesgue measure on \mathbb{R}^d ,

$$\bar{P}(0, y, t, \cdot) \hat{y}(\xi) = 1 + \int_0^t (\bar{L}^* \bar{P}(0, y, s, \cdot)) \hat{y}(\xi) ds$$

Inverting the Fourier transform , we get back equation (5.3).

Theorem 5.6 — *Suppose that $y \in \mathcal{S}_p$, $\sigma_{ij}, b_i \in \mathcal{S}_{-p}$, $i, j = 1, \dots, d$ and in addition are such that $\bar{\sigma}_{ij}(\cdot, y), \bar{b}_i(\cdot, y)$ are bounded continuous functions. Let $q > \frac{d}{4}$. Then for any $x \in \mathbb{R}^d$, the map $t \rightarrow \bar{P}(x, y, t, \cdot) : [0, \infty) \rightarrow \mathcal{S}_{-q}$ is differentiable and satisfies the forward equation (5.2) with $\psi = \delta_x$.*

PROOF : Let $(X(x, y, t))$ be the unique solution of (2.1). Then by our assumptions on $\bar{\sigma}_{ij}, \bar{b}_i$, the moments of all orders of $(X(x, y, t))$ exist and are locally bounded functions of t . Further since $\delta_z \in \mathcal{S}_{-q}$ if and only if $q > \frac{d}{4}$ (see [14]), we have $\delta_{X(x, y, t)} \in \mathcal{S}_{-q}$. In particular, as in the proof of the previous theorem, $E\|\delta_{X(x, y, t)}\|_{-q} < \infty$. It follows that $\bar{P}(x, y, t, \cdot) = E\delta_{X(x, y, t)}$, where the right hand side is an element of \mathcal{S}_{-q} and the equality holds there. Using the Ito formula in [12] we get,

$$\delta_{X(x, y, t)} = \delta_x + \int_0^t \bar{L}(s)(\delta_{X(x, y, s)}) ds + \int_0^t \bar{A}(s)(\delta_{X(x, y, s)}) \cdot dB_s \quad (5.4)$$

where the operator valued processes $(\bar{L}(s, \omega))$ and $(\bar{A}(s, \omega))$ are defined for fixed x and y as follows. $\bar{A}(s, \omega) := (\bar{A}_1(s, \omega), \dots, \bar{A}_d(s, \omega))$ and for $\phi \in \mathcal{S}'$,

$$\bar{A}_i(s, \omega)\phi := - \sum_{j=1}^d \bar{\sigma}_{ji}(X(x, y, s, \omega), y) \partial_j \phi.$$

Similarly

$$\begin{aligned} \bar{L}(s, \omega)\phi &:= \frac{1}{2} \sum_{i, j=1}^d (\bar{\sigma}(X(x, y, s, \omega), y) \bar{\sigma}^t(X(x, y, s, \omega), y))_{ij} \partial_{ij}^2 \phi \\ &\quad - \sum_{i=1}^d \bar{b}_i(X(x, y, s, \omega), y) \partial_i \phi. \end{aligned}$$

As in the proof of Theorem 5.4, we can take expectations in equation (5.4) to get

$$\bar{P}(x, y, t, \cdot) = \delta_x + \int_0^t E(\bar{L}(s)\delta_{X(x, y, s)}) ds. \quad (5.5)$$

Note that we have the representation $\bar{P}(x, y, t, \cdot) = \int_{\mathbb{R}^d} \delta_z(\cdot) \bar{P}(x, y, t, dz)$, where the right hand side is a Bochner integral in \mathcal{S}_{-q} . Given a bounded continuous function ϕ we can use this representation to define the product $\phi \bar{P}(x, y, t, \cdot)$ as an element of \mathcal{S}_{-q} as follows :

$$\phi \bar{P}(x, y, t, \cdot) = \int_{\mathbb{R}^d} \phi(z) \delta_z(\cdot) \bar{P}(x, y, t, dz)$$

where the right hand side is a Bochner integral in \mathcal{S}_{-q} . Hence $\bar{L}^* \bar{P}(x, y, t, \cdot)$ is a well defined tempered distribution in \mathcal{S}_{-q-1} . It is now easy to see, by acting on test functions, that $\bar{L}^{*,y} \bar{P}(x, y, t, \cdot) = E(\bar{L}(t) \delta_{X(x,y,t)})$. In particular from (5.5) we get

$$\bar{P}(x, y, t, \cdot) = \delta_x + \int_0^t \bar{L}^* \bar{P}(x, y, s, \cdot) ds. \quad (5.6)$$

Since the moments of $X(x, y, t)$ are locally bounded functions of t , it follows using the dominated convergence theorem that the integrand in the right hand side in equation (5.6) is continuous in t . The desired conclusion now follows from equation (5.6). \square

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